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Short Communication

The Rayleigh–Plesset equation in terms of volume with explicit shear losses

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Abstract

The most common nonlinear equation of motion for the damped pulsation of a spherical gas bubble in an infinite body of liquid is the Rayleigh–Plesset equation, expressed in terms of the dependency of the bubble radius on the conditions pertaining in the gas and liquid (the so-called 'radius frame'). However over the past few decades several important analyses have been based on a heuristically derived small-amplitude expansion of the Rayleigh–Plesset equation which considers the bubble volume, instead of the radius, as the parameter of interest, and for which the dissipation term is not derived from first principles. So common is the use of this equation in some fields that the inherent differences between it and the 'radius frame' Rayleigh–Plesset equation are not emphasised, and it is important in comparing the results of the two equations to understand that they differ both in terms of damping, and in the extent to which they neglect higher order terms. This paper highlights these differences. Furthermore, it derives a 'volume frame' version of the Rayleigh–Plesset equation which contains exactly the same basic physics for dissipation, and retains terms to the same high order, as does the 'radius frame' Rayleigh–Plesset equation. Use of this equation will allow like-with-like comparisons between predictions in the two frames.

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1. Introduction

The most popular nonlinear equation for describing the nonlinear response of a gas bubble in liquid to a driving pressure field is the Rayleigh–Plesset equation. This can be derived from first principles using the bubble radius R as the dynamic parameter (which will here be termed the 'radius frame' approach):

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(p_{\mathrm{L}} - \frac{4\eta\dot{R}}{R} - p_\infty\right) \quad (1)$$

where ρ_0 is the unperturbed liquid density, η is the shear viscosity of the liquid, and p_∞ is the liquid pressure far from the bubble, which is here assumed to consist of a static pressure p_0 and an applied acoustic field $P(t)$, such that $p_\infty = p_0 + P(t)$ [1]. When a polytropic gas law is used to evaluate the liquid pressure at the bubble wall (p_{L}), and the contributions of surface tension (σ) and vapour pressure (p_{v}) are included, Eq. (1) becomes

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(\left(p_0 + \frac{2\sigma}{R_0} - p_{\mathrm{v}}\right)\left(\frac{R_0}{R}\right)^{3\kappa} + p_{\mathrm{v}} - \frac{2\sigma}{R} - \frac{4\eta\dot{R}}{R} - p_0 - P(t)\right) \quad (2)$$

where R_0 is the unperturbed bubble radius. It is noted that use of the polytropic index (κ) adjusts the gas stiffness for reversible heat flow across the bubble wall, but does not describe any net thermal losses. The only dissipation present in (2) occurs through viscous losses.

However there exist heuristic formulations based on a form of the Rayleigh–Plesset equation in which the bubble

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volume V is used as the dynamic parameter (which will be termed the 'volume frame' approach), where the damping is not derived from first principles. Furthermore, the 'volume frame' form of the damped Rayleigh–Plesset equation which is commonly quoted neglects higher order terms which are present in the 'radius frame' version (2). Therefore the two equations are not equivalent on two counts.

The predictions of the two approaches do not always agree, and before any differences can be attributed to another factor, it is important to ensure that one is comparing 'like-with-like'. It is not immediately apparent that this is being done, given the differences in the way dissipation is formulated, and the manner in which higher order terms are neglected in the 'volume frame' form which is widely used [2]. Therefore this study was undertaken to derive a 'volume frame' form of the Rayleigh–Plesset equation where the physics describing the dissipation is identical to that used when the Rayleigh–Plesset equation is cited in the radius frame (2), and where the higher order terms have not been assumed to be negligible.

This paper will proceed by using the following common assumptions: The bubble exists in an infinite medium. The bubble stays spherical at all times during the pulsation. Spatially uniform conditions exist within the bubble. The bubble radius is much smaller than the wavelength of the driving sound field. There are no body forces acting (e.g. gravity). Bulk viscous effects can be ignored. The density of the surrounding fluid is much greater than that of the internal gas. The gas content is constant.

2. Background

Of the various 'volume frame' equations for bubble dynamics [3], the form given by Zabolotskaya and Soluyan [2] has been most valuable and influential, and featured as the starting point in several notable studies. These include the bubble-mediated generation of difference frequencies when bubbles are insonified by two acoustic frequencies for a range of purposes, including bubble detection [4], the use of bubbles to enhance parametric sonar [5,6], and the acoustic characterization of gassy seabeds [7]. Biomedical investigations which have used the 'volume frame' include studies of contrast agent [8] and HIFU [9]. If the predictions of these important 'volume frame' studies are to be reconciled with those obtained using the 'radius frame' Rayleigh–Plesset Eq. (2), it is important to ensure that the comparison is of 'like-with-like', specifically that the equations of motion in each case contain the same physics and the same degree of approximation. This is the purpose of this paper.

The influential analysis of Zabolotskaya and Soluyan [2], which underpins the majority of studies of nonlinear bubble dynamics in the 'volume frame', begins with a statement (not derived) of the Rayleigh equation in the volume frame. The Rayleigh equation is the undamped form of the Rayleigh–Plesset equation, and the volume frame description given by Zabolotskaya and Soluyan [2] is

$$\left(\frac{\ddot{V}}{V^{1/3}} - \frac{\dot{V}^2}{6V^{4/3}}\right) = \frac{4\pi}{\rho_0}\left(\frac{3}{4\pi}\right)^{1/3}(p_g - p_\infty) \tag{3}$$

where p_g is the pressure in the bubble gas (assumed to be air in [2]). Understandably for the time, given the limited computing abilities then available, Zabolotskaya and Soluyan do not calculate output from this equation directly, but rather proceeded to generate a small amplitude expansion based on volume perturbations $V_\varepsilon(t)$ about an equilibrium bubble volume V_0

$$V = V_0 + V_\varepsilon(t) \quad V_\varepsilon \ll V_0 \tag{4}$$

with an adiabatic gas law

$$p_g = p_0(V_0/V)^\gamma \tag{5}$$

where γ is the ratio of the specific heat capacity of the gas at constant pressure, to its value at constant volume. The effects of surface tension and vapour pressure are neglected. This expansion generated the following expression

$$\ddot{V}_\varepsilon + \omega_M^2 V_\varepsilon - \alpha_{ZS} V_\varepsilon^2 - \beta_{ZS}(2\ddot{V}_\varepsilon V_\varepsilon + \dot{V}_\varepsilon^2) + F_{ZS}\dot{V}_\varepsilon = -\left(\frac{4\pi R_0}{\rho_0}\right)P(t) \tag{6}$$

where $V_\varepsilon(t)$ is the perturbation in bubble volume, and where

$$\omega_M = \sqrt{\frac{3\gamma p_0}{\rho_0 R_0^2}} \tag{7}$$

is the Minnaert frequency of the bubble and the parameters α_{ZS} and β_{ZS} represent the following groupings

$$\begin{aligned} \alpha_{ZS} &= 3\beta_{ZS}(\gamma + 1)\omega_M^2 \\ \beta_{ZS} &= 1/(8\pi R_0^3) \end{aligned} \tag{8}$$

The term F_{ZS} was introduced in an *ad hoc* fashion to include dissipation. It was assumed to be frequency dependent.

The achievement of Zabolotskaya and Soluyan in generating this analysis should not be underestimated. Its timing perceptively heralded and facilitated a wealth of investigations which employed their findings (ranging from biomedical therapy to seabed exploration [4–9]), yet did so in a way which provided equations that were appropriate not only for the computing power of the day, but also over the decades that followed. Furthermore, this analysis provided a framework in which the physical influences of the various terms are transparent.

Over the thirty years and more since Eq. (6) was published by Zabolotskaya and Soluyan, its popularity has increased. It is now important to revisit the assumptions inherent in the formulation, and ask whether the assumptions required for its derivation in 1973 are still necessary, given increased computing power, and to highlight the implications of the continued use of those assumptions. This is particularly so in light of two issues, both of which relate to the impression which can be given that Eq. (6) is an equivalent 'volume frame' expression to the familiar 'radius frame' Rayleigh–Plesset Eq. (2).

The first is an issue of terminology. Over these thirty years, Eq. (6) has colloquially come to be known as 'the Rayleigh–Plesset equation in terms of the bubble volume'; or, if F_{ZS} is set equal to zero to produce the form without dissipation, then it is sometimes referred to as 'Rayleigh's equation in terms of the bubble volume'. Both descriptions are imprecise: Eq. (6) has within it an implicit assumption of small-amplitude perturbations, with an accompanying neglect of some terms in the expansion, and in listing the assumptions inherent in Eq. (6), this fact should not be overlooked. Moreover, when such expansions are applied to calculations based on input parameters given for specific sound fields and bubbles, either in the radius frame or the volume frame, their validity should be addressed: pulsation amplitudes may be great enough to make significant some of the terms neglected in small amplitude expansions. Furthermore, the tolerances required when computing output from the differential equations need to be critically examined. This is particularly important when calculations from the differential equation are made in the volume frame, since the $V \alpha R^3$ relationship means that $\mathrm{d}V \sim (3V/R)\mathrm{d}R$. As a result, unless normalised variables are used, insufficient care with precision can lead to numerical errors if, for example, pressures of 10^6 Pa and volumes of 10^{-19} m^3 are calculated together by a single routine.

The second issue relates to the fact that the physics of the dissipation in Eq. (6) is different from that in the 'radius frame' Rayleigh–Plesset equation. In Eq. (6), the description of dissipation is not derived from the basic physics, but rather must be fitted to experiment or some other model in order to determine its form. This creates particular difficulties. If for example the parameter is to be fitted to another model, then both models must be comparing like-with-like, in that the dissipation in the second model must be proportional to $\dot{V}$ only, when expressed in the form of Eq. (6). If instead the parameter is to be determined by fitting to experiment, then it must be recognised that there can be many forms of loss contributing to damping in the experiment, and the fit will try to encompass all of these, again with the assumption that they will be proportional to $\dot{V}$ when expressed in the form of Eq. (6).

This paper derives a 'volume frame' form of the Rayleigh Plesset equation from first principles, using the same dissipation physics as is present in the familiar 'radius frame' form of the Rayleigh–Plesset Eq. (2). It should be noted that a series of correspondence has addressed the different predictions for sound radiation by the bubble, as generated by the Rayleigh–Plesset equation in the radius and volume frames [10–14]. This is distinct from the current paper, which addresses differences in the predictions of the underlying bubble pulsation, and the importance of appropriately describing the nonlinearities, before any consideration of the resulting radiation.

3. Derivation of the Rayleigh–Plesset equation in terms of the bubble volume

In the following derivation, the use of the dot notation indicates the use of the material derivative [1, Section 2.2.2], i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \tag{9}$$

where $\vec{u}$ is the liquid particle velocity. For the discussion of the pulsation of a single bubble whose centre remains fixed in space, as occurs in this paper, the convective term (the second term on the right) is zero.

Consider a spherical gas bubble which pulsates in an incompressible liquid as a result of an insonifying field (the long wavelength limit being assumed). The fluid velocity $u(r,t)$ falls off as an inverse square law with range r as a result of the assumption of liquid incompressibility. The mass of liquid which moves at the bubble wall, in time Δt, is $\rho_0 \dot{V}(t)\Delta t$, whilst that at range r is $\rho_0 4\pi r^2 u(t)\Delta t$, which implies that:

$$u(r,t) = \frac{\dot{V}(t)}{4\pi r^2} \tag{10}$$

where the bubble has volume $V(t)$ and wall volume velocity $\dot{V}(t)$.

Within the body of the liquid, Bernoulli's equation follows from the integration of a suitably reduced form of the Navier Stokes equation [1, Section 4.2.1b] to relate the pressure at a boundary within the fluid (p') to the velocity potential (Φ) and velocity:

$$\frac{p'}{\rho_0} = \frac{p_\infty}{\rho_0} - \frac{\partial \Phi}{\partial t} - \frac{u^2}{2} \tag{11}$$

An appropriate velocity potential can be calculated from (10) using the relevant boundary conditions and the definition $\vec{u} = \vec{\nabla}\Phi$:

$$\Phi(r,t) = -\frac{\dot{V}(t)}{4\pi r} \tag{12}$$

When this is substituted with (10) into (11), the latter can be evaluated at the bubble wall ($r = R$):

$$\begin{aligned} \frac{p'}{\rho_0} &= \frac{p_\infty}{\rho_0} - \left.\frac{\partial \Phi(r,t)}{\partial t}\right|_{r=R} - \frac{1}{2}\left(\left.\frac{\dot{V}(t)}{4\pi r^2}\right|_{r=R}\right)^2 \Rightarrow \\ \frac{p'}{\rho_0} &= \frac{p_\infty}{\rho_0} + \frac{\ddot{V}}{4\pi R} - \frac{\dot{V}^2}{2(4\pi R^2)^2} \end{aligned} \tag{13}$$

Elimination of p' can be achieved by neglecting all forms of loss except viscous losses (an assumption which is often not valid). An expression for the liquid pressure can be obtained by dynamically matching normal stresses across at the bubble wall [1, Section 4.2.1b, 15]. Finite shear modifies the liquid pressure at the bubble wall (p_L) such that it differs from the pressure at a boundary within the fluid (p') by an amount proportional to the principle rate of strain in the radial direction ($\varepsilon_r' = \partial u/\partial r$) as follows [1, Section 4.2.1b, Eq. (4.73)]:

$$p_L = p' - 2\eta\varepsilon_r' = p' - 2\eta\frac{\partial u}{\partial r} \tag{14}$$

This can readily be converted into a form relevant to the volume frame, using the incompressibility relation (10):

$$\frac{\partial u(r,t)}{\partial r} = -\frac{\dot{V}}{2\pi r^3} \tag{15}$$

Substitution of (15) into (14) gives the pressure in the liquid at the bubble wall

$$p_L = p' + \frac{\eta\dot{V}}{\pi R^3} \tag{16}$$

By combining this expression with (13), it is possible to derive a 'volume frame' description that is equivalent (in terms of the physics of dissipation and the content at higher order) to the 'radius frame' Rayleigh–Plesset Eq. (2).

$$\begin{aligned} &\frac{\ddot{V}}{4\pi R} - \frac{\dot{V}^2}{2(4\pi R^2)^2} = \frac{1}{\rho_0}\left(p_L - p_\infty - \frac{4\eta\dot{V}}{3V}\right) \Rightarrow \\ &\frac{1}{4\pi}\left(\frac{4\pi}{3V}\right)^{1/3}\left(\ddot{V} - \frac{\dot{V}^2}{6V}\right) = \frac{1}{\rho_0}\left(p_L - p_\infty - \frac{4\eta\dot{V}}{3V}\right) \end{aligned} \tag{17}$$

Evaluation of p_L for use in the equations of motion then becomes a question of calculating p_i, the pressure inside the bubble. This can be expressed in terms of the sum of the gas (p_g) and vapour (p_v) pressures, correcting for the Laplace pressure introduced through the effect of surface tension (p_σ):

$$\begin{aligned} &p_i = p_g + p_v = p_L + p_\sigma \Rightarrow \\ &p_L = p_g + p_v - p_\sigma \end{aligned} \tag{18}$$

where

$$p_\sigma = 2\sigma\left(\frac{4\pi}{3V}\right)^{1/3} \tag{19}$$

Evaluation of (18) when the bubble is at equilibrium size, and the pressures take values appropriate for that size in the absence of any driving pressure ($R = R_0$; $V = V_0$; $p_i = p_{i,e}$; $p_g = p_{g,e}$; see [1, Section 2.1]), gives

$$p_{i,e} = p_{g,e} + p_v = p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} \tag{20}$$

By far the most common way of calculating p_g is to appeal to a polytropic law. It involves calculating the pressure in the gas at a given bubble size by comparing it with the pressure at equilibrium. In this way, when the bubble has volume V, then from (20) the pressure in the gas will be

$$p_g = p_{g,e}\left(\frac{V_0}{V}\right)^\kappa = \left(p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} - p_v\right)\left(\frac{V_0}{V}\right)^\kappa \tag{21}$$

Substitution of (21) into (18) gives

$$\begin{aligned} p_L &= p_g + p_v - p_\sigma \\ &= \left(p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} - p_v\right)\left(\frac{V_0}{V}\right)^\kappa + p_v - 2\sigma\left(\frac{4\pi}{3V}\right)^{1/3} \end{aligned} \tag{22}$$

Substitution of this expression for p_L into (17) gives

$$\begin{aligned} \left(\ddot{V} - \frac{\dot{V}^2}{6V}\right) = \frac{4\pi}{\rho_0}\left(\frac{3V}{4\pi}\right)^{1/3}&\left(\left(p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} - p_v\right)\right. \\ &\left.\times\left(\frac{V_0}{V}\right)^\kappa + p_v - 2\sigma\left(\frac{4\pi}{3V}\right)^{1/3} - p_0 - P(t)\right) \\ &- \left(\frac{4\pi}{\rho_0}\left(\frac{3V}{4\pi}\right)^{1/3}\frac{4\eta\dot{V}}{3V}\right) \end{aligned} \tag{23}$$

which is the Rayleigh–Plesset equation in terms of the bubble volume, including viscous damping.

4. Discussion

It has been possible to derive a form of the Rayleigh–Plesset equation from first principles in the volume frame which incorporates viscous damping. This has been done using the Navier Stokes approach to generate Eq. (23). Other approaches are possible [16].

However the form of Eq. (23) creates a dilemma. If the viscosity term in Eq. (23) is neglected, then the undamped result is the Rayleigh equation in the 'volume frame', and this agrees with Eq. (3). However (noting that $\ddot{V}_\varepsilon = \ddot{V}$ and $\dot{V}_\varepsilon = \dot{V}$), the form with viscous damping given in (23) differs from the small-amplitude expansion of Zablotskaya and Soluyan [2] (Eq. (6)) which has since sometimes been taken to represent the damped form (the Rayleigh–Plesset equation) in the volume frame. When the equation of motion is expressed in the form $\ddot{V} + \ldots$ as in Eq. (23), the relevant damping term (the final bracketed term) is proportional to $\dot{V}V^{-2/3}$, not $\dot{V}$. Of course it may be argued that the parameter F_{ZS} can incorporate a $V_0^{-2/3}$ dependency, but this is not exactly the same as having a time-varying $V^{-2/3}$ dependency, and amounts to the use of a small-amplitude expansion (i.e. $V^{-2/3} \approx V_0^{-2/3}$) with the corresponding assumption that certain terms may be neglected. Such assumptions have physical implications (for example, if in place of $\dot{V} = \mathrm{d}(4\pi R^3/3)\mathrm{d}t$ the approximation $\dot{V} \approx 4\pi R_0^2\dot{R}$ is used, the bubble will be modelled as a rigid pulsator [1, Section 3.3.1] and the contribution to radiation of the oscillating gas pressure within the bubble will be ignored). Such assumptions, though they may be transparent in the original derivation, are sometimes understated and untested when the equations are used years later to calculate numerical output by other authors. In this study, Eq. (23) will allow the quantification of such effects when the traditional volume frame Eq. (6) is used.

Of course the Rayleigh–Plesset equation in either frame ((2) or (23)) only incorporates shear losses from first principles. There have been attempts artificially to enhance the viscosity to include radiation and thermal losses [17–20]. However these cannot give results that are as satisfactory as the inclusion of radiation losses by deriving the equation of motion to first-order Mach number in terms of the speed of the bubble wall [21–26]; and the inclusion of thermal effects by combining the continuity and energy relations for a perfect gas with spatially uniform pressure to provide an exact expression for the velocity field in terms of the temperature gradient [18,27–33]. The *ad hoc* manner of including dissipation as an adjustable parameter offers a way of incorporating all mechanisms of loss that are apparent in the data to which the fit is being made, but there is a limitation which must be borne in mind. The term which incorporates damping in this heuristic approach, if the equation of motion is expressed in the form $\ddot{V} + \ldots$, varies as $F_{ZS}\dot{V}$. The *ad hoc* approach would only produce a truly valid fit if all the significant forms of damping varied as $F_{ZS}\dot{V}$. As Eq. (23) shows, the shear component alone is proportional to $\dot{V}V^{-2/3}$.

5. Conclusions

By providing a form of the Rayleigh–Plesset equation in the volume frame (23) which encompasses exactly the same physics as the familiar form in the radius frame (2), it is possible to make like-with-like comparison of the predictions of these two equations. Any discrepancy between the two sets of predictions can then be attributed to factors other than differences in the damping, such as the requirement for greater numerical precision when perturbations are expressed in the volume frame (because $V \propto R^3$) or, if asymptotic expansions of these two equations are compared, to differences in truncating series expansions of $V^2, \dot{V}, \ddot{V}$ and $R^2, \dot{R}, \ddot{R}$ and any cross-terms etc.

Whilst Eq. (6) has provided the foundation for many important studies over the last decade, from acoustical oceanography to biomedical ultrasonics, the inherent small amplitude assumption within it means that terms are neglected which may become important. Whilst small amplitude expansions can give physical insight into the importance of terms, their validity needs to be critically examined in each case. Such examination is now possible with current computing facilities for nonlinear bubble dynamics, and schemes for translating the outputs of these calculations into nonlinear propagation models [34] provide alternative routes if greater accuracy is required.

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