EFFECTS OF COMPRESSIBILITY ON THE RADIATION AND VISCOUS DAMPING TERMS IN THE SCATTERING AND EXTINCTION CROSS-SECTIONS OF A SINGLE SPHERICAL BUBBLE: A PUZZLE SOLVED AND A PUZZLE POSED

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Abstract: *In [M. A. Ainslie & T. G. Leighton, Underwater Acoustic Measurements (Heraklion, Crete, 2007), pp 571-576], the authors described a discrepancy between the radiation damping coefficients in the models due to Weston and to Medwin describing the scattering cross-section of a single spherical bubble. The resolution of that discrepancy [M. A. Ainslie & T. G. Leighton, J. Acoust. Soc. Am. 126, 2163-2175 (2009)] is summarised, and a new question posed related to viscous damping, as follows. The usual derivation of bubble damping due to viscosity assumes an incompressible medium; in that derivation, dilatational viscosity is neglected on the grounds that there is no compression. Modern theoretical treatments of scattering and attenuation through bubble clouds permit a compressible medium for radiation damping, but do not revisit the effect of this compressibility on the viscous damping. This raises as yet unanswered questions about the validity of the currently accepted expressions for the viscous damping factor used for calculating scattering and extinction cross-sections.*

Keywords: *Scattering cross-section, radiation damping, bulk viscosity, compressibility*

1. INTRODUCTION

Modern interest in the acoustic properties of individual bubbles began with Minnaert’s pioneering work on bubble resonance [1]. The understanding of the response of a small spherical bubble to an incident plane wave was further increased by Refs. [2-5]. Now the problem of small spherical gas bubble undergoing linear spherically symmetric pulsations, and radiating sound into a compressible viscous liquid is generally considered to be solved, but is it? We describe two discrepancies that have come to light recently.

The accuracy of the expression for radiation damping published by Medwin [6] was questioned by Hampton and Anderson [7] and more recently by Ainslie and Leighton [8], pointing out that the work of Weston [9, 10] leads to the opposite frequency dependence to that of Ref. [6]. Weston’s model, though hardly ever used other than by Weston himself [9-11], has since been shown to be the correct one [12]. The second discrepancy, concerning the effect of medium compressibility on viscous damping, for which the effect of bulk viscosity is usually omitted without explanation, is not yet resolved.

In Sec. 2, the scattering and extinction cross-sections in an incompressible medium are described, followed by an expression for the resonance frequency. In Sec. 3, two effects of a compressible medium are discussed.

2. INCOMPRESSIBLE MEDIUM

2.1. Scattering cross-section

The scattering cross-section of a small spherical bubble of radius R_0 in an incompressible medium and ensonified by a plane propagating wave of angular frequency ω is [13]

$$\sigma_\mathrm{s} = \frac{4\pi R_0^{\,2}}{\left(\omega_0^{\,2}/\omega^2 - 1\right)^2 + \left(2\beta_0/\omega\right)^2}, \tag{1}$$

where, for gas pressure P_a in the bubble, liquid density ρ, surface tension τ, and (complex) polytropic index Γ, the parameter ω_0, closely related to the resonance frequency, is

$$\omega_0\left(R_0,\omega\right) = \frac{3P_\mathrm{a}}{\rho R_0^{\,2}}\sqrt{\mathrm{Re}\,\Gamma - \frac{2\tau}{3P_\mathrm{a}R_0}} \tag{2}$$

and the damping factor, including contributions from thermal and viscous damping, is

$$\beta_0 = \beta_\mathrm{th} + \beta_\mathrm{vis}. \tag{3}$$

The thermal and viscous damping factors are

$$\beta_{\mathrm{th}}\left(R_0,\omega\right)=\frac{3P_{\mathrm{a}}}{\rho R_0^{\;2}}\frac{\operatorname{Im}\Gamma}{2\omega} \tag{4}$$

and, for shear viscosity η_{S}, (independent of frequency)

$$\beta_{\mathrm{vis}}=\frac{2\eta_{\mathrm{S}}}{\rho R_0^{\;2}}\,. \tag{5}$$

2.2. Extinction cross-section

The extinction cross-section is [12, 14]

$$\sigma_{\mathrm{e}}=2\frac{\beta_0}{\omega\varepsilon}\sigma_{\mathrm{s}}\,. \tag{6}$$

This expression is valid when $\varepsilon \ll 1$ and $\varepsilon\,\omega/\beta_0 \ll 1$. It implies that in a nearly incompressible medium for which both inequalities hold, a far greater proportion of the incident sound is absorbed than is scattered.

2.3. Resonance frequency

If the only form of damping were due to viscosity, the resonance frequency, defined as the frequency that maximises the scattering cross-section, would be [12] (assuming ω_0 to be independent of frequency)

$$\omega_{\mathrm{vis}}=\frac{\omega_0}{\sqrt{1-2\beta_{\mathrm{vis}}^{\;2}/\omega_0^{\;2}}}\,. \tag{7}$$

If a small amount of thermal damping is also present, this becomes

$$\omega_{\mathrm{res}}=\omega_{\mathrm{vis}}\left[1+\alpha+O\!\left(\alpha^2\right)\right] \tag{8}$$

where

$$\alpha=\frac{\omega_{\mathrm{vis}}^{\;2}}{\omega_0^{\;2}}\frac{\beta_{\mathrm{th}}\left(\omega_{\mathrm{vis}}\right)}{\omega_0^{\;2}}\left(2\beta_{\mathrm{vis}}+\beta_{\mathrm{th}}\left(\omega_{\mathrm{vis}}\right)\right). \tag{9}$$

It follows from (7) that ω_{vis} increases with increasing viscosity. which seems to conflict with Eq. (29) from [15], which states that the resonance frequency *decreases* with increasing viscosity. The apparent inconsistency is resolved (see [16]) by noting a difference in the definition of resonance frequency between the present work, which considers a resonance in the (far-field) scattered pressure, and that of [15], which considers a resonance in the bubble wall displacement.

3. COMPRESSIBLE MEDIUM

Two effects of compressibility of the surrounding medium are considered next. First, the synchronised compressions and rarefactions around the bubble result in density perturbations that radiate outwards from the surface of the bubble in the form of a spherical wave, travelling at the speed of sound in the compressible medium. The radiation sound field is driven by the bubble pulsations, and the energy required to radiate the sound is taken from these pulsations. This form of energy loss is known as ‘radiation damping’.

The second effect considered arises from bulk viscosity of the liquid medium. If the compression of an acoustic particle in the liquid occurs without loss, the bulk viscosity is zero. If not, the question arises of whether a term representing bulk viscosity is needed in the viscous damping factor.

In the following, the effect of liquid compressibility is considered on the scattering cross-section, the extinction cross-section and the resonance frequency.

3.1. Scattering cross-section

3.1.1. Radiation damping: a puzzle solved

For a compressible medium, the scattering cross-section is [12]

$$\sigma_{\mathrm{s}} = \frac{4\pi R_0^{\,2}}{\left(\dfrac{\omega_0^{\,2}}{\omega^2} - 1 - \dfrac{2\varepsilon\beta_0}{\omega}\right)^2 + \left(2\dfrac{\beta_0}{\omega} + \delta_{\mathrm{rad}}\right)^2}, \tag{10}$$

where

$$\delta_{\mathrm{rad}} = \frac{\omega_0^{\,2}}{\omega^2}\varepsilon \tag{11}$$

and $\varepsilon \equiv \varepsilon(\omega) = \omega R_0 / c$. The frequency dependence for the radiation damping in (10) confirms Weston’s result [9-11], namely that the radiation damping coefficient (i.e., the parameter δ_{rad} that appears in (10)) is inversely proportional to frequency, and not proportional to frequency as often stated.

The viscous damping factor is

$$\beta_{\mathrm{vis}} = \frac{2\eta_{\mathrm{tot}}}{\rho R_0^{\,2}}, \tag{12}$$

where for an incompressible medium η_{tot} would be equal to η_{S}. The form taken for a compressible medium is the subject of Sec. 3.1.2.

3.1.2. Bulk viscosity: a puzzle posed

In this section we discuss the effect of medium compressibility on viscous damping. In particular we question the usual assumption that bulk or dilatational viscosities may be neglected (both cannot simultaneously be zero unless the shear viscosity is also zero) [22], and consider the contribution to the problem made by Love's pioneering work on the effect of viscosity of fish flesh [17].

One derivation [18] starts with the assumption of an incompressible liquid and points out that if there is no compression there can be no loss associated with dilatational viscosity. This leads to the standard expression for viscous damping proportional to η_{S}. When compression is re-introduced in this derivation, the need for a term representing either bulk viscosity (η_{B}) or dilatational viscosity (η_{D}) is not considered.

Some light is shed on the problem by the pioneering work of Love, who introduced a term representing bulk viscosity in his calculations of the scattering cross-section of a fish bladder. Specifically, Love derived an expression for viscous damping that amounts to substituting $\eta_{\mathrm{tot}} = \eta_{\mathrm{Love}}$ in (12), with

$$\eta_{\mathrm{Love}} = \tfrac{2}{3}\left(\eta_{\mathrm{S}} + \tfrac{3}{4}\eta_{\mathrm{B}}\right). \tag{13}$$

Using [21]

$$\eta_{\mathrm{B}} = \tfrac{2}{3}\eta_{\mathrm{S}} + \eta_{\mathrm{D}} \tag{14}$$

we obtain

$$\eta_{\mathrm{Love}} = \eta_{\mathrm{S}} + \tfrac{1}{2}\eta_{\mathrm{D}}, \tag{15}$$

which seems consistent with the incompressible case if $\eta_{\mathrm{D}} = 0$. However, in general η_{D} is not zero. In particular, consider a completely reversible compression, with all potential energy invested in the compression being returned to the incident field as (acoustic) kinetic energy. For a medium that permits such a compression the bulk viscosity is zero, which means from (13) that

$$\eta_{\mathrm{Love}} = \tfrac{2}{3}\eta_{\mathrm{S}}. \tag{16}$$

This result is for a compressible medium. If the limit of infinite bulk modulus is considered, there is an apparent discrepancy with the expectation that η_{tot} should be equal to η_{S} in this limit. We suggest four alternative (but not mutually exclusive) explanations for this discrepancy:

- 1. There might be an error in Devin's derivation for the viscous damping factor;
- 2. There might be an error in Love's derivation for the viscous damping factor;
- 3. Love's and Devin's derivations, while both correct, might make different physical assumptions;
- 4. Love's and Devin's derivations might both be correct, and made consistent by the possible existence of a physical rule that requires dilatational viscosity to vanish in the incompressible limit.

A separate question is whether one or other of η_D or η_B might be so small that it may be neglected compared with η_S. According to Liebermann [19] the dilatational viscosity η_D is $\eta_D \approx 2.2\eta_S$ and therefore $\eta_B \approx 2.9\eta_S$. Liebermann's valueof η_D , which is confirmed by modern measurements [20], suggests that neither η_D nor η_B is small relative to η_S. Therefore, if the bubble is small enough for β_{vis} to make a significant contribution to the damping, and unless it can be shown that the bulk viscosity term cancels, the contribution from η_B seems likely to be similar in magnitude to, and possibly greater than that of η_S.

3.2. Extinction cross-section

In a compressible medium, the extinction cross-section is [12, 14]

$$\sigma_e = \left(1 + 2\frac{\beta_{th} + \beta_{vis}}{\omega\varepsilon}\left(1+\varepsilon^2\right)\right)\sigma_s . \tag{17}$$

Equations (10) and (17) are sometimes both written in terms of the same dimensionless damping coefficient $\delta_{tot} = 2\ \beta_0/\omega + \delta_{rad}$, incorporating the combined effects of viscosity, thermal conductivity and acoustic radiation. This practice is avoided here because it requires a different functional form for δ_{rad} for each of the two equations [12].

3.3. Resonance frequency

In a compressible medium, the resonance frequency in the presence of viscous and radiation damping only is

$$\omega'_{vis} = \frac{\omega_0}{\sqrt{1 - \frac{2\beta_{vis}{}^2}{\omega_0{}^2} - \frac{\varepsilon(\omega_0)^2}{2}}} . \tag{18}$$

Including thermal damping (assumed small), this resonance frequency becomes

$$\omega_{res} = \omega'_{vis}\left[1 + \alpha' + O\left(\alpha'^2\right)\right] \tag{19}$$

where

$$\alpha' = \frac{{\omega'_{\mathrm{vis}}}^2}{{\omega_0}^2}\frac{\beta_{\mathrm{th}}(\omega'_{\mathrm{vis}})}{{\omega_0}^2}\left(2\sqrt{{\beta_{\mathrm{vis}}}^2 + \frac{{\omega_0}^2\varepsilon(\omega_0)^2}{4}} + \beta_{\mathrm{th}}(\omega'_{\mathrm{vis}})^2\right). \tag{20}$$

The various damping mechanisms considered all result in an increase to the resonance frequency relative to ω_0, which for the purpose of the present ω_{res} calculation is assumed to be independent of frequency.

4. SUMMARY AND CONCLUSIONS

Expressions for scattering cross-section, extinction cross-section and resonance frequency of a gas bubble in a liquid medium are presented, first for an incompressible medium and then for a compressible one. Conclusions of this work are:

- The discrepancy related to radiation damping is resolved by Ref. [12]. The scattering and extinction cross-sections are given by (10) and (17), respectively. The confusion that can arise by use of the dimensionless damping coefficient δ is avoided by expressing the extinction coefficient in terms of the damping factor β.
- The effect of bulk viscosity on bubble damping is not understood. In particular, a discrepancy between Love's result in the limit of zero bulk viscosity (lossless compression) differs by a factor 2/3 from Devin's model, which was derived for an incompressible medium.

ACKNOWLEDGEMENTS

The authors have benefited from discussions about bulk viscosity with K. Baik, A. A. Doinikov and R. H. Love. In particular, R. H. Love first pointed out to us that (15) follows from (13) and (14).

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