Institute of Sound and Vibration Research

Vibrations of a Rail Coupled to a Foundation Beam Through a Series of Discrete Elastic Supports

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

Vibrations of a Rail Coupled to a Foundation Beam Through a Series of Discrete Elastic Supports

by

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October 2001

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Group Chairman

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1. INTRODUCTION

When trains pass over bridges and elevated structures, particularly steel structures, an increase in their radiated noise can occur compared to the case of ballasted track [1]. This is due principally to the additional component of noise radiated by the bridge structure caused by vibrations transmitted from the rails. However, an increased component of noise from the rails themselves can also occur due to the use of softer rail fasteners on bridges than in conventional ballasted track [1]. Both components of noise are influenced by the coupling between the rails and the bridge structure.

In order to represent the coupling between the rails and the bridge, a model has been developed [2] in which the rail and the bridge were both considered to be infinite Euler beams, coupled by a continuous flexible layer. The upper beam is excited by a point force. Using this model it is found that the continuous connection can be replaced by a single equivalent connection, the stiffness of which is equal to that of the continuous layer within a length of 0.45 times the shorter of the bending wavelengths in the two beams. This model is reproduced briefly in Section 2. A major short-coming of this model appears to be that it predicts that, if the damping of the upper beam (the rail) is increased, the response of the lower beam (the bridge) is not affected. This seems contrary to experience. Moreover, in comparison with measurements, the model appears to under-predict the power flow into the bridge. To overcome this, the model in [1] introduced a number of equivalent connection points between the rail and the bridge rather than a single equivalent connection for each wheel load.

The above model is idealised in a number of aspects. (a) It was assumed that both beams are infinite whereas the bridge is finite and the rail may be. (b) It was assumed that the bridge is a uniform beam whereas in practice it contains many inhomogeneities. (c) It was assumed that the connection is a uniform flexible layer whereas it consists of discrete fasteners, the stiffness of which will vary randomly between each connection point [3], and will tend to be higher beneath wheel loads than between wheels [4]. (d) The rail and the bridge were modelled as Euler beams. For the rail, a Timoshenko beam including shear deformation and rotational inertia is more appropriate above about 500 Hz [5]. For the bridge, a beam is only appropriate at low frequencies, more complicated effects such as shear of the vertical members becoming important from frequencies as low as 250 Hz [1].

In this work a new model of a rail on a bridge is developed which takes into account several of these effects. The rail and the bridge are modelled by beams which may include shear deformation and rotational inertia. These are connected by a series of discrete damped springs, the stiffness and position of which may be varied randomly. A random mass may also be added to the bridge at each connection point. The modelling method is an extension of that used in [3, 4] in which a rail is modelled by a series of Green's functions (transfer receptances) and the supports under the rail are replaced by reaction forces.

2. PREVIOUS MODELS

2.1 Two beams connected by a continuous elastic layer

The first model treated [2] consists of two beams connected by a continuous elastic layer (considered either un-damped as a first approximation, or damped) modelling the dynamic behaviour of a railway connected to a metal bridge. The upper beam is considered the source of the vibration, while the lower one is only considered a receiver structure. This model is an extension of one published by Pinnington [6] for a finite source beam on an infinite receiver. The solution is obtained for the system consisting of the two Euler beams plus the connection layer. The equations for the system are:

$$B_s \frac{\partial^4 u}{\partial x^4} + \mu_s \frac{\partial^2 u}{\partial t^2} + s(u-v) = F\delta(x)e^{i\omega t} \tag{1}$$

$$B_r \frac{\partial^4 v}{\partial x^4} + \mu_r \frac{\partial^2 v}{\partial t^2} - s(u-v) = 0 \tag{2}$$

where B_s and B_r are the bending stiffness of source and receiver beams, μ_s and μ_r are their mass per unit length, s is the stiffness per unit length of the elastic layer and u and v are the displacements. Damping can be introduced in the elastic layer by making s complex:

$$s \rightarrow s(1+i\eta) \tag{3}$$

Similarly, damping can be introduced in the beams by setting:

$$B_s \rightarrow B_s(1+i\eta_s) \quad B_r \rightarrow B_r(1+i\eta_r) \tag{4}$$

The external force is applied at the point x=0. It is helpful to define:

$$\omega_s^{\ 2} = \frac{s}{\mu_s} \quad \omega_r^{\ 2} = \frac{s}{\mu_r} \quad k_s^{\ 4} = \omega^2 \frac{\mu_s}{B_s} \quad k_r^{\ 4} = \omega^2 \frac{\mu_r}{B_r} \tag{5}$$

The solution is found in the classical form with separated variables:

$$u, v(x,t) \propto e^{i\omega t} e^{kx} \tag{6}$$

substitution of which into (1) and (2) in the absence of the applied force yields the free wavenumber (see also [7]):

$$k_{I,II}{}^4 = \frac{1}{2}\left[k_s{}^4\left(1-\frac{\omega_s{}^2}{\omega^2}\right)+k_r{}^4\left(1-\frac{\omega_r{}^2}{\omega^2}\right)\right] \pm \frac{1}{2}\left\{\left[k_s{}^4\left(1-\frac{\omega_s{}^2}{\omega^2}\right)+k_r{}^4\left(1-\frac{\omega_r{}^2}{\omega^2}\right)\right]^2 + 4k_s{}^4 k_r{}^4\left(\frac{\omega_s{}^2+\omega_r{}^2}{\omega^2}-1\right)\right\}^{\frac{1}{2}} \tag{7}$$

From considerations on (7) it is possible to deduce that ω_0, the decoupling frequency, corresponds to a resonance in which the mass of the two beams move in anti-phase on the elastic foundation:

$$\omega_0{}^2 = \omega_s{}^2 + \omega_r{}^2 = s\left(\frac{1}{\mu_s}+\frac{1}{\mu_{rs}}\right) \tag{8}$$

A solution at this frequency exists with k=0.

Above ω_0 there are four solutions k_I representing forward and backward travelling waves and near-field decaying waves. Similarly there are four solutions k_{II}. Below ω_0 only one pair of travelling wave solutions exists.

The waves corresponding to $k_{I,II}$ represent the vibration of the two beams acting in a fixed ratio for each given wavenumber. In this way it is possible to define the quantities [7]:

$$\varepsilon_I = \frac{v_I}{u_I} = \frac{\left(\frac{\omega_r}{\omega}\right)^2}{\left(\frac{k_I}{k_r}\right)^4 - 1 + \left(\frac{\omega_r}{\omega}\right)^2} \qquad -\varepsilon_{II} = \frac{u_{II}}{v_{II}} = \frac{\left(\frac{\omega_s}{\omega}\right)^2}{\left(\frac{k_{II}}{k_s}\right)^4 - 1 + \left(\frac{\omega_s}{\omega}\right)^2} \tag{9}$$

See Appendix B and [2] for further details on the solution procedures.

The response of the model to a point force applied to the point x=0 is obtained by imposing the standard boundary conditions for an infinite beam. This yields

$$\begin{aligned} u(x) &= \frac{-F}{4B_s(1+\varepsilon_I\varepsilon_{II})}\left\{\frac{1}{k_I{}^3}(e^{-k_I x}+ie^{-ik_I x})+\frac{\varepsilon_I\varepsilon_{II}}{k_{II}{}^3}(e^{-k_{II}x}+ie^{-ik_{II}x})\right\} \\ v(x) &= \frac{-F}{4B_s(1+\varepsilon_I\varepsilon_{II})}\left\{\frac{\varepsilon_I}{k_I{}^3}(e^{-k_I x}+ie^{-ik_I x})-\frac{\varepsilon_I}{k_{II}{}^3}(e^{-k_{II}x}+ie^{-ik_{II}x})\right\} \end{aligned} \tag{10}$$

By comparing the response of the lower beam with that of an equivalent system joined only by a single spring of stiffness K located at $x=0$, it is possible to show that, for $k_s>k_r$, the response is identical if K is given by:

$$K = \frac{2\sqrt{2}\,s}{k_s} \approx 0.45 s\lambda_s \tag{11}$$

This can be stated in the form that the equivalent point stiffness is the stiffness of the elastic layer within approximately half a bending wavelength of the excitation point at the source beam. Similarly for $k_s<<k_r$ it is found that

$$K = \frac{\omega}{\sqrt{2}\,\mathrm{Re}(Y_r)}\left(\frac{\omega_s}{\omega}\right)^2 = \frac{2\sqrt{2}\,s}{k_r} \approx 0.45 s\lambda_r \tag{12}$$

which means that the equivalent stiffness can be expressed as the stiffness of the elastic layer within half a bending wavelength (on the receiver beam) of the excitation point.

2.2 Rail on discrete supports

In this section a model is considered [4] that is composed of only one beam representing the rail, constrained to a rigid ground through a series of discrete elastic supports. These represent the pad and the ballast supporting the railway at each connection point. The elastic connections are modelled as two damped springs of stiffness K_p and K_b connected in series with a mass M_s representing the sleepers. In this way, the scheme for the general model is described in Fig. 1. The beam includes shear deformation and rotational inertia.

The solution for the system described is obtained taking into account the Green's functions, which represent the solution for an infinite Timoshenko beam at one point due to a unit force at another point. The general Green's function for a Timoshenko beam is:

$$G(x,x') = (u_1 e^{-ik_1|x-x'|} + u_2 e^{-k_2|x-x'|})e^{i\omega t} \tag{13}$$

which represents the displacement of the point x when a unit load acts at x'.

The parameters introduced in the expression are:

$$k_{1,2} = \left(\frac{\omega}{\sqrt{2}}\right)\left\{\pm\left(\frac{\rho}{E}+\frac{\rho}{G\kappa}\right)+\left[\left(\frac{\rho}{E}+\frac{\rho}{G\kappa}\right)^2+\frac{4\rho A}{EI\omega^2}\right]^{1/2}\right\}^{1/2} \tag{14}$$

$$u_1 = \frac{i}{IEG\kappa}\frac{\rho I\omega^2 - G\kappa A - EIk_1^{\,2}}{2Ak_1(k_1^{\,2}+k_2^{\,2})} \qquad u_2 = \frac{1}{IEG\kappa}\frac{\rho I\omega^2 - G\kappa A + EIk_2^{\,2}}{2Ak_2(k_1^{\,2}+k_2^{\,2})} \qquad (15)^{1,2}$$

where E, G, and κ are respectively the Young's modulus, the shear modulus and the shear coefficient, A and I are the area and second moment of area of the rail and ρ is the mass density.

The dynamic stiffness for the generic support at position n is:

$$Z_n = \frac{K_{pn}(1+i\eta_p)[K_{bn}(1+i\eta_b)-M_s\omega^2]}{K_{pn}(1+i\eta_p)+K_{bn}(1+i\eta_b)-M_s\omega^2} \tag{16}$$

where η_p and η_b are the loss factor for the pad and the ballast.

The displacements at x due to an external harmonic force F at x_F are:

$$u(x) = -\sum_{n=-N}^{N} Z_n u(x_n) G(x,x_n) + F\,G(x,x_F) \tag{17}$$

and for $x=x_m$:

[1] Note: I is missing in the denominator in [1].
[2] Note: the term EIk_2^2 is considered as $-EIk_2^2$ in [1].

$$u(x_m) = -\sum_{n=-N}^{N} Z_n\, u(x_n)\, G(x_m, x_n) + F\, G(x_m, x_F) \qquad m = -N, \ldots, N \tag{18}$$

The solution to the problem can be obtained by solving the system of *(2N+1)* linear equations (18) in the variables $u(x_m)$ and substituting them in the expression for the general $u(x)$.

The technical data used for the numerical evaluation are those corresponding to a UIC 60 rail [4]:

$E = 2 \cdot 10^{11}$ N/m^2, $G = 0.77 \cdot 10^{11}$ N/m^2, $\eta_r = 0.01$, $\rho = 8000$ kg/m^3, $A = 0.75 \cdot 10^{-2}$ m^2

$I = 3.2 \cdot 10^{-5}$ m^4, $\kappa = 0.4$, $\eta_{pr} = 0.25$, $\eta_b = 0.6$, $M_s = 150$ kg, $A = 0.75 \cdot 10^{-2}$ m^2, $d = 0.6$ m, $N = 40$

Concerning the values of stiffness of the supports, various different conditions are considered in [4]. For the purpose of this report it is sufficient to consider only constant pad stiffness. The value of K_p and K_b used in this case are $K_p = 68.8$ MN/m and $K_b = 151$ MN/m for the soft stiffness model and K_p=551.1 MN/m and $K_b = 460.1$ MN/m for the stiff model when the force is acting above the sleeper, while if the driving point is placed at a midspan the values for the stiffness are $K_p = 68.8$ MN/m and $K_b = 151$ MN/m for the soft support and $K_p = 375.2$ MN/m and $K_b = 434.7$ MN/m for the stiff model.

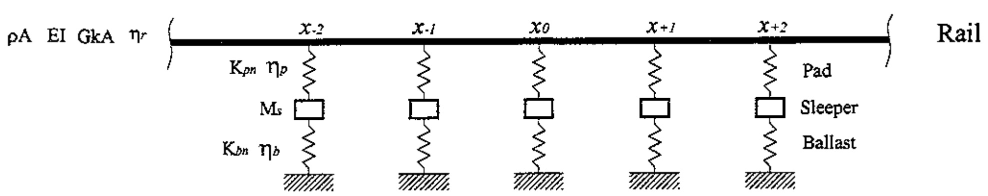


Fig. 1 Scheme derived from the Wu-Thompson model

The results for the receptance are illustrated in Fig. 2(a) and Fig. 2(b) where load at mid-span and load above the sleeper are considered. These results are identical to those in [4].

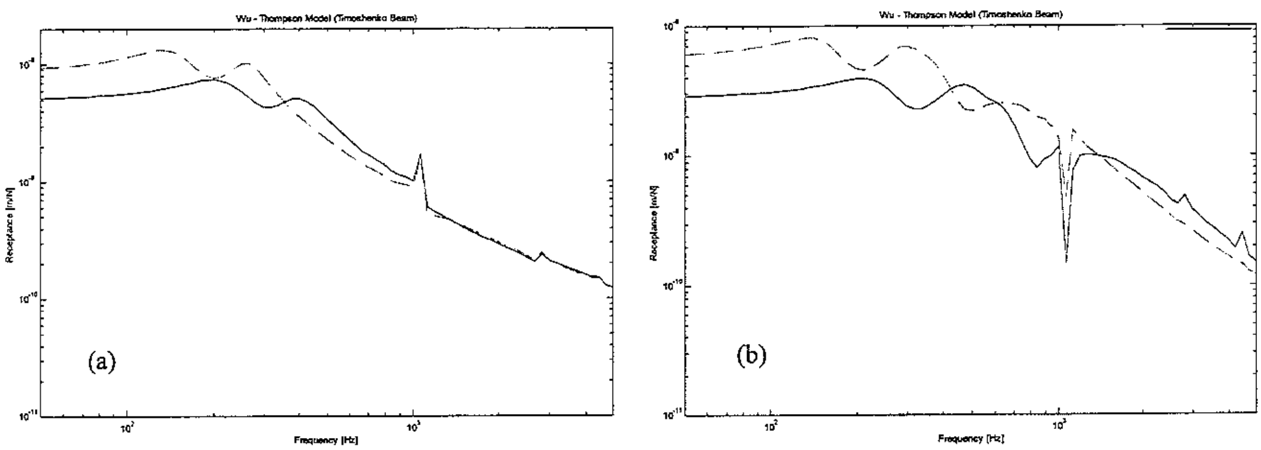


Fig. 2 Results-of the simulation for discretely supported track model. (a) Force above the sleeper. (b) Force at midspan. The solid lines represents the results for a set of stiff loaded supports, while the dashed lines are the response for soft loaded supports.

3. MODEL FOR TWO BEAMS CONNECTED BY DISCRETE SPRINGS

3.1 Formulation

By combining the features of the two models described above, a model is produced of a rail connected to the bridge beam through a series of discrete elastic supports. While in Wu and Thompson's model the elastic supports also had a mass in series with the springs, in this case the springs are taken as mass-less. The distance between two consecutive supports is considered equal to *d* and the equivalent stiffness of each support is $K_s = s \cdot d$ where *s* is the stiffness per unit length. Considering a constant distance *d* between each support, the relationship between the two structures consists in the force equilibrium :

$$F_n = K^n{}_s\,(u^s(x_n) - u^r(x_n)) \tag{19}$$

where $u^s(x)$ is the displacement of the source beam at the point *x* and $u^r(x)$ is that of the receiver beam. The system of equations obtained in this way is:

$$\left[\begin{array}{l} u^s(x_n) = -\sum_{m=-N}^{N} G^s(x_m, x_n) F_m + G^s(x_F, x_n) F \\ u^r(x_n) = \sum_{m=-N}^{N} G^r(x_m, x_n) F_m \\ F_m = K_s{}^m (u^s(x_m) - u^r(x_m)) \end{array}\right. \tag{20}$$

where G^s is the Green's function for the source beam (rail) and G^r is that for the receiver (bridge). Substituting for F_m and writing $u_n{}^s$ and $u_n{}^r$ for $u^s(x_n)$ and $u^r(x_n)$ we obtain:

$$\left[\begin{array}{l} u_n{}^s = -\sum_{m=-N}^{N} G^s(x_m, x_n) \cdot K_s{}^m (u_m{}^s - u_m{}^r) + G^s(x_F, x_n) F \\ u_n{}^r = \sum_{m=-N}^{N} G^r(x_m, x_n) \cdot K_s{}^m (u_m{}^s - u_m{}^r) \end{array}\right. \tag{21}$$

Equation (21) can be written in matrix form:

$$\begin{bmatrix} \underline{\underline{a}}^s + \underline{\underline{I}}(2N+1) & -\underline{\underline{a}}^s \\ -\underline{\underline{a}}^r & \underline{\underline{a}}^r + \underline{\underline{I}}(2N+1) \end{bmatrix} \cdot \begin{bmatrix} \underline{u}^s \\ \underline{u}^r \end{bmatrix} = F \begin{bmatrix} \underline{G}^s(0) \\ \underline{0} \end{bmatrix} \tag{22}$$

where:

$$a^s(i,j) = \left[G^s(x_i, x_j)\right] \cdot K_s(i) \qquad \text{and} \quad K_s(i) = K_s{}^i(1 + \mathrm{i}\eta_p) \tag{23}$$
$$a^r(i,j) = \left[G^r(x_i, x_j)\right] \cdot K_s(i)$$

while

$$G^s(0)(i) = G^s(x_i, 0) \tag{24}$$

This system of equations is solved in Matlab. The program is in Appendix A.

3.2 Euler Beam

The procedure described can be applied both to Euler beams and to Timoshenko beams. In the case of an Euler Beam there is a simplification for the expression of the complex wavenumber and the wave amplitude.

The general Green's function in equations (18) is:

$$G(x,x') = (u_1 e^{-ik_1|x-x'|} + u_2 e^{-k_1|x-x'|})e^{i\omega t} \tag{25}$$

where the parameters k_1 and $u_{1,2}$ become:

$$k_1 = \left(\frac{\rho\omega^2 A}{EI}\right)^{\frac{1}{4}} \tag{26}$$

$$u_1 = -\frac{i}{4IEk_1^3} \quad u_2 = -\frac{1}{4IEk_1^3} \tag{27}$$

For the algorithm of the solution see Appendix B.

4. VALIDATION OF THE MODEL

In this section a receiver beam with infinite bending stiffness is considered. This simplified model (coming from the previous section) is compared with two existing models both consisting of a source beam connected by a resilient layer to a rigid foundation, in order to verify the behaviour of the coupled system in "limit" conditions.

4.1 Euler Beam on elastic layer

An Euler Beam is considered vibrating on an elastic layer, being constrained to a rigid foundation. Damping factors are introduced in the beam and in the layer. For the elastic layer a stiffness per unit length s is introduced equal to $\frac{K_s}{d}$. The rigidity of the lower beam is simulated by defining:

$$\frac{EI_r}{EI_s} = 10^7 \qquad (28)$$

The other technical values used are:

$E = 2.1 \cdot 10^{11}$ N/m², $\eta_r = 0.01$, $\eta_p = 0.25$, $\rho = 8000$ kg/m³, $K_s = 6.68 \cdot 10^6$ N/m, $d = 0.6$, $M_s = 1.5 \cdot 10^{-4}$ kg

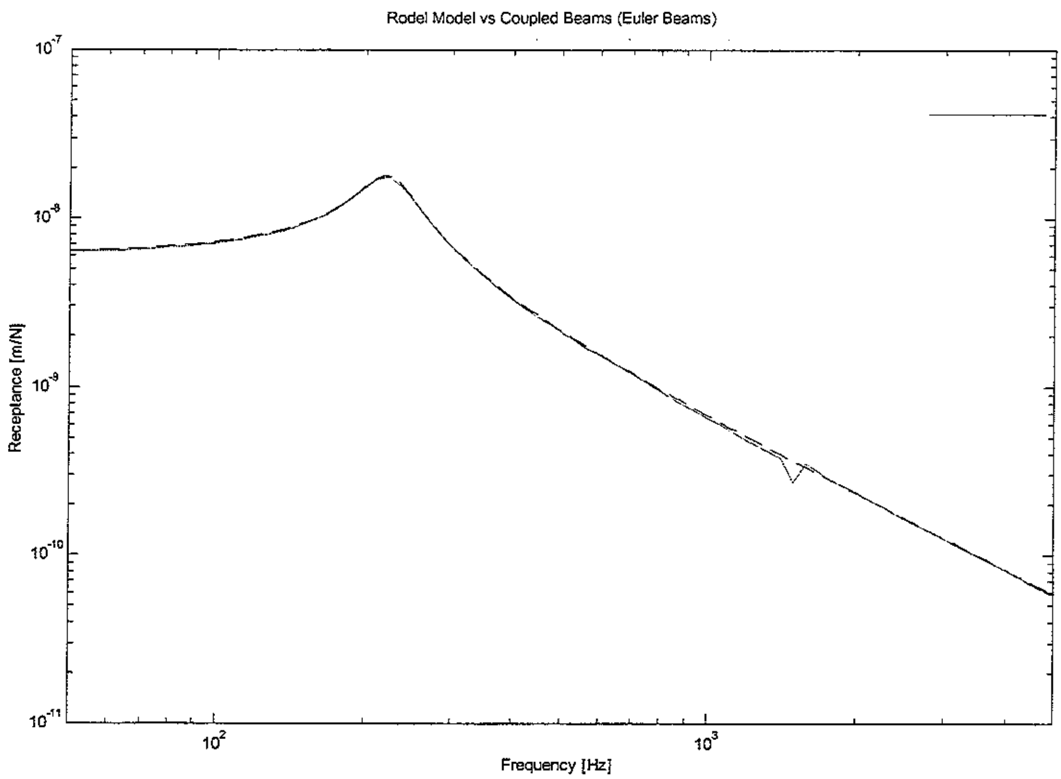


Fig.3: Point receptance for an Euler beam on an elastic layer.
--- rodel model, continuous line —— present model

The results of the present model are compared with those obtained using a model of a beam on an elastic foundation. This is obtained by using a model "rodel" from the TWINS package [9].
It is used here with an Euler beam and a single layer resilient support. The results are shown in Figure 3. As can be seen the point receptance is the same for the whole range of frequency considered. The dip for the coupled model at 1500 Hz is due to the pinned-pinned resonance for the discretely supported beam.

4.2 Timoshenko Beam on discrete elastic supports.

The simplified model with an infinite bending stiffness for the receiver beam and a negligible mass for the sleepers is considered in the case that the source beam includes shear deformation (Timoshenko beam). The system is compared with the one described in [4] using the following data:

$\frac{EI_r}{EI_s} = 10^7$ $M_s = 1.5 \cdot 10^{-4}$ kg $K_p = 68.8$ MN/m $K_b = 151.0$ MN/m

$E = 2 \cdot 10^{11}$ N/m^2, $G = 0.77 \cdot 10^{11}$ N/m^2, $\kappa = 0.4$, $\rho = 8000$ kg/m^3, $\eta_r = 0.01$, $\eta_p = 0.25$

$I_s = 3.2 \cdot 10^{-5}$ m^4, $A_s = 0.75 \cdot 10^{-2}$ m^2, $d = 0.6$ m, $N = 40$

The comparison between the results of the two models is plotted in the Fig. 4; the agreement is very good.

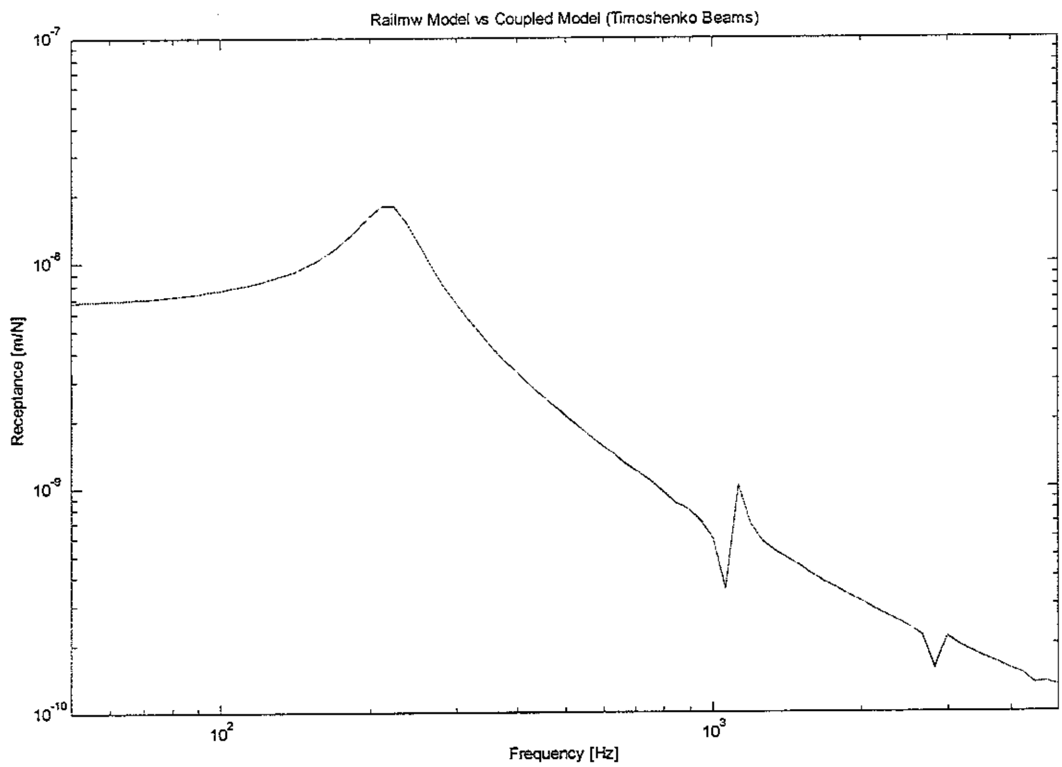


Fig.4: Point Receptance for a Timoshenko beam on discrete mass-less supports
——— present model, ---- model from [4]

4.3 Two Euler beams connected together

The models described in sections 2.1 and 3.1 are compared in this section.
The technical data used in this model are those from [2], where the material used is steel and the source and the receiver beams are the UIC 54 section and an equivalent section for a steel bridge [2].
The values used are:

$$E = 2.1 \cdot 10^{11}\ \text{N/m}^2,\ \rho = 8000\ \text{kg/m}^3,\ K_s = 1.98 \cdot 10^8\ \text{N/m}^2,\ \eta_p = 0.1$$

Beam	$A\ (m^2)$	$I\ (m^4)$	$\mu_{s,r}$ (kg/m)	$\eta_{s,r}$ (kg/m)
Source Beam	$6.75 \cdot 10^{-3}$	$2.346 \cdot 10^{-5}$	54	0.01
Receiver Beam	$81.25 \cdot 10^{-3}$	$5.652 \cdot 10^{-2}$	650	0.1

The procedure followed in order to obtain the solution is based on the calculation of Green's functions for both beams, so the system of equations describing the problem is doubled.
The results using the model from 2.1 for the same beams are represented in Fig. 5 as a comparison to the model here introduced here.

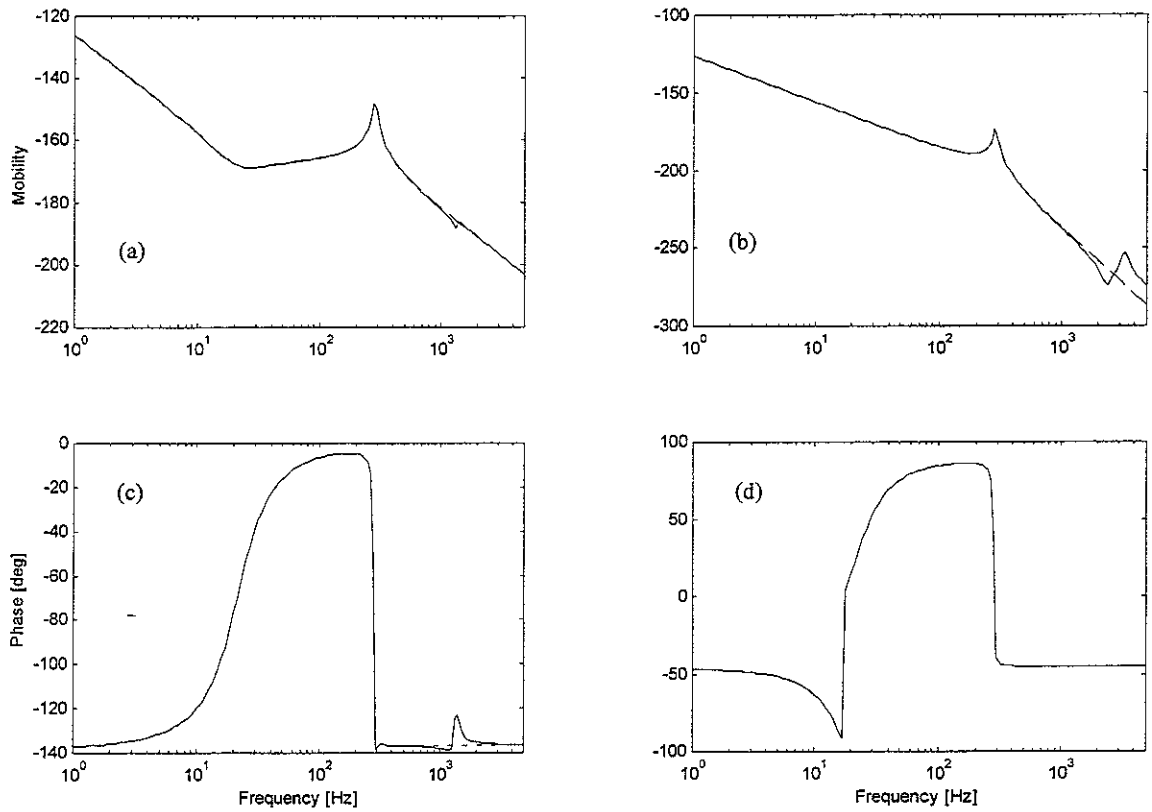


Fig.5 Point mobility (dB re 1 m/sN) and phase angle.

——— Discretely connected Euler beams, ---- continuously connected Euler beams
(a) and (c) Source, (b) and (d) Receiver

As can be observed the results are similar. The main difference between the two models is the dip in the mobility at ~1.4 kHz which is due to the pinned-pinned effect. This corresponds to a standing wave in the rail with nodes at the support points. The peak in the receiver beam response at ~3.5 kHz is due to a similar phenomenon in the receiver beam. The discrete model is excited above supports.

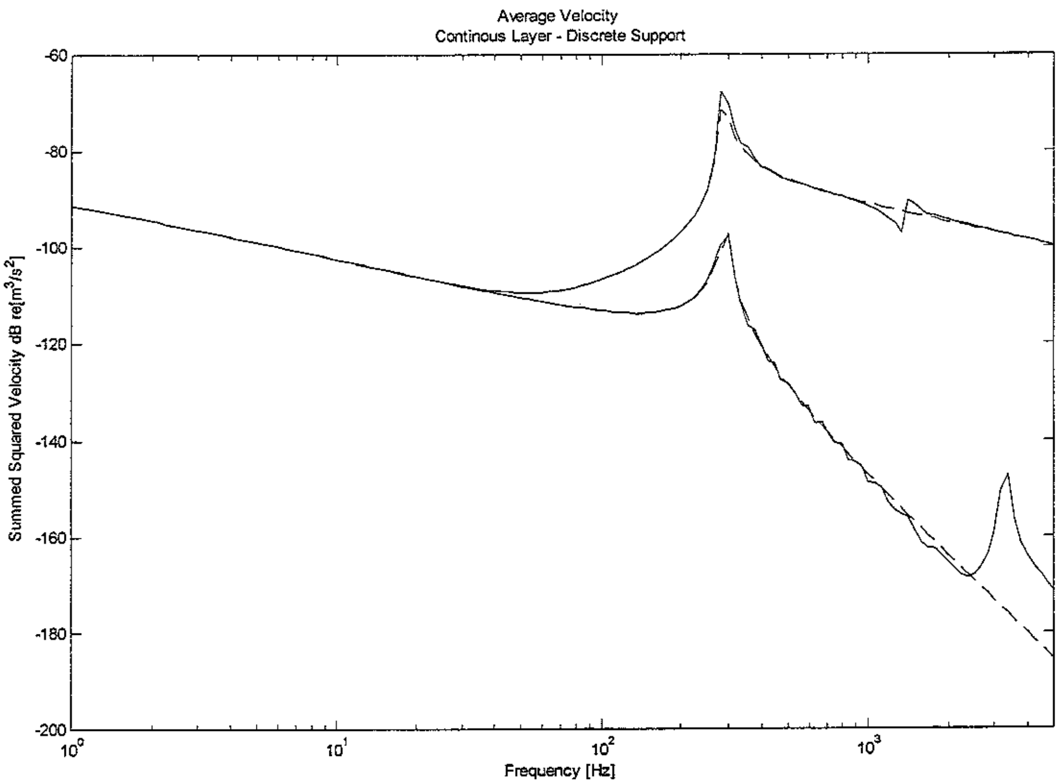
5. FURTHER MODEL DEVELOPMENT

5.1 Average vibration

It is necessary to introduce an index for the global response of the upper (or lower) beam to a point excitation. This quantity can be defined as in [3]:

$$R = \int_{-L}^{L} |\omega u(x)|^2 \, dx \qquad (29)$$

where the range of integration is the whole length of the beam.



——— Discretely coupled beams, ---- Continuously coupled beams

The Fig. 6 presents the value R evaluated for the continuously connected model and that for the model coupled through a series of springs. The models agree reasonably well except for a pinned peak at 3000 Hz in the receiver beam of the discrete model and an oscillation of the amplitude in this beam for frequency above the decoupling value. Both these features can be considered to be caused by the presence of a finite number of springs placed periodically. When the average global excitation from the upper structure matches the wavelength of the lower beam, a secondary resonance occurs.

The theoretical total average velocity is obtained for $L=$, however, the effects of the decaying waves suggests that a finite length of the beam is sufficient. In order to evaluate the influence of L on the integral (29) an analysis of the convergence of the quantity R has been carried out.
It has been possible, increasing of the number of supports included in the calculation and (in order to accelerate the convergence of the procedure) the damping factor for the upper beam, to find the optimal value of N and η for which R can be considered reasonably close to its ideal value. Fig. 7 shows the results obtained for η=0.03 and N=60,120. The set of values η=0.03, N=60 has been chosen as a reference for all the models described in the following sections.

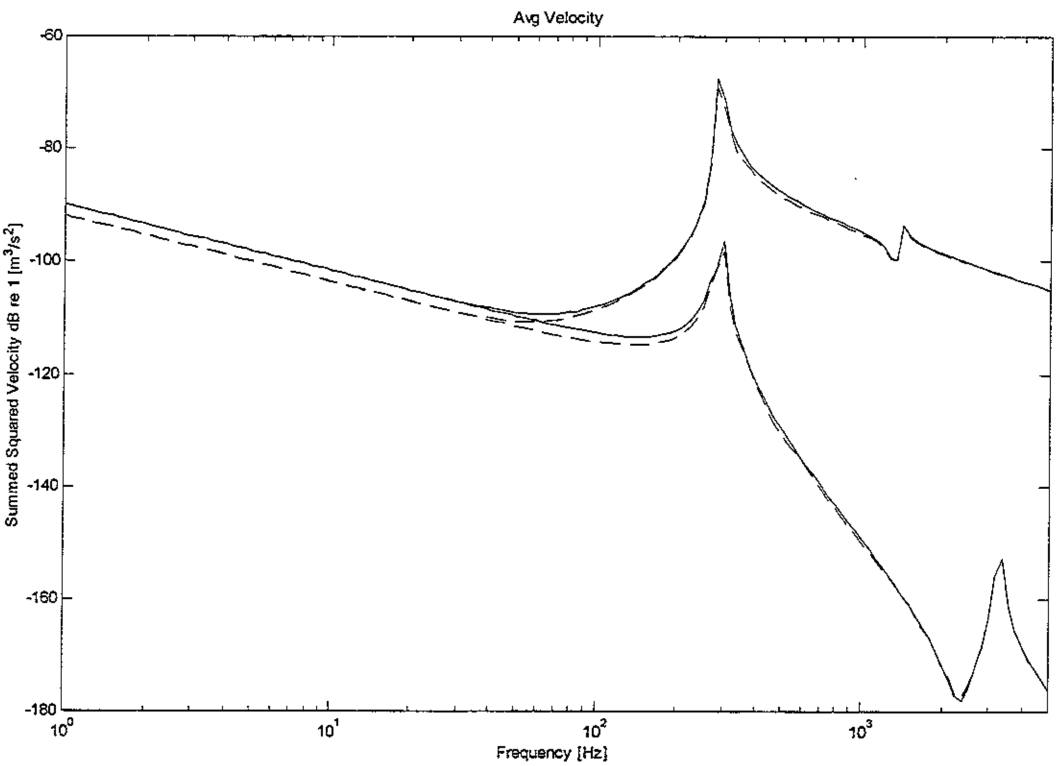


Fig. 7 Effect of number of supports on squared average velocity

——— N=60, η=0.03, ---- N=120, η=0.03

5.2 Random effects

The next step is to introduce a random variation in the stiffness of the connections between the source and the receiver beams. In order to analyse the effects of random properties in the track and the bridge, the following models have been developed:

1. Discretely connected beams with random distribution of stiffness
2. Discretely connected beams with random distribution of sleeper spacing
3. Discretely connected beams with discrete masses joined to the receiver beam
4. Discretely connected beams with random mass distribution

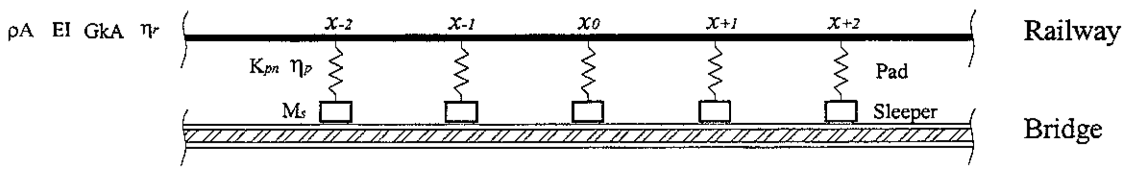


Fig. 8 Scheme of the generic coupled model

The scheme of Fig. 8 represents the general model adopted in this section. The masses are considered joined to the bridge and rigid while the springs represents the flexible connections of the pad. It should be noted that the model is still considering both the beams as infinite Euler beams, this is because the wavelengths involved are as usually much shorter than the real length of the structures analysed. The results are given for Euler beams (solutions for the Timoshenko beams are included in the codes) to allow a direct comparison with the model of section 2.1.

5.2.1 Random Stiffness

As can be seen from [3] the problem of a random distribution of stiffness in the supports has already been developed for the model [4] of a beam resiliently constrained to the rigid foundation. The random distribution for the series of supports has been considered having a normal probability distribution. The generation of random data has been introduced in Matlab through the command "`randn(m,n)`" which generates an $m \times n$ array of random numbers having 0 as initial *mean value* and 1 as *standard deviation* (see Appendix B).

Two sets of distribution for the stiffness have been considered, the first having a relative standard deviation 0.36 and the second having a relative standard deviation of 0.18. Both models have been considered with a mean value K_s=100 MN/m. The results for the calculations are plotted in Fig. 9. The parameter σ is in this case the relative standard deviation. The random stiffness model is compared to the constant stiffness model. While at low frequency the results of the two models are the same, for the frequency range from 300 Hz (after the decoupling frequency of the two beams) to 3000 Hz the lower beam shows a difference with a maximum deviation of 6 dB in average, the random stiffness tending to increase the response. The response of the upper beam is not greatly affected.

Fig. 10 show the effects of damping on the randomly stiffened model. The results are presented in a dB scale relative to those of the model of section 2.1, in the latter case with η_s is fixed at 0.03. The quantities plotted are the differences between the discrete model and the continuous model with mean values for the stiffness equal to 25 MN/m and 100 MN/m. The structural damping of the source is increased with respect to the original in the discrete model. The main difference between the stiffer model ((a)-(b) 100 MN/m) and the soft one ((c)-(d) 25 MN/m) is the shift of the decoupling peak from ~300 Hz to ~130 Hz. This result can be expected from equation (7). Of more interest here is the effect of damping on the discrete model. The increase of η_s generates a reduction of the amplitude of vibrations at high frequencies both in the source and in the *receiver*. This is not true for the continuous layer model for which only the source beam response is reduced, as can be verified by the plots on Fig. 11 where the effects of damping are presented for the model of continuously connected beams.

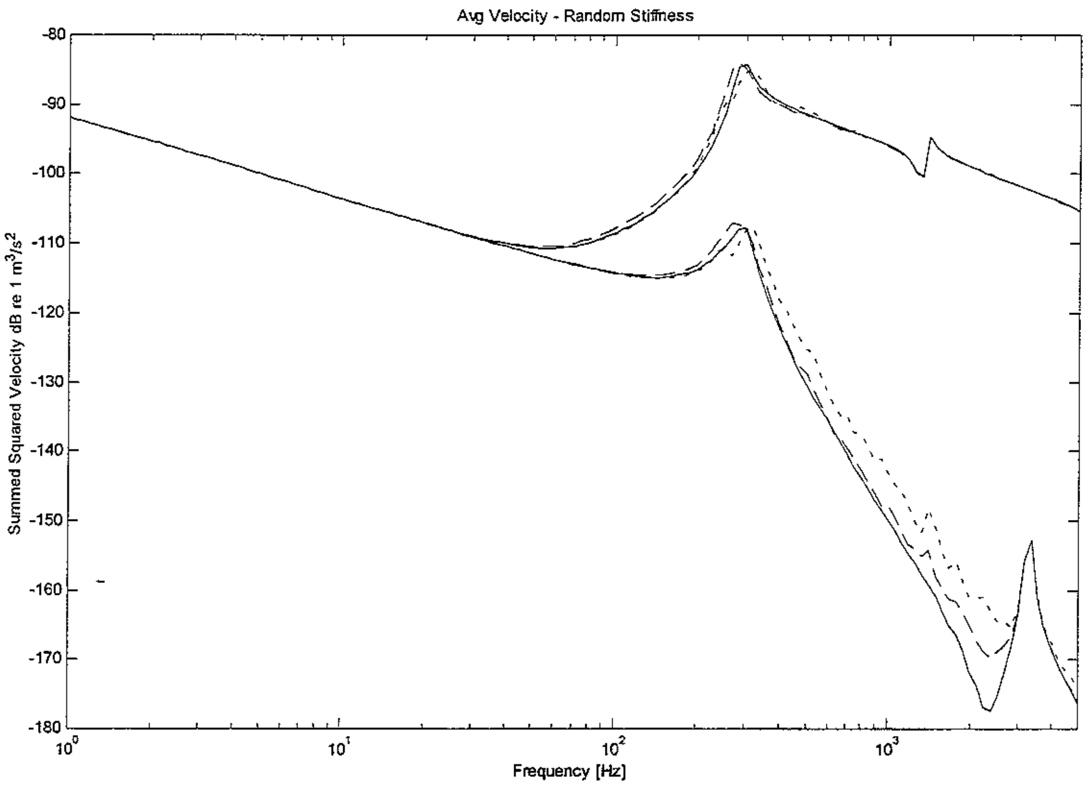


Fig. 9: Effect of random stiffness on response of the rail (upper curves) and bridge (lower curves)

——— Constant stiffness (σ=0, K_s=100 MN/m)
.... Random stiffness (σ=0.36, K_s=100 MN/m)
- - - - Random stiffness (σ=0.18, K_s=100 MN/m)

The average reduction in the beam response for the different values of loss factor in the range of frequencies 200 to 2000 Hz are quantified in the table below.

Table 1

Effect of changing damping of source beam from η_s=0.03 to the value indicated for random stiffness with σ=0.36 from 200 to 2000 Hz

Damping	Soft Model (25 MN/m^2)	Stiff Model (100 MN/m^2)	
η_s=0.1	5.0 dB	5.5 dB	Source Beam
η_s=0.3	12.0 dB	11.5 dB	
η_s=0.1	2.0 dB	3.0 dB	Receiver Beam
η_s=0.3	10.0 dB	8.0 dB	

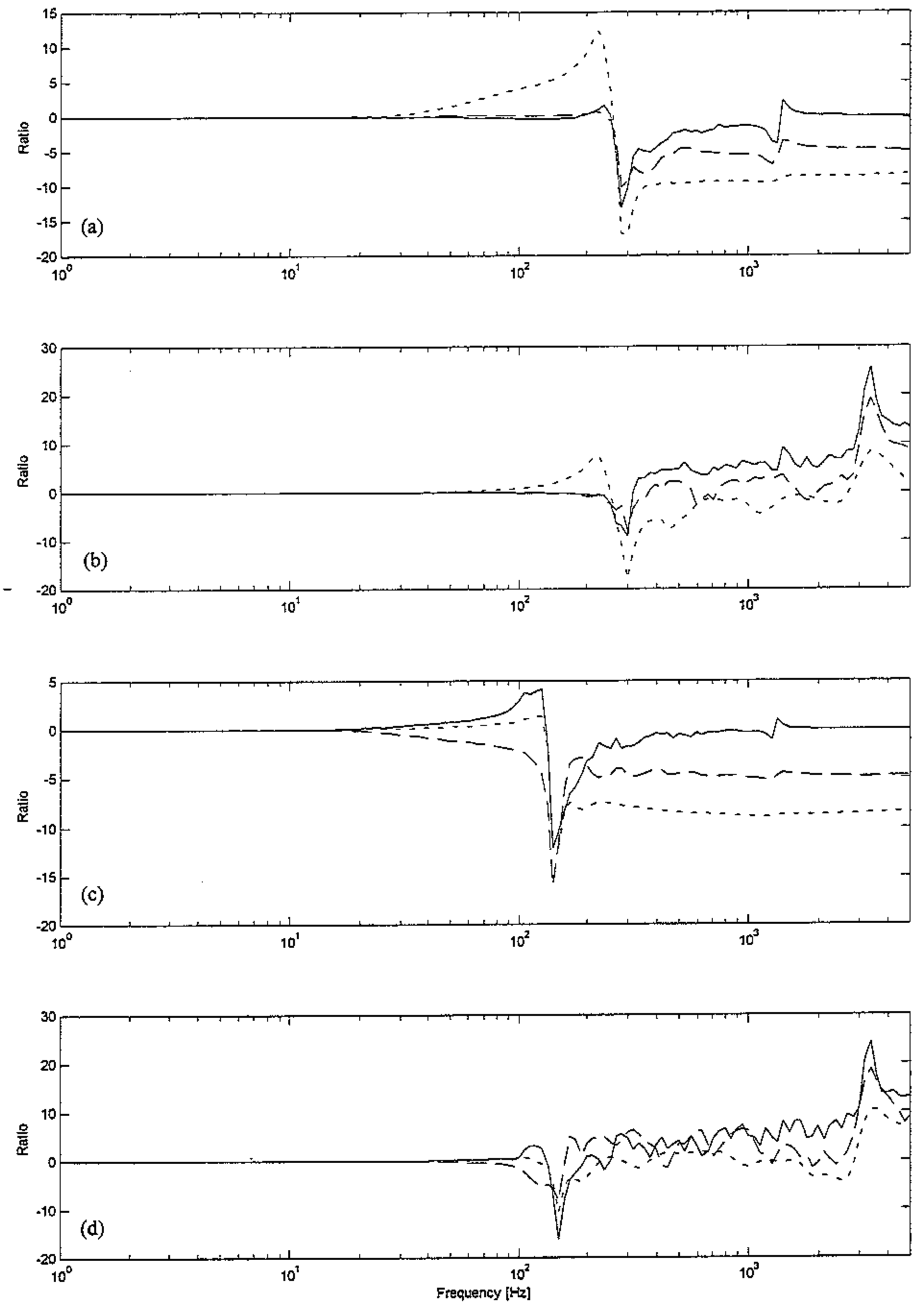


Fig. 10: Ratio between the squared summed velocities expressed in dB scale. Results for the discretely connected beams with different levels of damping are shown relative to the result for the continuous connection with η_s=0.03 in each case

(a), (b) K_s=100 MN/m (a), (c) upper beam

(c), (d) K_s=25 MN/ m (b) (d) lower beam

——— η=0.03

.... η=0.1

– – – η=0.3

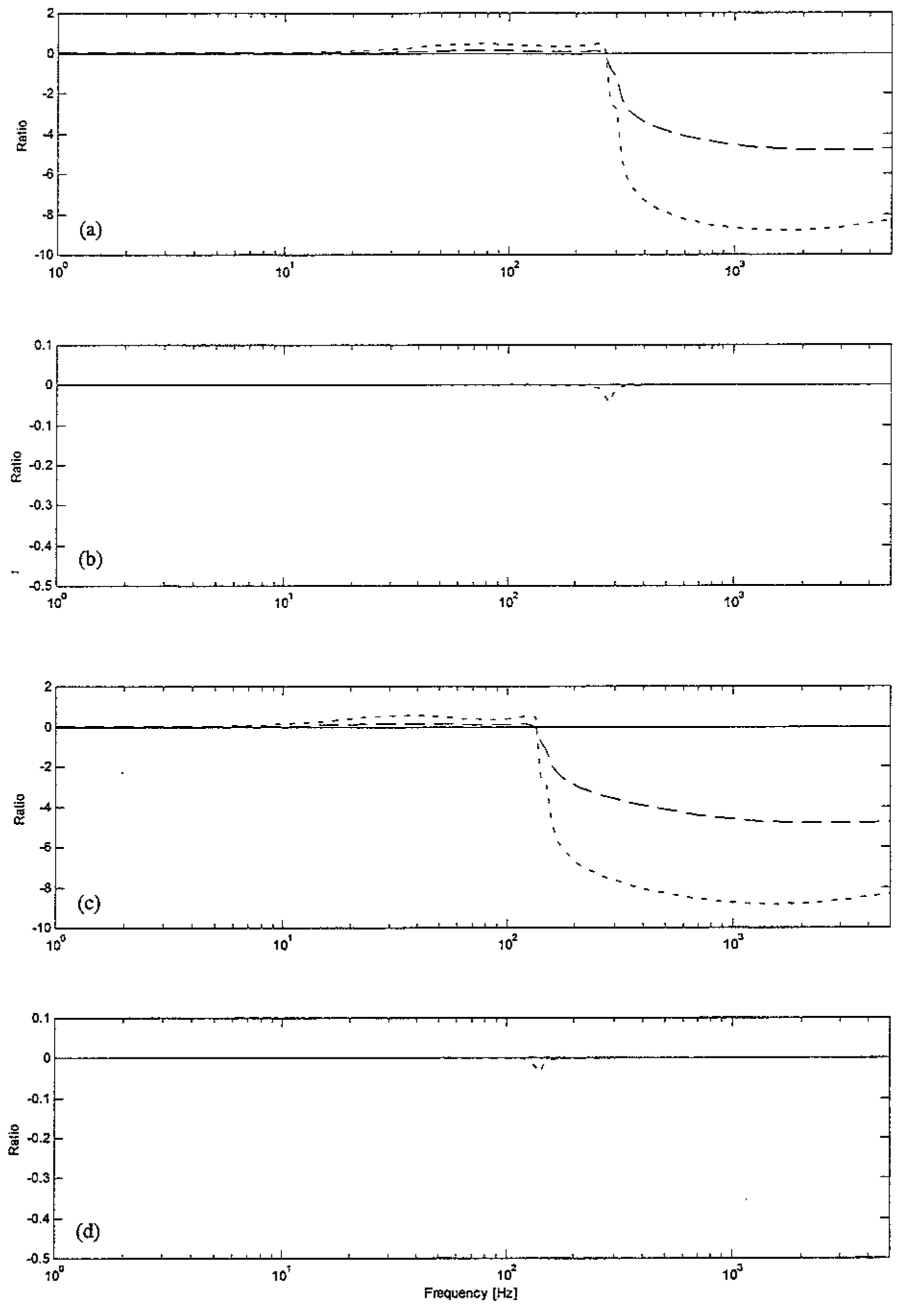


Fig. 11: Ratio between the squared summed velocities expressed in dB scale. Results for the continuously connected beams with different levels of damping are shown relative to the result for the same continuous connection with η_s=0.03

(a), (b) K_s=100 MN/m (a), (c) upper beam

(c), (d) K_s=25 MN/ m (b), (d) lower beam

——— η=0.03

.... η=0.1

– – – η=0.3

5.2.2 Pre-load effects

The reaction of the supports is strongly dependent on the load acting on them. The behaviour of the pad as a support under the effects of local preload (modelling the weight of the train) has already been studied in [4,8] and a non-linear stiffness model has been derived from experimental measurements. For the range of loads acting when the train wheel crosses a railway bridge, a good approximation of the reaction law is:

$$u = \frac{P\beta}{2k} e^{-\beta x} (\cos(\beta x) + \sin(\beta x)) \tag{28}$$

where $\beta = \sqrt[4]{\frac{s}{4E_s I_s}}$, s is the constant modulus of the support stiffness of the bridge and E_sI_s is the bending stiffness of the source beam (the rail).

The reaction force per unit length for the rail loaded by a train of wheels is assumed to change from a peak value of 200 MN/m for the support under the wheel to the standard value of 100 MN/m. As can be seen from the Fig. 12 based on equation (28), the influence of the preload is significant over a range of ±2 m, which corresponds to 5 supports (because of the symmetry of the load) only these supports will experience a change in the stiffness. If two consecutive wheels are spaced less than 2 m apart, the mutual interaction between the loads is assumed to act in such a way that all the supports between the two point loads can be considered to have a constant stiffness K_s= 200 MN/m [4].

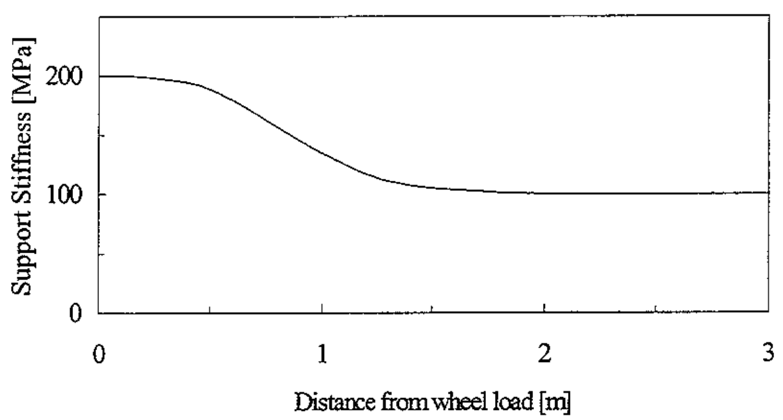


Fig. 12 Space variation of stiffness
for a single wheel load

Fig. 13 shows the average squared velocity for the receiver and the source beam with different values of the stiffness. The soft support model has a constant value for the stiffness of 100 MN/m. The stiff support model presents a uniform stiffness of 200 MN/m, while the pre-loaded model is derived considering a train acting on the source beam and presenting the load distribution of Fig.14. The variation of stiffness generates a shift in the decoupling peak (300-400 Hz) but does not affect the source beam pinned-pinned peak at 1400 Hz or the receiver beam pinned-pinned peak which occurs at 3000 Hz. The higher value of the stiffness increases the amplitude of the receiver vibrations at frequencies above 400 Hz by about 5dB. The result for the preloaded supports follows that for all stiff supports in this region.

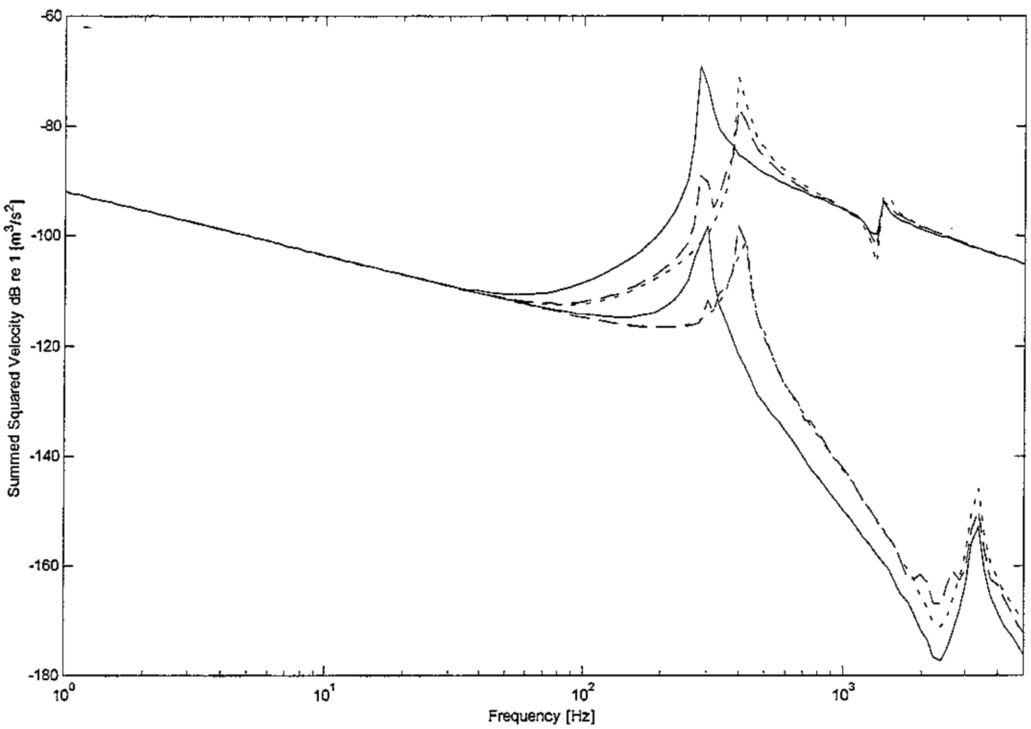


Fig. 13: Mean Square Velocity for pre-loaded supports

.... Stiff Supports
----- Preloaded Supports
——— Soft Supports

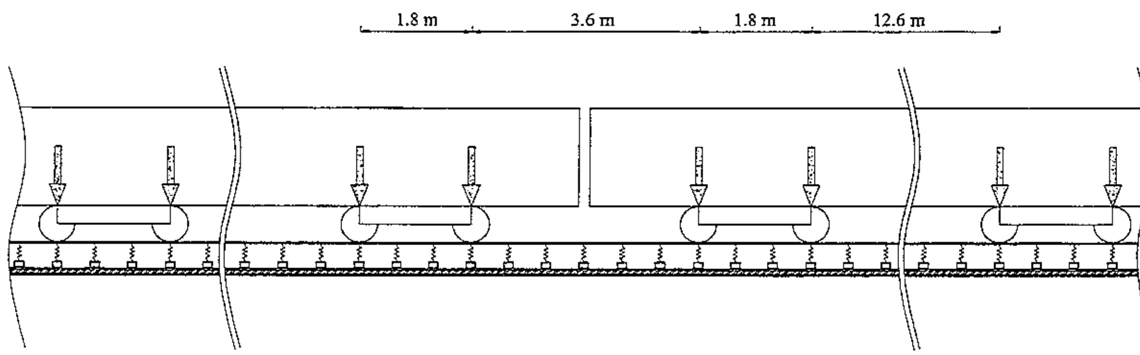


Fig. 14: Scheme of the loadings for a train passing on the bridge

5.2.3 Random Spacing

In order to consider the random spacing of the support points, a special probability density function has been developed in order to represent a similar shape to the normal distribution, but with finite extremes of the range of probability. While the normal distribution presents a probability density of zero for $x=\pm\infty$, the support point spacing has to be defined in a finite range and, in this sense, it is necessary to avoid the spacing, as a random variable, tending towards either negative or positive infinity (see [3] for further explanation). A good approximation for the properties required is given by the probability density function (see Fig. 15):

$$p(x) = \begin{cases} 2\sin^2(\pi x) & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \tag{29}$$

Using an algorithm known in the signal processing programming it has been possible to select a random set of values in the range $x \in [0.3, 0.9]$ having a probability distribution equal to (29). It is possible to demonstrate that the standard deviation of x is $\sigma = 0.06$.

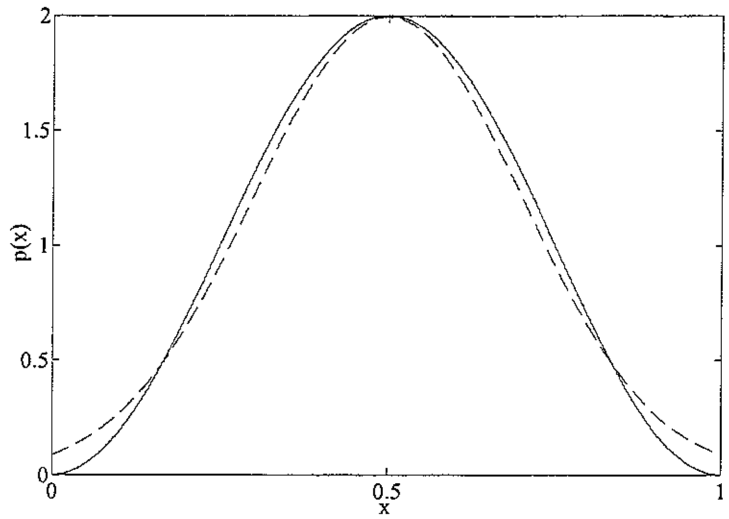


Fig. 15 Probability density for the ad hoc function.

The dashed line represents a Gaussian distribution in [0,1] with mean 0.5 and σ=0.11.

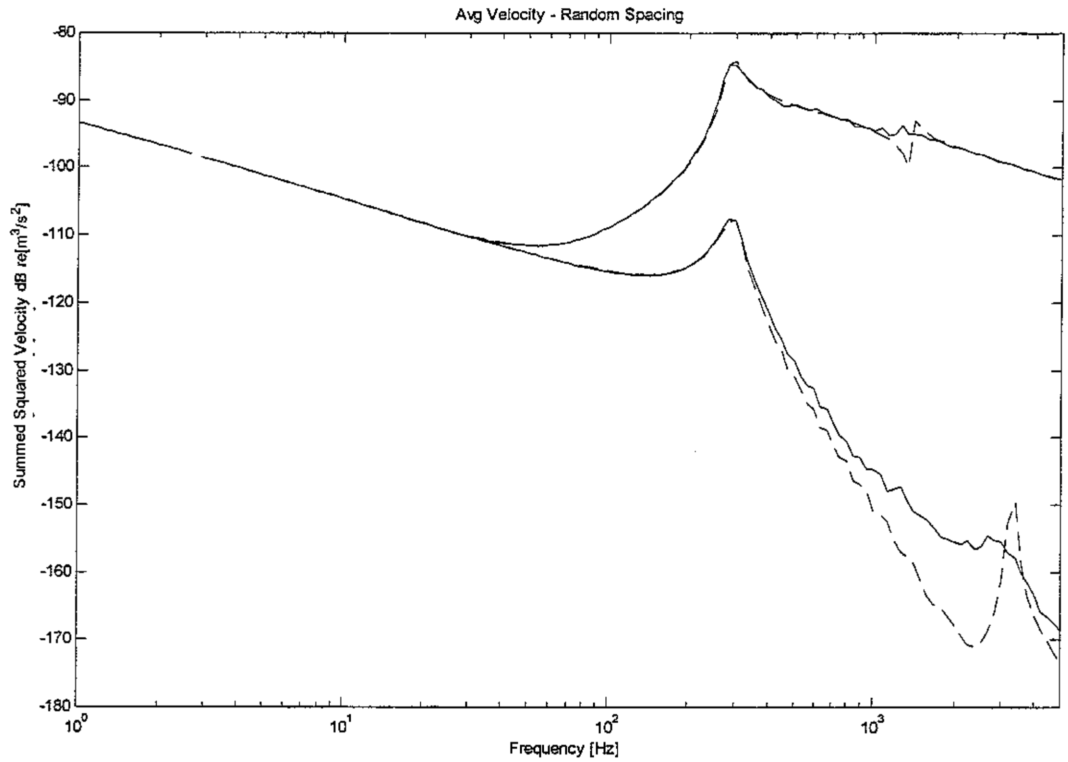


Fig. 16: Effect of random stiffness on response of the rail (upper curves) and bridge (lower curves)

——— Constant spacing (σ=0, d=0.6 m)

- - - Random spacing (σ=0.06, $d \in [0.3, 0.9]$ m)

As can be seen from the Fig. 16, the main difference between the uniformly spaced model and the random distribution of supports is an increased amplitude in the average squared velocity of the receiver in the frequency range 500 Hz to 3000 Hz. The elimination of the pinned-pinned peak is due to the random distribution of reactions: while in a symmetric structure every span is excited by a force aligned to the wave length, this does not happen in the random model.

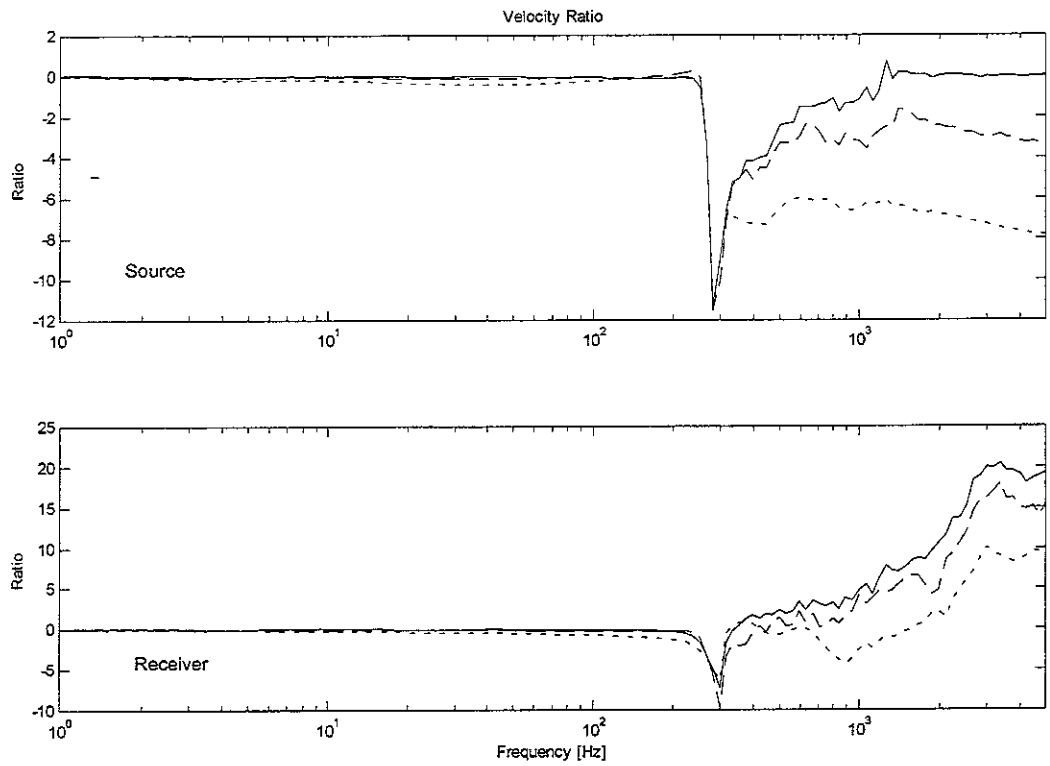


Fig.17: Ratio between the squared summed velocities expressed in dB scale:
random spacing $d \in [0.3,0.9]$ m minus result for continuous connected beam and η=0.03

——— η=0.03
.... η=0.1
---· η=0.3

Fig. 17 presents the difference between the results of the randomly spaced model described in this section and that from [2]. It is possible to observe again the effects of increasing damping in the source beam on the receiver. When the damping is changed from η=0.03 the reduction for the frequency range 300 to 4000 Hz is 3.5 dB and 8 dB in the source and 5 dB and 10 dB in the receiver respectively for loss factors 0.1 and 0.3.

5.2.4 Discrete mass model

A series of masses $M_s^{\,n}$ can be considered joined to the lower beam at the support positions see Fig. 7. In this way the force equilibrium (17) of section 3 changes because of the inertial forces $M_s^{\,n}\cdot\frac{\partial^2}{\partial t^2}(u^r(x_n))$, and, for a harmonic time dependence, it becomes:

$$F_n^{\,s} = K^n{}_s\,(u^s(x_n)-u^r(x_n)) \tag{30}$$

$$F_n^{\,r} = K^n{}_s\,(u^r(x_n)-u^s(x_n)) - M_s^{\,n}(\omega^2\cdot u^r(x_n)) \tag{31}$$

where again $u^s(x)$ is the displacement of the source beam at the point x and $u^r(x)$ is that of the receiver beam. The system of equations obtained in this way is:

$$\begin{cases} u^s(x_n) = -\sum_{m=-N}^{N} G^s(x_m,x_n)F^s{}_m + G^s(x_F,x_n)F \\ u^r(x_n) = \sum_{m=-N}^{N} G^r(x_m,x_n)F^r{}_m \\ F_n^{\,s} = K^n{}_s\,(u^s(x_n)-u^r(x_n)) \\ F_n^{\,r} = K^n{}_s\,(u^r(x_n)-u^s(x_n)) - M_s^{\,n}(\omega^2\cdot u^r(x_n)) \end{cases} \tag{32}$$

Equation (32.iv) represents a difference in the (2,2) component of the assembled matrix. The additional term is defined as follows:

$$b^s(i,j) = \left[G^r(x_i,x_j)\right]\cdot M_s^{\,i} \tag{33}$$

and equation (21) becomes:

$$\begin{bmatrix} \underline{\underline{a}}^s + \underline{\underline{I}}(2N+1) & -\underline{\underline{a}}^s \\ -\underline{\underline{a}}^r & \underline{\underline{a}}^r + \underline{\underline{I}}(2N+1) - \omega^2\cdot\underline{\underline{b}}^s \end{bmatrix}\cdot\begin{bmatrix} \underline{u}^s \\ \underline{u}^r \end{bmatrix} = F\begin{bmatrix} \underline{G}^s(0) \\ \underline{0} \end{bmatrix} \tag{34}$$

the other terms being unchanged.

The plots in Fig. 18 show the changes in the averaged squared velocity for two sets of masses added, the dashed lines represent the response of the model for a value of M_s=20 kg. The dotted lines are for a value of Ms=200 kg. The increase of the total mass in the structure influences the general behaviour of the model in the terms of a reduction of the amplitude of the vibrations at low frequencies (in the range 1 to 300 Hz) and in a growth of the peak at the decoupling resonance.

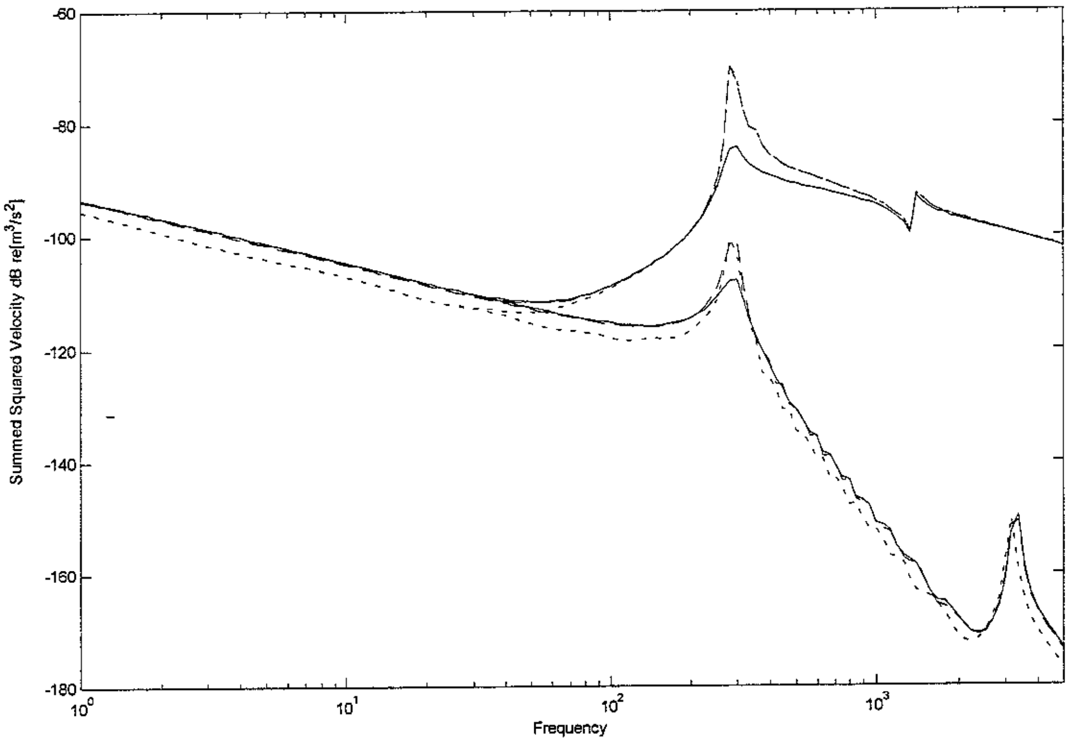


Fig. 18: Effects of discrete masses on the model.

Source beam (upper plots) and receiver (lower plots)

——— M_s=0 kg
- - - M_s=20 kg
······· M_s=200 kg

The effects of a random variation of the discrete masses have been considered, and there is not a substantial difference in the response of the model compared to that with a constant mass.
The results shown in Fig. 19 are developed for the scheme of Fig. 8 considering a random distribution of masses with the same probability density already adopted for the spacing. Two different sets of masses are considered: (a) the first is referred to as a random distribution of masses having M_s=20 kg as a mean value and [10,30] kg as a range of variation, these quantities are adopted in order to simulate a random distribution of the sleeper masses. (b) The second set takes into account a set of masses varying in the range [100,300] kg with a mean value M_s=200 kg. Such a variation is to consider the variability of the masses in the bridge structure. The changes in the resulting vibration can be considered negligible from an engineering point of view.

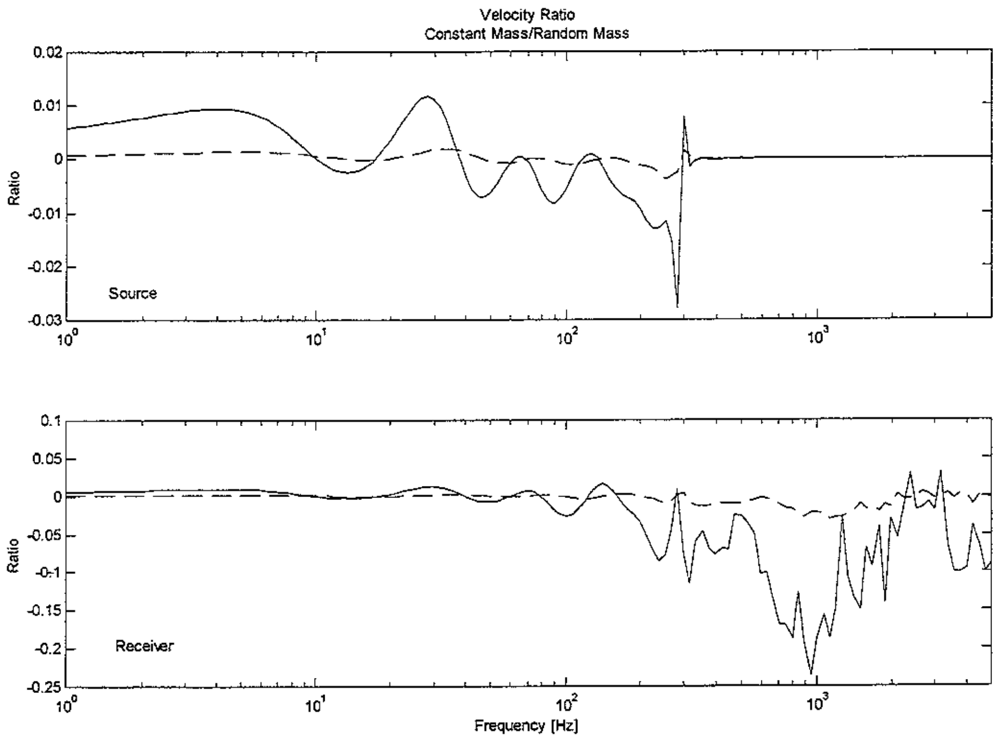


Fig. 19: Difference of summed squared velocity between the random mass model and the corresponding constant mass model

——— M_s=200 kg
.... M_s=20 kg

6. CONCLUSIONS

A model for the high frequency vibration of a rail connected to a bridge has been developed. The effects of a variation of the stiffness in the connection points between the rail and the bridge have been analysed in terms of the influence of the local preload. A variation of about 10 dB has been found in the amplitude of the response for frequencies above the decoupling resonance. At the same time the random variation of the stiffness introduced an increase in the vibration levels of the structure, particularly for the lower beam, where the difference between a constant stiffness model and a variable one reaches an average of about 7 dB for the frequency range 800 to 2000 Hz.

The effects of a random spacing have been considered and a variation of the average levels has been found in the range 200-2000 Hz. It appears that a regular arrangement of supports on the bridge structure should be considered for a better noise design, particularly when higher frequency excitations can act on the deck.

The main difference between the presented model and the one described in [2] is the evidence of the influence of the rail-bridge interactions on the average noise levels of the bridge itself, for which a reduction (8 to 12 dB) can be reached by increasing the damping on the rail by a factor of 10 in the presence of random support stiffness or spacing.

The variation of the global mass of the model obtained by adding discrete masses to the lower beam has been considered. This affects the results since the average total mass of it changes, but a random variation of the set of masses, even for strong variation of mass, does not affect the results significantly.

Further work should consider studying the effects of the Timoshenko beam formulation (included in the program) and the effects of finite length beams. Experimental validation should also be performed.

ACKNOWLEDGEMENTS

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APPENDICES

APPENDIX A

APPENDIX B

APPENDIX A

1.1 Matlab Codes

In this section the codes of the solution of every algorithm are listed.

1.1.1 greens.m

the program evaluate the point receptance for the model described in [4].

```
%% Definition of Green's Functions %%
%% Receptance %%
%% Data loading %%


load c:/carlone/greens/E;
load c:/carlone/greens/G;
load c:/carlone/greens/etr;
load c:/carlone/greens/etp;
load c:/carlone/greens/etb;
load c:/carlone/greens/ro;
load c:/carlone/greens/A;
load c:/carlone/greens/I;
load c:/carlone/greens/k;
load c:/carlone/greens/Ms;
load c:/carlone/greens/d;
load c:/carlone/greens/Kp1;
load c:/carlone/greens/Kb1;
load c:/carlone/greens/N1;
load c:/carlone/greens/N2;
load c:/carlone/greens/O;

x1=d*linspace(-N1,N1,2*N1+1);
oo=10.^[1.7:0.025:3.7]';
o=2*pi*oo';

save x1 x1;

      %% Green's Function %%

k1=(o./sqrt(2)).*((ro/E)+(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;
k2=(o./sqrt(2)).*(-(ro/E)-(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;

u1=(1.i)*((ro*I*o.^2)-(G*k*A)-(E*I*k1.^2))./(E*I*G*k*2*A*k1.*(k1.^2+k2.^2));
u2=(1)*((ro*I*o.^2)-(G*k*A)-(E*I*k2.^2))./(E*G*I*k*2*A*k2.*(k1.^2+k2.^2));

%% DEFINITION OF GREEN FUNCTION %%

%Gp=zeros(N1,N1,81);
Gs=zeros(2*N1+1,2*N1+1,81);
for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      g=(u1.*exp(-1.i*k1.*abs(x1(ii)-x1(jj))))+(u2.*exp(-1*k2.*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj,:)=g;
   end
   gz=(u1.*exp(-1.i*k1.*abs(0-x1(jj))))+(u2.*exp(-1*k2.*abs(0-x1(jj))));
```

```
   Gz(jj,:)=gz;
end

save Gs Gs
save Gz Gz


KP=zeros(2*N1+1,3);
KB=zeros(2*N1+1,3);

%% Choise of use of constant or variable stiffness of the supports %%

   KP(N1+1-8:N1,:)=Kp1;
   KB(N1+1-8:N1,:)=Kb1;

   KP(N1+1+1:N1+1+8,:)=Kp1;
   KB(N1+1+1:N1+1+8,:)=Kb1;

   KP(N1+1,:)=Kp1(1,:);
   KB(N1+1,:)=Kb1(1,:);

   %KP(1:N1,:)=ones(N1,1)*Kp1(8,:);
   %KB(1:N1,:)=ones(N1,1)*Kb1(8,:);


   KP(1:32,:)=ones(N1+1-9,1)*Kp1(8,:);
   KB(1:32,:)=ones(N1+1-9,1)*Kb1(8,:);

   KP(50:2*N1+1,:)=ones(N1+1-9,1)*Kp1(8,:);
   KB(50:2*N1+1,:)=ones(N1+1-9,1)*Kb1(8,:);


save KP KP;
save KB KB;

%% Definition of dynamic stiffness (Alternative) %%

%Z=zeros(2*N1,1,81,3);
 %for jj=1:81
  % for ii=1:2*N1
   %  z=[(1+1.i*etp)*KP(1,1)*[((1+1.i*etb)*KB(1,1))-
Ms*(o(jj)).^2]]./[((1+1.i*etp)*KP(1,1))+((1+1.i*etb)*KB(1,1))-Ms*(o(jj)).^2];
    % Z(ii,:,jj,:)=z;
   %end
%end

%% Definition of dynamic stiffness %%

Z=zeros(2*N1+1,1,81,3);
 for jj=1:81

   for ii=1:2*N1+1
       z=[(1+1.i*etp)*KP(ii,:).*[((1+1.i*etb)*KB(ii,:))-
Ms*(o(jj)).^2]]./[((1+1.i*etp)*KP(ii,:))+((1+1.i*etb)*KB(ii,:))-Ms*(o(jj)).^2];
     Z(ii,:,jj,:)=z;
    end
end

 save Z Z
```

```
%% Definition of the Receptance Matrix %%
%% Data loading %%

load c:/carlone/greens/Z;
load c:/carlone/greens/Gs;
load c:/carlone/greens/Gz;
load c:/carlone/greens/stiff;

Maa=zeros(2*N1+1,81);

for ii=1:81
   for jj=1:2*N1+1
      for kk=1:2*N1+1

         a=(Z(kk,1,ii,stiff))*(Gs(jj,kk,ii));
         aa(jj,kk)=a;

      end
      aa(jj,jj)=aa(jj,jj)+1;
   end
   %Ma=inv(aa);
   %ii
   Maa(:,ii)=aa\Gz(:,ii);


   Um(ii)=-1*(Z(:,1,ii,stiff).*Maa(:,ii))'*(Gz(:,ii))+Gz(N1+1,ii);
end
```

1.1.2 Coup01.m - Coup02.m:

The programme evaluates the point receptance for the coupled beams. It's possible to set both the Timoshenko and the Euler Beam.

```
%% COUPLED MODEL VERSION 2_B   GENERAL MODEL
%% Definition of Green's Functions %%
%% Receptance %%
%% Data loading %%

% IMPORTANT NOTE: EL(1) IS FOR SOURCE EL(2) IS FOR RECEIVER IN THE VECTORS OF
CHARCTERISTICS
load c:/carlone/coupling4/E;
load c:/carlone/coupling4/ro;
load c:/carlone/coupling4/G;
load c:/carlone/coupling4/k;
load c:/carlone/coupling4/AA;
load c:/carlone/coupling4/II;
load c:/carlone/coupling4/Ks;
load c:/carlone/coupling4/etb;
load c:/carlone/coupling4/etp;
load c:/carlone/coupling4/d;
load c:/carlone/coupling4/N1;
load c:/carlone/coupling4/N2;

E=(1+1.i*etb)*E;
Ks=(1+1.i*etp)*Ks;
x1=d*linspace(-N1,N1,2*N1+1);
save x1 x1;
```

```
oo=10.^[1.7:0.025:3.7]';
o=2*pi*oo';

%% DEFINITION OF GREEN FUNCTION %%  Both for the SOURCE and for the RECEIVER

Gs=zeros(2*N1+1,2*N1+1,81);
Gr=zeros(2*N1+1,2*N1+1,81);
Gzs=zeros(2*N1+1,81);
Gzr=zeros(2*N1+1,81);

k1=zeros(81);
k2=zeros(81);
u1=zeros(81);
u2=zeros(81);

%Source%
A=AA(1);
I=II(1);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
k1=(o./sqrt(2)).*((ro/E)+(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;
k2=(o./sqrt(2)).*(-(ro/E)-(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;

u1=(1.i)*((ro*I*o.^2)-(G*k*A)-(E*I*k1.^2))./(E*I*G*k*2*A*k1.*(k1.^2+k2.^2));
u2=(1)*((ro*I*o.^2)-(G*k*A)-(E*I*k2.^2))./(E*G*I*k*2*A*k2.*(k1.^2+k2.^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
%k1=[(ro*(o.^2)*A./(E*I))].^.25;
%k2=[(ro*(o.^2)*A./(E*I))].^.25;

%u1=(-1.i)./(E*I*4*(k1.^3));
%u2=(-1)./(E*I*4*(k1.^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gs=(u1.*exp(-1.i*k1.*abs(x1(ii)-x1(jj))))+(u2.*exp(-1*k2.*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj,:)=gs;
   end
      gzs=(u1.*exp(-1.i*k1.*abs(0-x1(jj))))+(u2.*exp(-1*k2.*abs(0-x1(jj))));
      Gzs(jj,:)=gzs;
end

save Gs Gs
save Gzs Gzs

%Receiver%

A=AA(2);
I=II(2);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o./sqrt(2)).*((ro/E)+(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;
```

```
%k2=(o./sqrt(2)).*(-(ro/E)-(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A./(E*I*o.^2))].^.5).^.5;

%u1=(1.i)*((ro*I*o.^2)-(G*k*A)-(E*I*k1.^2))./(E*I*G*k*2*A*k1.*(k1.^2+k2.^2));
%u2=(1)*((ro*I*o.^2)-(G*k*A)-(E*I*k2.^2))./(E*G*I*k*2*A*k2.*(k1.^2+k2.^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o.^2)*A./(E*I))].^.25;
k2=[(ro*(o.^2)*A./(E*I))].^.25;

u1=(-1.i)./(E*I*4*(k1.^3));
u2=(-1)./(E*I*4*(k1.^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gr=(u1.*exp(-1.i*k1.*abs(x1(ii)-x1(jj))))+(u2.*exp(-1*k2.*abs(x1(ii)-
x1(jj))));
      Gr(ii,jj,:)=gr;
   end
      gzr=(u1.*exp(-1.i*k1.*abs(0-x1(jj))))+(u2.*exp(-1*k2.*abs(0-x1(jj))));
      Gzr(jj,:)=gzr;
end

save Gr Gr
save Gzr Gzr


%KP=zeros(2*N1+1,3);
%KB=zeros(2*N1+1,3);

%% Choise of use of constant or variable stiffness of the supports %%
   %KP(1:8,:)=Kp1;
   %KB(1:8,:)=Kb1;

   %KP(1:N1,:)=ones(N1,1)*Kp1(8,:);
   %KB(1:N1,:)=ones(N1,1)*Kb1(8,:);


   %KP(9:N1,:)=ones(N1-8,1)*Kp1(8,:);
   %KB(9:N1,:)=ones(N1-8,1)*Kb1(8,:);

%save KP KP;
%save KB KB;

%% Definition of dynamic stiffness %% N1 elements of Z are for neg G and the
following N1 are for pos G

Z=Ks*ones(2*N1+1,1,81,3);
 %for jj=1:81

  % for ii=1:N1
     %z=[(1+1.i*etp)*KP(ii,:).*[((1+1.i*etb)*KB(ii,:))-
Ms*(o(jj)).^2]]./[((1+1.i*etp)*KP(ii,:))+((1+1.i*etb)*KB(ii,:))-Ms*(o(jj)).^2];
    % Z(ii,:,jj,:)=Ks;
   %end
%end
 save Z Z
```

```
%% Definition of the Receptance Matrix %%
%% Data loading %%

load c:/carlone/coupling4/Z;
load c:/carlone/coupling4/Gs;
load c:/carlone/coupling4/Gzs;
load c:/carlone/coupling4/Gr;
load c:/carlone/coupling4/Gzr;
load c:/carlone/coupling4/stiff;

Ma=zeros(4*N1+2,4*N1+2);
Maa=zeros(4*N1+2,81);
aas=zeros(2*N1+1,2*N1+1);
aar=zeros(2*N1+1,2*N1+1);
aas1=zeros(2*N1+1,2*N1+1);
aar1=zeros(2*N1+1,2*N1+1);
tn=zeros(4*N1+2,81);
tn(1:2*N1+1,:)=Gzs;

Um=zeros(81,1);
%Um30=zeros(81,1);

%% Source ****
for ii=1:81
   for jj=1:2*N1+1
      for kk=1:2*N1+1

         as=(Z(kk,1,ii,stiff)).*Gs(jj,kk,ii);
%         if kk==1; as=as./2; end;
         aas(jj,kk)=as;
         aas1(jj,kk)=as;

         ar=(Z(kk,1,ii,stiff)).*Gr(jj,kk,ii);
%         if kk==1; ar=ar./2; end;
         aar(jj,kk)=ar;
         aar1(jj,kk)=ar;



      end
      aas1(jj,jj)=aas(jj,jj)+1;
      aar1(jj,jj)=aar(jj,jj)+1;
   end
   Ma=[aas1,-aas;-aar,aar1];

   Maa(:,ii)=Ma\tn(:,ii);


   Um(ii)=(-1*(Z(1,1,1,stiff)*Maa(1:2*N1+1,ii)).'*(tn(1:2*N1+1,ii)))-
Gzs(N1+1,ii);

   %definition of decay rate

   %Um30(ii)=-2*(Z(1,1,1,stiff)*Maa(1:N1,ii)).'*(tn30(1:N1,ii));

end
```

1.1.3 IndexKload.m:

The programme evaluates the point receptance and the squared average velocity for the beams considering different settings of stiffness for the supports. It is possible to define either a constant or a variable stiffness, in order to consider the effects of local pre-loads [3,4]. The programme can also consider the response both for Euler beams and for Timoshenko Beams.

```
%%%Note
%In order to analyse the behaviour of the lower beam I'm going to introduce the
technical data from TNO_Report
% for the lower beam
% E=2.1e+11 etr=0.01 rho=8000 Ks=1.98e+8 etp=0.25
% Ibridge=5.652e-2 Abridge=81.25e-3
%
%A MASS FOR THE SLEEPER IS NOW INTRODUCED MS=20Kg

%*******************************************************************************
******************************


% Data loading %%

% IMPORTANT NOTE: EL(1) IS FOR SOURCE EL(2) IS FOR RECEIVER IN THE VECTORS OF
CHARCTERISTICS


%%% IT'S INTRODUCED A VALUE FOR THE STIFFNESS KS


N1=60;
Z=ones(2*N1+1,1,3);

% soft stiffness
Ks=1.00e+8;

% hard stiffness
%Ks=1.98e+8;

% preloaded rail stiffness A VARIABLE VALUE FOR THE STATIC STIFFNESS IS
INTRODUCED
%Ks=1.00e+8;

Z(10:13,1,:)=2;
Z(19:22,1,:)=2;
Z(49:52,1,:)=2;
Z(58:61,1,:)=2;
Z(88:91,1,:)=2;
Z(97:100,1,:)=2;


Z(8,1,:)=1.17;
Z(15,1,:)=1.17;
Z(17,1,:)=1.17;
Z(24,1,:)=1.17;
Z(47,1,:)=1.17;
Z(54,1,:)=1.17;
Z(56,1,:)=1.17;
Z(63,1,:)=1.17;
```

```
Z(86,1,:)=1.17;
Z(93,1,:)=1.17;
Z(95,1,:)=1.17;
Z(102,1,:)=1.17;


Z(9,1,:)=1.79;
Z(14,1,:)=1.79;
Z(18,1,:)=1.79;
Z(23,1,:)=1.79;
Z(48,1,:)=1.79;
Z(53,1,:)=1.79;
Z(57,1,:)=1.79;
Z(62,1,:)=1.79;
Z(87,1,:)=1.79;
Z(92,1,:)=1.79;
Z(96,1,:)=1.79;
Z(101,1,:)=1.79;

Z=Ks*Z;

E=2.07e+11;
ro=8000;
G=7.7e+10;
k=0.4;
AA=[.00675,.0812];
II=[.00002346,.05652];
etb=.03;
etbl=.1;
etp=.1;
d=.6;
%N1=40;
p=1;
stiff=1;

Ms=0;
MMs=Ms*ones(2*N1+1,1);
%%%%%%%%%%%%%



Eu=(1+1.i*etb)*E;
Gu=(1+1.i*etb)*G;
El=(1+1.i*etbl)*E;
Gl=(1+1.i*etbl)*G;
Ks=(1+1.i*etp)*Ks;

x1=d*linspace(-N1,N1,2*N1+1);
oo=10.^[0:0.025:3.7]';
o=2*pi*oo';

N2=p*N1;
x2=(d/p)*linspace(-N2,N2,2*N2+1);


%% DEFINITION OF GREEN FUNCTION %%  Both for the SOURCE and for the RECEIVER

Gs=zeros(2*N1+1,2*N1+1);
```

```
Gr=zeros(2*N1+1,2*N1+1);
Gzs=zeros(2*N1+1,1);
Gzr=zeros(2*N1+1,1);

Ma=zeros(4*N1+2,4*N1+2);
Maa=zeros(4*N1+2,1);
aas=zeros(2*N1+1,2*N1+1);
aar=zeros(2*N1+1,2*N1+1);
aas1=zeros(2*N1+1,2*N1+1);
aar1=zeros(2*N1+1,2*N1+1);
bbs=zeros(2*N1+1,2*N1+1);
bbr=zeros(2*N1+1,2*N1+1);


tn=zeros(4*N1+2,1);
%tn30=zeros(2*N1);

Ums=zeros(2*N1+1,1);
Umr=zeros(2*N1+1,1);

Um2s=zeros(149,1);
Um2r=zeros(149,1);


Um0s=zeros(149,1);
Um0r=zeros(149,1);

Ges=zeros(2*N1+1,2*N2+1);
Ger=zeros(2*N1+1,2*N2+1);
Gezs=zeros(2*N2+1,1);
Gezr=zeros(2*N2+1,1);

Nor=zeros(149,1);
Norr=zeros(149,1);

Leo=zeros(149,2);

%%%%%%%% FREQUENCY LOOP %%%%%%%%

for kk=1:149;

%Source%

A=AA(1);
I=II(1);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/Eu)+(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;
%k2=(o(kk)/sqrt(2))*(-(ro/Eu)-(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(G*k*A)-(Eu*I*k1^2))/(Eu*I*G*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(G*k*A)+(Eu*I*k2^2))/(Eu*G*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;
```

```
u1=(-1.i)/(Eu*I*4*(k1^3));
u2=(-1)/(Eu*I*4*(k1^3));

Leo(kk,1)=k1;
Leo(kk,2)=u1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gs=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj)=gs;

   end

      gzs=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzs(jj,1)=gzs;

end


%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%%
%                                      SOURCE BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ges=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ges(ii,jj)=ges;

   end

      gezs=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezs(jj,1)=gezs;

end
%%%%%%%%%%%%%%%%%%%%%%%

%Receiver%

A=AA(2);
I=II(2);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/El)+(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;
%k2=(o(kk)/sqrt(2)).*(-(ro/El)-(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(Gl*k*A)-(El*I*k1^2))/(El*I*Gl*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(Gl*k*A)+(El*I*k2^2))/(El*Gl*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(El*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(El*I))]^.25;
```

```
u1=(-1.i)/(El*I*4*(k1^3));
u2=(-1)/(El*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gr=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gr(ii,jj)=gr;

   end
      gzr=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzr(jj,1)=gzr;

end

%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%
%                                      RECEIVER BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ger=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ger(ii,jj)=ger;

   end

      gezr=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezr(jj,1)=gezr;

end
%%%%%%%%%%%%%%%%%%%%%%%

%% Definition of dynamic stiffness %% N1 elements of Z are for neg G and the
following N1 are for pos G

%Z=Ks*ones(2*N1+1,1,3);

  % for ii=1:N1
     %z=[(1+1.i*etp)*KP(ii,:).*[((1+1.i*etb)*KB(ii,:))-
Ms*(o(kk)).^2]]./[((1+1.i*etp)*KP(ii,:))+((1+1.i*etb)*KB(ii,:))-Ms*(o(kk)).^2];
    % Z(ii,:,:)=z;
   %end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Definition of the Receptance Matrix %%

%% Data loading %%
%% ********** %%

tn(1:2*N1+1,:)=Gzs;

   for jj=1:2*N1+1
      for ll=1:2*N1+1

         as=(Z(ll,1,stiff))*Gs(jj,ll);
         bs=MMs(ll,1)*Gs(jj,ll);
```

```
        aas(jj,ll)=as;
        aas1(jj,ll)=as;
        bbs(jj,ll)=bs;

        ar=(Z(ll,1,stiff))*Gr(jj,ll);
        br=MMs(ll,1)*Gr(jj,ll);
        aar(jj,ll)=ar;
        aar1(jj,ll)=ar;
        bbr(jj,ll)=br;


    end
    aas1(jj,jj)=aas(jj,jj)+1;
    aar1(jj,jj)=aar(jj,jj)+1;
  end
  Ma=[aas1,-aas;-aar,aar1-(o(kk)^2)*bbr];

  Maa(:,1)=Ma\tn(:,1);


   Um0s(kk,1)=Maa(N1+1,1);
   Um0r(kk,1)=Maa(3*N1+2,1);

   Ums(:,1)=Maa(1:2*N1+1,1);
   Umr(:,1)=Maa(2*N1+2:4*N1+2,1);

 %%%%%%%%%%SOURCE BEAM %%%%%%%%%%%%%%%%%%%%

  u1m=-1*(((Ges(:,:)).')*((Z(:,1,stiff).*(Ums-Umr))))+Gezs(:,1);
  u1m=abs(u1m);
  Um2s(kk,1)=u1m(N2+1,1);
  u1m=u1m.^2;

  su1m=o(kk)^2*(d/p)*sum(u1m,1);

  Nor(kk,1)=su1m;

  %%%%%%% RECEIVER BEAM %%%%%%%%%%%%%%%%%%%%%%

  u1mr=-1*(((Ger(:,:)).')*((Z(:,1,stiff).*(-Ums+Umr))-((o(kk)^2)*(MMs.*Umr))));
  u1mr=abs(u1mr);
  Um2r(kk,1)=u1mr(N2+1,1);
  u1mr=u1mr.^2;

  su1mr=o(kk)^2*(d/p)*sum(u1mr,1);

  Norr(kk,1)=su1mr;

end
```

1.1.4 IndexKrand.m

The programme evaluates the point receptance and the squared average velocity for a random set of stiffness of the supports.
The line
`Z=sig*Ks*randn(2*N1+1,1,3)+Ks*ones(2*N1+1,1,3);`
is the definition of the siffness matrix with a normal distribution.
`sig` is the relative standard deviation and `Ks` is the mean value.

```
%%%Note
%In order to analyse the behaviour of the lower beam I'm going to introduce the
technical data from TNO_Report
% for the lower beam
% E=2.1e+11 etr=0.01 rho=8000 Ks=1.98e+8 etp=0.25
% Ibridge=5.652e-2 Abridge=81.25e-3

% IN ADDITION A RANDOM DISTRIBUTION OF THE STIFFNESS IS INTRODUCED
% THE DEFINITION OF THE STIFFNESS VECTOR FOR THE SUPPORTS IS INTRODUCED BEFORE
THE FREQUENCY LOOP STARTS AND THE
% STIFFNESS IS CONSIDERED STATIC AT THIS LEVEL OF ANALYSIS

%*******************************************************************************
******************************


%% Receptance %%

%% Data loading %%

% IMPORTANT NOTE: EL(1) IS FOR SOURCE EL(2) IS FOR RECEIVER IN THE VECTORS OF
CHARCTERISTICS

RandK=zeros(149,4,10);

for rr=1:5


E=2.07e+11;
ro=8000;
G=7.7e+10;
k=0.4;
AA=[.00675,.0812];
II=[.00002346,.05652];
Ks=1.00e+8;
etb=.01;
etbl=.1;
etp=.1;
d=.6;
N1=40;
p=1;
stiff=1;

% sig is the standard deviation
sig=.36;
```

```
Eu=(1+1.i*etb)*E;
Gu=(1+1.i*etb)*G;
El=(1+1.i*etbl)*E;
Gl=(1+1.i*etbl)*G;
Ks=(1+1.i*etp)*Ks;

x1=d*linspace(-N1,N1,2*N1+1);
oo=10.^[0:0.025:3.7]';
o=2*pi*oo';

N2=p*N1;
x2=(d/p)*linspace(-N2,N2,2*N2+1);


%% DEFINITION OF GREEN FUNCTION %%  Both for the SOURCE and for the RECEIVER

Gs=zeros(2*N1+1,2*N1+1);
Gr=zeros(2*N1+1,2*N1+1);
Gzs=zeros(2*N1+1,1);
Gzr=zeros(2*N1+1,1);

Ma=zeros(4*N1+2,4*N1+2);
Maa=zeros(4*N1+2,1);
aas=zeros(2*N1+1,2*N1+1);
aar=zeros(2*N1+1,2*N1+1);
aas1=zeros(2*N1+1,2*N1+1);
aar1=zeros(2*N1+1,2*N1+1);

tn=zeros(4*N1+2,1);


Ums=zeros(2*N1+1,1);
Umr=zeros(2*N1+1,1);

Um2s=zeros(149,1);
Um2r=zeros(149,1);


Ges=zeros(2*N1+1,2*N2+1);
Ger=zeros(2*N1+1,2*N2+1);
Gezs=zeros(2*N2+1,1);
Gezr=zeros(2*N2+1,1);

Nor=zeros(149,1);
Norr=zeros(149,1);

%%%%%%%%%%%%%%%%STIFFNESS MATRIX%%%%%%%%%%%%%%%%%%
%% Definition of stiffness %% N1 elements of Z are for neg G and the following
N1 are for pos G

Z=sig*Ks*randn(2*N1+1,1,3)+Ks*ones(2*N1+1,1,3);
%Z=Ks*ones(2*N1+1,1,3);

%%%%%%%% FREQUENCY LOOP %%%%%%%%

for kk=1:149;

%Source%
```

```
A=AA(1);
I=II(1);

%%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/Eu)+(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;
%k2=(o(kk)/sqrt(2))*(-(ro/Eu)-(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(G*k*A)-(Eu*I*k1^2))/(Eu*I*G*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(G*k*A)+(Eu*I*k2^2))/(Eu*G*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;

u1=(-1.i)/(Eu*I*4*(k1^3));
u2=(-1)/(Eu*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gs=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj)=gs;

   end

      gzs=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzs(jj,1)=gzs;

end


%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%
%                                       SOURCE BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ges=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ges(ii,jj)=ges;

   end

      gezs=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezs(jj,1)=gezs;

end
%%%%%%%%%%%%%%%%%%%%%%%

%Receiver%

A=AA(2);
I=II(2);
```

```
%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/El)+(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;
%k2=(o(kk)/sqrt(2)).*(-(ro/El)-(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(Gl*k*A)-(El*I*k1^2))/(El*I*Gl*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(Gl*k*A)+(El*I*k2^2))/(El*Gl*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(El*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(El*I))]^.25;

u1=(-1.i)/(El*I*4*(k1^3));
u2=(-1)/(El*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gr=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gr(ii,jj)=gr;

   end
      gzr=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzr(jj,1)=gzr;

end

%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%
%                                  RECEIVER BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ger=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ger(ii,jj)=ger;

   end

      gezr=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezr(jj,1)=gezr;

end
%%%%%%%%%%%%%%%%%%%%%%%


%% Definition of the Receptance Matrix %%

%% Data loading %%



%% ********** %%

tn(1:2*N1+1,:)=Gzs;

   for jj=1:2*N1+1
```

```
        for ll=1:2*N1+1

            as=(Z(ll,1,stiff))*Gs(jj,ll);
            aas(jj,ll)=as;
            aas1(jj,ll)=as;

            ar=(Z(ll,1,stiff))*Gr(jj,ll);
            aar(jj,ll)=ar;
            aar1(jj,ll)=ar;


        end
        aas1(jj,jj)=aas(jj,jj)+1;
        aar1(jj,jj)=aar(jj,jj)+1;
    end

    Ma=[aas1,-aas;-aar,aar1];
    Maa(:,1)=Ma\tn(:,1);

    Um0s(kk,1)=Maa(N1+1,1);
    Um0r(kk,1)=Maa(3*N1+2,1);

    Ums(:,1)=Maa(1:2*N1+1,1);
    Umr(:,1)=Maa(2*N1+2:4*N1+2,1);


    %%%%%%%%%%SOURCE BEAM %%%%%%%%%%%%%%%%%%%

    u1m=-1*(((Ges(:,:)).')*(Z(:,1,stiff).*(Ums-Umr)))+Gezs(:,1);
    u1m=abs(u1m);
    Um2s(kk,1)=u1m(N2+1,1);
    u1m=u1m.^2;

    su1m=o(kk)^2*(d/p)*sum(u1m,1);

    Nor(kk,1)=su1m;

    %%%%%%% RECEIVER BEAM %%%%%%%%%%%%%%%%%%%%%%

    u1mr=-1*(((Ger(:,:)).')*(Z(:,1,stiff).*(-Ums+Umr)));
    u1mr=abs(u1mr);
    Um2r(kk,1)=u1mr(N2+1,1);
    u1mr=u1mr.^2;

    su1mr=o(kk)^2*(d/p)*sum(u1mr,1);

    Norr(kk,1)=su1mr;

end

RandK(:,:,rr)=[Um0s Um0r Nor Norr];

end
```

1.1.5 IndexDrand

The programme evaluates the point receptance and the squared average velocity for a random set of spacing among the supports.
The ad hoc probability density distribution is defined by an external function `custompdf.m` (see 1.1.6).

```
%
%A RANDOM DISTRIBUTION OF THE SLEEPER SPACING IS INTRODUCED.
%AT THIS LEVEL A NORMALDISTRIBUTION IS USED AND THE VECTOR X1 GIVING THE
POSITION OF THE SUPPORTS IS CONIDERED
%SEPARATED BY X2 USED TO DEFINE THE GREEN FUNCTIONS IN A HIGHER NUMBER OF POINTS
FOR EACH BEAM


% A PROB FUNCTION IS DEFINED FOR THE SLEEPER SPACING
% AN INSIDE_LOOP IS INTRODUCED

******************************

var1=clock;

%% Receptance %%

%% Data loading %%

% IMPORTANT NOTE: EL(1) IS FOR SOURCE EL(2) IS FOR RECEIVER IN THE VECTORS OF
CHARCTERISTICS

RandD=zeros(149,4,10);

for rr=1:3

E=2.07e+11;
ro=8000;
G=7.7e+10;
k=0.4;
AA=[.00675,.0812];
II=[.00002346,.05652];
Ks=1.00e+8;
etb=.1;
etbl=.1;
etp=.1;
d=.6;
N1=40;
p=1;
stiff=1;

% sig is the standard deviation
%sig=.35;


E=(1+1.i*etb)*E;
G=(1+1.i*etb)*G;
El=(1+1.i*etbl)*E;
Gl=(1+1.i*etbl)*G;
Ks=(1+1.i*etp)*Ks;
```

```
%x1=d*linspace(-N1,N1,2*N1+1);

%RANDOM SPACING

%Random 1
%%%%%%%%%%%%%%%%%%%%%%%%%%
%x1=d*linspace(-N1,N1,2*N1+1)+sigs*randn(1,2*N1+1);
%%%%%%%%%%%%%%%%%%%%%%%%%%

%Random 2
%%%%%%%%%%%%%%%%%%%%%%%%%%
% Loop to calculate single arrays with customized distribution

%% RANGE OF DISTRIBUTION OF SPACING AND PROBABILITY FUNNCTION

CU=N1;
x=linspace(0.3,0.9,1000);
xx=linspace(0,1,1000);
fctn=2*(sin(xx*pi)).^2;

cust=zeros(1,CU);
xr=zeros(1,CU+1);
xl=zeros(1,CU+1);


for kk=2:CU+1,

   cust(:,kk-1)=custompdf(x,fctn,1);

   xr(kk)=xr(kk-1)+custompdf(x,fctn,1);
   xl(kk)=xl(kk-1)-custompdf(x,fctn,1);


end;

xl=fliplr(xl(1,2:CU+1));
x1=[xl,xr];
%%%%%%%%%%%%%%%%%%%%%%%%%%


oo=10.^[0:0.025:3.7]';
o=2*pi*oo';

N2=p*N1;
%x2=(d/p)*linspace(-N2,N2,2*N2+1);

x2=x1;

%% DEFINITION OF GREEN FUNCTION %%  Both for the SOURCE and for the RECEIVER

%%%%%%%%%%%%%%%%%%%%%%%
Gs=zeros(2*N1+1,2*N1+1);
Gr=zeros(2*N1+1,2*N1+1);
Gzs=zeros(2*N1+1,1);
Gzr=zeros(2*N1+1,1);

Ma=zeros(4*N1+2,4*N1+2);
Maa=zeros(4*N1+2,1);
aas=zeros(2*N1+1,2*N1+1);
```

```
aar=zeros(2*N1+1,2*N1+1);
aas1=zeros(2*N1+1,2*N1+1);
aar1=zeros(2*N1+1,2*N1+1);

tn=zeros(4*N1+2,1);


Ums=zeros(2*N1+1,1);
Umr=zeros(2*N1+1,1);

Um2s=zeros(149,1);
Um2r=zeros(149,1);


Ges=zeros(2*N1+1,2*N2+1);
Ger=zeros(2*N1+1,2*N2+1);
Gezs=zeros(2*N2+1,1);
Gezr=zeros(2*N2+1,1);

Nor=zeros(149,1);
Norr=zeros(149,1);

%%%%%%%%%%%%%%%%%STIFFNESS MATRIX%%%%%%%%%%%%%%%%%%%
%% Definition of stiffness %% N1 elements of Z are for neg G and the following
N1 are for pos G

% RANDOM STIFFNESS
%Z=sig*Ks*randn(2*N1+1,1,3)+Ks*ones(2*N1+1,1,3);

% CONSTANT STIFFNESS
Z=Ks*ones(2*N1+1,1,3);


%%%%%%%% FREQUENCY LOOP %%%%%%%%

for kk=1:149;

   % DYNAMIC STIFFNESS
  % for ii=1:N1
     %z=[(1+1.i*etp)*KP(ii,:).*[((1+1.i*etb)*KB(ii,:))-
Ms*(o(kk)).^2]]./[((1+1.i*etp)*KP(ii,:))+((1+1.i*etb)*KB(ii,:))-Ms*(o(kk)).^2];
     %Z(ii,:,:)=z;
   %end

%Source%

A=AA(1);
I=II(1);

%%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/E)+(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A/(E*I*o(kk)^2))]^.5)^.5;
%k2=(o(kk)/sqrt(2))*(-(ro/E)-(ro/(k*G))+[((ro/E)-
(ro/(k*G)))^2+(4*ro*A/(E*I*o(kk)^2))]^.5)^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(G*k*A)-(E*I*k1^2))/(E*I*G*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(G*k*A)+(E*I*k2^2))/(E*G*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(E*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(E*I))]^.25;

u1=(-1.i)/(E*I*4*(k1^3));
u2=(-1)/(E*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gs=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj)=gs;

   end

      gzs=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzs(jj,1)=gzs;

end


%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%
%                                      SOURCE BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ges=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ges(ii,jj)=ges;

   end

      gezs=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezs(jj,1)=gezs;

end
%%%%%%%%%%%%%%%%%%%%%%%

%Receiver%

A=AA(2);
I=II(2);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/El)+(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;
%k2=(o(kk)/sqrt(2)).*(-(ro/El)-(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(Gl*k*A)-(El*I*k1^2))/(El*I*Gl*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(Gl*k*A)+(El*I*k2^2))/(El*Gl*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(El*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(El*I))]^.25;
```

```
u1=(-1.i)/(El*I*4*(k1^3));
u2=(-1)/(El*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gr=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gr(ii,jj)=gr;

   end
      gzr=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzr(jj,1)=gzr;

end

%%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%%
%                                        RECEIVER BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ger=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ger(ii,jj)=ger;

   end

      gezr=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezr(jj,1)=gezr;

end
%%%%%%%%%%%%%%%%%%%%%%%


%% Definition of the Receptance Matrix %%

%% Data loading %%


%% ********** %%

tn(1:2*N1+1,:)=Gzs;

   for jj=1:2*N1+1
      for ll=1:2*N1+1

         as=(Z(ll,1,stiff))*Gs(jj,ll);
         aas(jj,ll)=as;
         aas1(jj,ll)=as;

         ar=(Z(ll,1,stiff))*Gr(jj,ll);
         aar(jj,ll)=ar;
         aar1(jj,ll)=ar;



      end
      aas1(jj,jj)=aas(jj,jj)+1;
```

```
        aar1(jj,jj)=aar(jj,jj)+1;
    end
    Ma=[aas1,-aas;-aar,aar1];
    Maa(:,1)=Ma\tn(:,1);

    Um0s(kk,1)=Maa(N1+1,1);
    Um0r(kk,1)=Maa(3*N1+2,1);

    Ums(:,1)=Maa(1:2*N1+1,1);
    Umr(:,1)=Maa(2*N1+2:4*N1+2,1);


    %%%%%%%%%%%SOURCE BEAM %%%%%%%%%%%%%%%%%%%%%
    u1m=-1*(((Z(1,1,stiff)*(Ges(:,:))).')*(Ums-Umr))+Gezs(:,1);
    u1m=abs(u1m);
    Um2s(kk,1)=u1m(N2+1,1);
    u1m=u1m.^2;

    su1m=o(kk)^2*(d/p)*sum(u1m,1);

    Nor(kk,1)=su1m;

    %%%%%%% RECEIVER BEAM %%%%%%%%%%%%%%%%%%%%%%%

    u1mr=-1*(((Z(1,1,stiff)*(Ger(:,:))).')*(-Ums+Umr));
    u1mr=abs(u1mr);
    Um2r(kk,1)=u1mr(N2+1,1);
    u1mr=u1mr.^2;

    su1mr=o(kk)^2*(d/p)*sum(u1mr,1);

    Norr(kk,1)=su1mr;

end

RandD(:,:,rr)=[Um0s Um0r Nor Norr];

end
var2=round(etime(clock,var1));
```

1.1.6 Custompdf.m

The programme allows to generate a set of values whose distribution is defined by the probability function `fctn` given as an input. The other input data are the range of definition of the values `xin` b and the number of points `nout` for the output set `xout`.

```
% function [xout]=custompdf(xin,fctn,nout)
%
% given an input set xin, and a function fctn of xin
% generate a new set xout whose pdf is fctn.
%
% nout is the size of output sequences
%
% an associated data sequence din is then mapped in
% an output data sequence dout.
%
% that is, if xin(:,j) is associated with din(:,j)
% then xout(:,j) will be associated with dout(:,j)
%
% this function uses cumulative function uniform sampling
% to generate the desired pdf
%


function [xout]=custompdf(xin,fctn,nout
[D,T]=size(xin);



% calculate normalized pdf

fctn=fctn/sum(fctn);


% calculate cumulative distribution function

FCTN=zeros(1,T);

for i=1:T,

   FCTN(1,i)=sum(fctn(1,1:i));

end;


% generate custom distribution
xout=zeros(D,nout);


for i=1:nout,

   n=min(find(FCTN>rand(1)));
   xout(:,i)=xin(:,n);

end;
```

1.1.7 IndexMrand.m

The programme evaluates the point receptance and the squared average velocity for a random set of masses of the sleepers.
The ad hoc probability density distribution is defined by an external function `custompdf.m` (see 1.1.6).

```
%A MASS FOR THE SLEEPER IS NOW INTRODUCED
% IN ADDITION A RANDOM DISTRIBUTION OF THE MASS OF SLEEPERS IS ADDED
% THE DISTRIBUTION USED ID THE SAME OF THE SLEEPER SPACING
%*******************************************************************************
******************************

var1=clock;
RandM=zeros(149,4,10);
for rr=1:10

% Data loading %%

% IMPORTANT NOTE: EL(1) IS FOR SOURCE EL(2) IS FOR RECEIVER IN THE VECTORS OF
CHARCTERISTICS


%%% IT'S INTRODUCED A VALUE FOR THE STIFFNESS KS

% soft stiffness
%Ks=1.00e+8;

% hard stiffness
%Ks=1.98e+8;
Ks=1.00e+8;
Z=ones(81,1,3);
Z=Ks*Z;



E=2.07e+11;
ro=8000;
G=7.7e+10;
k=0.4;
AA=[.00675,.0812];
II=[.00002346,.05652];
etb=.01;
etbl=.1;
etp=.1;
d=.6;
N1=40;
p=1;
stiff=1;

Ms=20;
MMs=zeros(2*N1+1,1);

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Random Mass Added

x=linspace(.5,1.5,1000);
xx=linspace(0,1,1000);
fctn=2*(sin(xx*pi)).^2;
```

```
MMs=Ms*(custompdf(x,fctn,2*N1+1)).';
%%%%%%%%%%%%%


Eu=(1+1.i*etb)*E;
Gu=(1+1.i*etb)*G;
El=(1+1.i*etbl)*E;
Gl=(1+1.i*etbl)*G;
Ks=(1+1.i*etp)*Ks;

x1=d*linspace(-N1,N1,2*N1+1);
oo=10.^[0:0.025:3.7]';
o=2*pi*oo';

N2=p*N1;
x2=(d/p)*linspace(-N2,N2,2*N2+1);


%% DEFINITION OF GREEN FUNCTION %%  Both for the SOURCE and for the RECEIVER

Gs=zeros(2*N1+1,2*N1+1);
Gr=zeros(2*N1+1,2*N1+1);
Gzs=zeros(2*N1+1,1);
Gzr=zeros(2*N1+1,1);

Ma=zeros(4*N1+2,4*N1+2);
Maa=zeros(4*N1+2,1);
aas=zeros(2*N1+1,2*N1+1);
aar=zeros(2*N1+1,2*N1+1);
aas1=zeros(2*N1+1,2*N1+1);
aar1=zeros(2*N1+1,2*N1+1);
bbs=zeros(2*N1+1,2*N1+1);
bbr=zeros(2*N1+1,2*N1+1);



tn=zeros(4*N1+2,1);
%tn30=zeros(2*N1);

Ums=zeros(2*N1+1,1);
Umr=zeros(2*N1+1,1);

Um2s=zeros(149,1);
Um2r=zeros(149,1);


Um0s=zeros(149,1);
Um0r=zeros(149,1);

Ges=zeros(2*N1+1,2*N2+1);
Ger=zeros(2*N1+1,2*N2+1);
Gezs=zeros(2*N2+1,1);
Gezr=zeros(2*N2+1,1);

Nor=zeros(149,1);
Norr=zeros(149,1);

Leo=zeros(149,2);
```

```
%%%%%%%% FREQUENCY LOOP %%%%%%%%

for kk=1:149;

%Source%

A=AA(1);
I=II(1);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/Eu)+(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;
%k2=(o(kk)/sqrt(2))*(-(ro/Eu)-(ro/(k*G))+[((ro/Eu)-
(ro/(k*G)))^2+(4*ro*A/(Eu*I*o(kk)^2))]^.5)^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(G*k*A)-(Eu*I*k1^2))/(Eu*I*G*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(G*k*A)+(Eu*I*k2^2))/(Eu*G*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(Eu*I))]^.25;

u1=(-1.i)/(Eu*I*4*(k1^3));
u2=(-1)/(Eu*I*4*(k1^3));

Leo(kk,1)=k1;
Leo(kk,2)=u1;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gs=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gs(ii,jj)=gs;

   end

      gzs=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzs(jj,1)=gzs;

end


%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%%
%                                      SOURCE BEAM

for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ges=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ges(ii,jj)=ges;

   end

      gezs=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezs(jj,1)=gezs;
```

```
end
%%%%%%%%%%%%%%%%%%%%%%%

%Receiver%

A=AA(2);
I=II(2);

%%%%%%%%%%%%%%%%%% Timoshenko Beam %%%%%%%%
%k1=(o(kk)/sqrt(2))*((ro/El)+(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;
%k2=(o(kk)/sqrt(2)).*(-(ro/El)-(ro/(k*Gl))+[((ro/El)-
(ro/(k*Gl)))^2+(4*ro*A./(El*I*o(kk)^2))].^.5).^.5;

%u1=(1.i)*((ro*I*o(kk)^2)-(Gl*k*A)-(El*I*k1^2))/(El*I*Gl*k*2*A*k1*(k1^2+k2^2));
%u2=(1)*((ro*I*o(kk)^2)-(Gl*k*A)+(El*I*k2^2))/(El*Gl*I*k*2*A*k2*(k1^2+k2^2));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%% Euler Beam %%%%%%%%%%%%%
k1=[(ro*(o(kk)^2)*A/(El*I))]^.25;
k2=[(ro*(o(kk)^2)*A/(El*I))]^.25;

u1=(-1.i)/(El*I*4*(k1^3));
u2=(-1)/(El*I*4*(k1^3));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

for jj=1:2*N1+1;
   for ii=1:2*N1+1;
      gr=(u1*exp(-1.i*k1*abs(x1(ii)-x1(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x1(jj))));
      Gr(ii,jj)=gr;

   end
      gzr=(u1*exp(-1.i*k1*abs(0-x1(jj))))+(u2*exp(-1*k2*abs(0-x1(jj))));
      Gzr(jj,1)=gzr;

end

%%%%%%%%%%%%%%%%%%% DEFINITION OF THE EXTENDED MATRIX %%%%%%%%%%%%%%%%
%                                      RECEIVER BEAM
for jj=1:2*N2+1;
   for ii=1:2*N1+1;
      ger=(u1*exp(-1.i*k1*abs(x1(ii)-x2(jj))))+(u2*exp(-1*k2*abs(x1(ii)-
x2(jj))));
      Ger(ii,jj)=ger;

   end

      gezr=(u1*exp(-1.i*k1*abs(0-x2(jj))))+(u2*exp(-1*k2*abs(0-x2(jj))));
      Gezr(jj,1)=gezr;

end
%%%%%%%%%%%%%%%%%%%%%%%
%% Definition of the Receptance Matrix %%

tn(1:2*N1+1,:)=Gzs;

   for jj=1:2*N1+1
```

```
        for ll=1:2*N1+1

            as=(Z(ll,1,stiff))*Gs(jj,ll);
            bs=MMs(ll,1)*Gs(jj,ll);
            aas(jj,ll)=as;
            aas1(jj,ll)=as;
            bbs(jj,ll)=bs;

            ar=(Z(ll,1,stiff))*Gr(jj,ll);
            br=MMs(ll,1)*Gr(jj,ll);
            aar(jj,ll)=ar;
            aar1(jj,ll)=ar;
            bbr(jj,ll)=br;



        end
        aas1(jj,jj)=aas(jj,jj)+1;
        aar1(jj,jj)=aar(jj,jj)+1;
    end
    Ma=[aas1,-aas;-aar,aar1-(o(kk)^2)*bbr];

    Maa(:,1)=Ma\tn(:,1);


     Um0s(kk,1)=Maa(N1+1,1);
     Um0r(kk,1)=Maa(3*N1+2,1);

     Ums(:,1)=Maa(1:2*N1+1,1);
     Umr(:,1)=Maa(2*N1+2:4*N1+2,1);

   %%%%%%%%%%SOURCE BEAM %%%%%%%%%%%%%%%%%%%%

    u1m=-1*(((Ges(:,:)).')*((Z(:,1,stiff).*(Ums-Umr))))+Gezs(:,1);
    u1m=abs(u1m);
    Um2s(kk,1)=u1m(N2+1,1);
    u1m=u1m.^2;

    su1m=o(kk)^2*(d/p)*sum(u1m,1);

    Nor(kk,1)=su1m;

    %%%%%%% RECEIVER BEAM %%%%%%%%%%%%%%%%%%%%%%

    u1mr=-1*(((Ger(:,:)).')*((Z(:,1,stiff).*(-Ums+Umr))-((o(kk)^2)*(MMs.*Umr))));
    u1mr=abs(u1mr);
    Um2r(kk,1)=u1mr(N2+1,1);
    u1mr=u1mr.^2;

    su1mr=o(kk)^2*(d/p)*sum(u1mr,1);

    Norr(kk,1)=su1mr;

end

RandM(:,:,rr)=[Um0s Um0r Nor Norr];

end
var2=round(etime(clock,var1));
```

APPENDIX B

1.1 TNO Report

1.1.1 Damping effects:

introduction of damping factor c_l effect in the elastic layer between the superior beam and the receiver ground. The variables introduced in the procedure change in the following way:

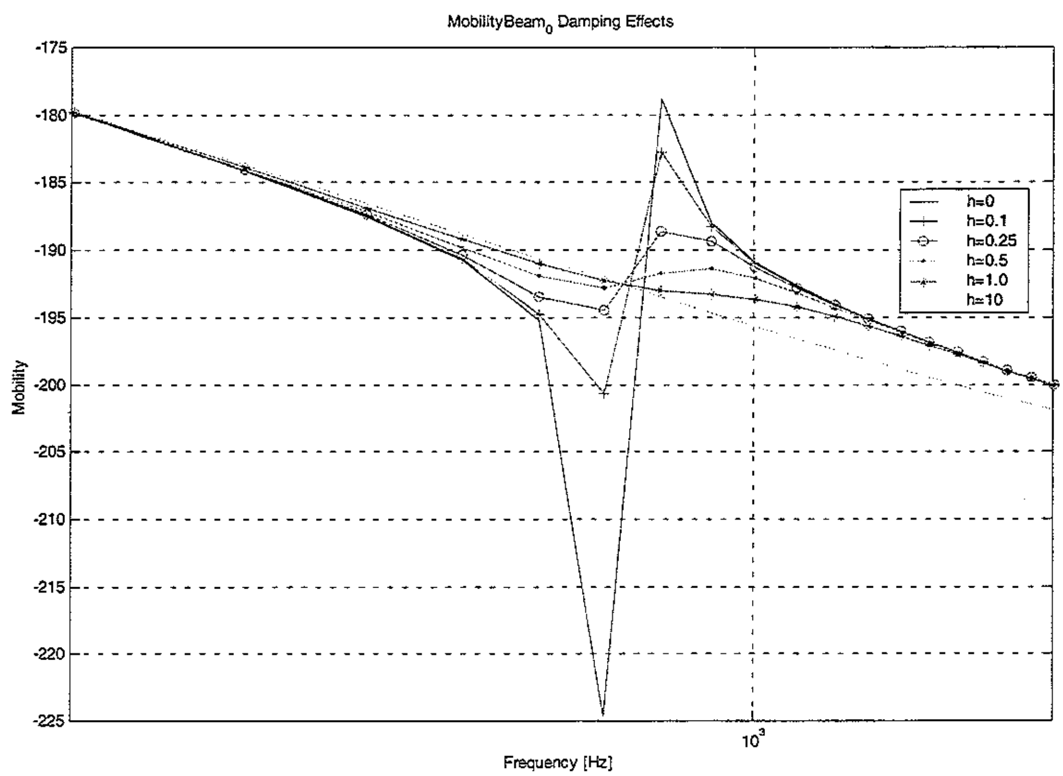
$$\omega^2{}_s = \frac{s(1+\eta i)}{\mu_s} \qquad \omega^2{}_r = \frac{s(1+\eta i)}{\mu_r}$$

In this sense the algorithm for the solution of the characteristic equation and the determination of the mobility and isolation of the system is equal to the undampded one.

Algorithm (Matlab 5.3.1)

```
         TNO_Report 10_Apr_2001   Damping in the layer


%%load c:/carlone/prova/kr1;
%load c:/carlone/prova/ks1;
%load ks4;
%load kr4;
load c:/carlone/damping/et;
%load c:/carlone/damping/osd;
%load c:/carlone/damping/ord;
load c:/carlone/damping/s;
load c:/carlone/damping/o;
load c:/carlone/damping/Bs;
load c:/carlone/damping/Br;
%load c:/carlone/damping/o1;
load c:/carlone/damping/mus;
load c:/carlone/damping/mur;



kr4=mur/Br
ks4=mus./Bs
Bs
Br

% Change of value of variable os and or introducing damping%%%%%%%%%%

et
os=sqrt(s*(1+1.i*et)./mus);
or=sqrt(s*(1+1.i*et)/mur);

os
or
%%%%%%%%%%%%%%%%%%%%%%%%%

for j=1:5
   for i=1:1000
```

```
      a=0.5*[((1-(os(j)/o(i))^2)*ks4(j)*o(i)^2)+((1-(or/o(i))^2)*kr4*o(i)^2)];

      b=0.5*([((1-(os(j)/o(i))^2)*ks4(j)*o(i)^2)+((1-
(or/o(i))^2)*kr4*o(i)^2)]^2+[(((os(j)^2+or^2)/o(i)^2)-
1)*4*kr4*ks4(j)*o(i)^4])^0.5;

      eps1=((or/o(i))^2)/[((a+b)/(kr4*o(i)^2))-1+(or/o(i))^2];

      eps2=-((os(j)/o(i))^2)/[((a-b)/(ks4(j)*o(i)^2))-1+(os(j)/o(i))^2];

   %   eps2=-((os(j)/o1(i))^2)/((((a+b))/((((o(i))^0.5)*ks1(j))^4))-
1+((os(j)/o1(i))^2));

   %   eps2=-
((((os(j))^2))/((o1(i))^2))/((((a+b)^4)/((((o(i))^0.5)*ks1(j)(j))^4))-
1+((((os(j))^2))/((o1(i))^2)));

      y=[(1-1.i)*o(i)/(4*Bs(j)*(1+eps1*eps2))]*[(1/(a+b)^0.75)+(eps1*eps2/(a-
b)^0.75)];


      k14(i,j,:)=a+b;
      k24(i,j,:)=a-b;
      e1(i,j,:)=eps1;
      e2(i,j,:)=eps2;
      Y(i,j,:)=y;

   end
end
```

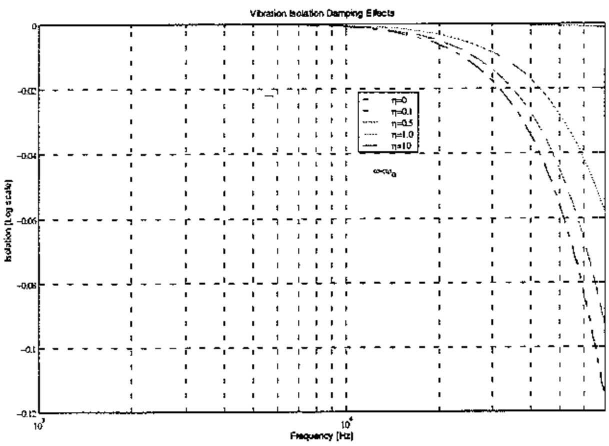


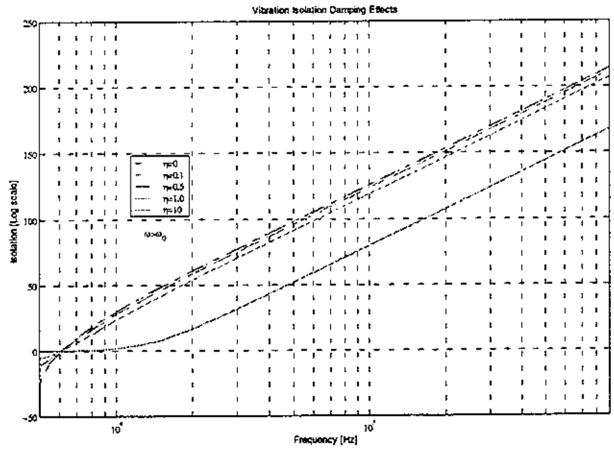
1.1.2 Isolation

the same procedure is taken for the determination of the damping effect in the isolation of the railway. It's possible to notice that the effects of damping are relevant for **high values** of the damping factor ($\eta \geq 1$). In the following plots it's represented the variation trough η of the index I_2 defined as:

$$I_2 = \frac{\bar{u}}{\bar{v}} = \frac{1}{|\varepsilon_I|} \quad \text{for} \quad \omega < \omega_0 \qquad \text{and} \qquad I_2 = \frac{\bar{u}}{\bar{v}} = \left(\frac{u_3^2 + u_4^2}{v_3^2 + v_4^2} \right) \quad \text{for} \quad \omega > \omega_0$$

it represents an index of isolation in terms of relative displacements between the railway and the receiver beam.



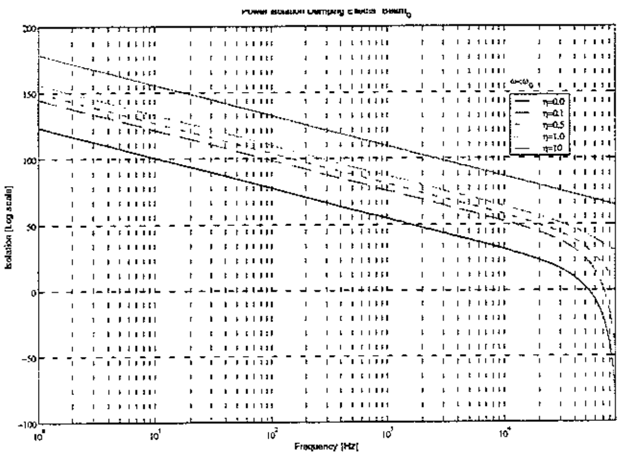


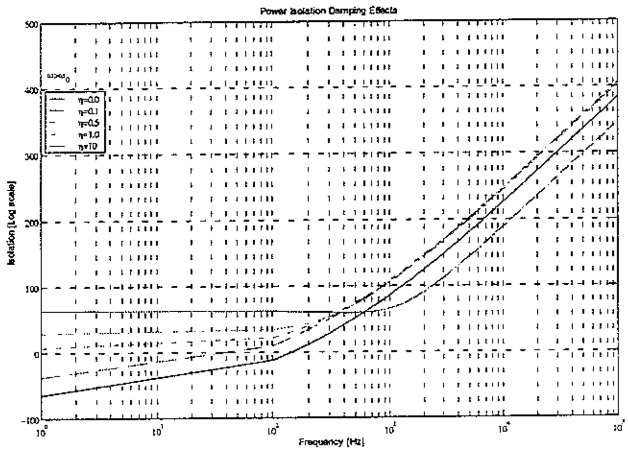
In the following plots the variation of Power Isolation against the whole range of frequency in analysis, trough the index I_3 defined as:

$$I_3^2 = \frac{P_{in}}{P_r} = \frac{\mathrm{Re}(Y)\mathrm{Re}(Y_r)}{\mathrm{Re}(Y_s)^2} \left(\frac{1}{\varepsilon_I} + \varepsilon_{II} \right)^2 \frac{k_I^3 k_r^3}{k_s^6} \quad \text{for} \quad \omega < \omega_0 \quad \text{and}$$

$$I_3^2 = \frac{P_{in}}{P_r} = \frac{\mathrm{Re}(Y)\mathrm{Re}(Y_r)}{\mathrm{Re}(Y_s)^2} \left(\frac{1}{\varepsilon_I} + \varepsilon_{II} \right)^2 \left(\frac{k_s^3}{k_I^3} + \frac{k_s^3}{k_{II}^3} \right)^{-1} \frac{k_r^3}{k_s^3} \quad \text{for} \quad \omega > \omega_0$$

It's possible to notice that the damping effects on the power transmission are more visible for all the values of the damping ratio η. In this sense, the damping is more efficient for the reduction of power transmitted than for the vibration isolation. The plots are about the technical values of the Beam 0.





The following text is the Matlab algorithm for the evaluation of Power Isolation after the introduction of the damping factor:

```
%      TNO_Report 12_Apr_2001     Damping effects in the layer Vibration
Isolation Power


load c:/carlone/damping/power/et;
load c:/carlone/damping/power/s;
load c:/carlone/damping/power/o;
load c:/carlone/damping/power/f0;
load c:/carlone/damping/power/Bs;
load c:/carlone/damping/power/Br;
load c:/carlone/damping/power/o2;
load c:/carlone/damping/power/mus;
load c:/carlone/damping/power/mur;


kr4=mur/Br
ks4=mus./Bs
Bs
Br

% Change of value of variable os and or introducing damping%%%%%%%%%%%

et
os=sqrt(s*(1+1.i*et)./mus);
or=sqrt(s*(1+1.i*et)/mur);

os
or

for j=1:5
    o=6.28*f0(j)*o2;

    for i=1:1000
```

```
%o=ones(1000);

      a=0.5*[((1-(os(j)/o(i))^2)*ks4(j)*o(i)^2)+((1-(or/o(i))^2)*kr4*o(i)^2)];

      b=0.5*([((1-(os(j)/o(i))^2)*ks4(j)*o(i)^2)+((1-
(or/o(i))^2)*kr4*o(i)^2)]^2+[(((os(j)^2+or^2)/o(i)^2)-
1)*4*kr4*ks4(j)*o(i)^4])^0.5;

      eps1=((or/o(i))^2)/[((a+b)/(kr4*o(i)^2))-1+(or/o(i))^2];

      eps2=-((os(j)/o(i))^2)/[((a-b)/(ks4(j)*o(i)^2))-1+(os(j)/o(i))^2];

      y=[(1-1.i)*o(i)/(4*Bs(j)*(1+eps1*eps2))]*[(1/(a+b)^0.75)+(eps1*eps2/(a-
b)^0.75)];


      ar=0.5*[((1-(os(j)/or)^2)*ks4(j)*or^2)];
      br=0.5*([((1-(os(j)/or)^2)*ks4(j)*or^2)]^2+[(((os(j)^2+or^2)/or^2)-
1)*4*kr4*ks4(j)*or^4])^0.5;
      as=0.5*[((1-(or/os(j))^2)*kr4*os(j)^2)];
      bs=0.5*([((1-(or/os(j))^2)*kr4*os(j)^2)]^2+[(((os(j)^2+or^2)/os(j)^2)-
1)*4*kr4*ks4(j)*os(j)^4])^0.5;
      eps1r=1/[((ar+br)/(kr4*or^2))];
      eps2r=-((os(j)/or)^2)/[((ar-br)/(ks4(j)*or^2))-1+(os(j)/or)^2];
      eps1s=((or/os(j))^2)/[((as+bs)/(kr4*os(j)^2))-1+(or/os(j))^2];
      eps2s=-1/[((as-bs)/(ks4(j)*os(j)^2))];


      yr=[(1-
1.i)*or/(4*Bs(j)*(1+eps1r*eps2r))]*[(1/(ar+br)^0.75)+(eps1r*eps2r/(ar-
br)^0.75)];
      ys=[(1-
1.i)*os(j)/(4*Bs(j)*(1+eps1s*eps2s))]*[(1/(as+bs)^0.75)+(eps1s*eps2s/(as-
bs)^0.75)];




p3=((real(y))*(real(yr))/(real(ys))^2)*(((1/eps1)+eps2)^2)*((((ks4(j)*(o(i))^2/(
a+b))^0.75)+((ks4(j)*(o(i))^2/(a-b))^0.75))^(-1))*(kr4/ks4(j))^0.75;


      k14(i,j,:)=a+b;
      k24(i,j,:)=a-b;
      e1(i,j,:)=eps1;
       e2(i,j,:)=eps2;
      Y(i,j,:)=y;
      I3(i,j,:)=p3;
      end

   end
 %   end
```

The following tables are the Excel sheets used as a support for the identification of the several kinds of beam described in the Report.

	s	mu	B	f0	omega0	f	f'
Sources	1160000	1.46E+01	320000	120	753.6	44.9	44.884142
	1160000	3.53	5003	144	904.32	91	91.281338
	1160000	0.912	75.5	211	1325.08	179	179.58577
	1160000	0.23	1.36	375	2355	357	357.60652
	1160000	0.057	0.0207	727	4565.56	718	718.3431
Receiver	1160000	2.31	186			112	112.84007

omega	omega2	omega4	k	k2	k4
281.87241	79452.055	6312629011	0.0821865	0.0067546	4.563E-05
573.2468	328611.9	1.07986E+11	0.1629807	0.0265627	0.0007056
1127.7987	1271929.8	1.61781E+12	0.3315217	0.1099066	0.0120795
2245.769	5043478.3	2.54367E+13	0.6412793	0.4112392	0.1691176
4511.1947	20350877	4.14158E+14	1.2881787	1.6594045	2.7536232
708.63566	502164.5	2.52169E+11	0.3338295	0.1114422	0.0124194