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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

The Influence of Boundary Conditions on the Mode Count and Modal Density of Two-Dimensional Systems

by

G. Xie, D.J. Thompson and C.J.C. Jones

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Authorised for issue by
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Group Chairman

ABSTRACT

The relationship between the mode count of two-dimensional systems and their boundary conditions is investigated, with particular reference to rectangular plates. The effects due to line boundary conditions are found using wavenumber space integration for flexural vibrations of a single plate. These effects are an extension of the corresponding one-dimensional boundary conditions but are not independent of frequency. Then the mode count of a multi-plate system in a single plane is studied, and it is shown that an intermediate line constraint has the same effect on the average mode count as the same type of constraint applied at an edge. Based on the results of these studies of the mode count, the modal density of two-dimensional systems is obtained. It is found that the modal density is also dependent on boundary conditions. An approximate method for modelling an extruded plate is developed based on the results from multi-plate systems. The result of the mode count and modal density for the extruded plate are finally compared with those from an FEM analysis.

CONTENTS

1 INTRODUCTION

Statistical Energy Analysis (SEA) is now one of the most commonly used tools used in the field of noise and vibration analysis at high frequencies. It deals with spatially averaged response levels within frequency bands rather than the precise details of mode shapes and resonance frequencies. Any continuous system or structure possesses an infinite number of natural modes of vibration. At low frequencies the system can be analysed by the method of modal summation but this becomes unwieldy at high frequencies because of larger calculation times, as many modes contribute to the response. By considering the average interaction between the modes of different subsystems, SEA can be used to estimate the high frequency vibration response levels. One of the parameters required to define a subsystem within SEA is the modal density, or average number of modes in a unit frequency interval. Related to this is the mode count, the average number of modes occurring below a certain frequency.

The mode count has the form of a 'staircase' function taking integer values that increase by 1 at each resonance frequency. However, it is often more appropriate to work in terms of an average mode count, which is a smooth line which approximates the staircase function. This can also be seen as the average number of modes below a certain frequency occurring within an ensemble of notionally similar structures. The derivative of this average mode count with respect to frequency is the modal density, which is also a statistical quality.

In order to apply SEA to a complicated structural response problem, it is necessary to know the modal densities of the subsystems. For such a composite structure, the simple additive over its components is utilised in the applications of SEA [1, 2], based on the knowledge of the basic structural elements such as rods, beams and plates. Hart and Shah [1] gave a systematic presentation of the modal density of many basic structural elements such as rods, beams and plates. Expressions for the mode count and modal density of these basic elements of structures are also given by Cremer, Heckl and Ungar [3] and Lyon and DeJong [2]. All these results have been extensively used in the applications of SEA for many years. In general, these expressions are based on the forms in which the modal density is independent of the boundary conditions and is proportional to the size of the system[†]. However, the effect of boundary conditions on the mode count and modal density has received much less attention, being seen as of secondary importance.

For an acoustic volume, the mode count is given by [4]

$$N(f) = \frac{4\pi V}{3c^3} f^3 + \frac{\pi S}{4c^2} f^2 + \frac{L}{8c} f \tag{1.1}$$

where V is the volume, S is the total surface area, L is the total length of edges, c is the sound speed in air and f is frequency. This expression was first obtained by Maa [5] for rigid wall boundaries. For zero pressure boundary conditions the above expression becomes [6]

$$N(f) = \frac{4\pi V}{3c^3} f^3 - \frac{\pi S}{4c^2} f^2 + \frac{L}{8c} f \tag{1.2}$$

In reference [2], the effect of boundary conditions on the mode count and modal density was indicated in terms of a parameter δ_{BC} for one-dimensional subsystems that is stated as being usually constant. An equivalent parameter Γ_{BC} is used for two-dimensional subsystems which are used to introduce a term proportional to the perimeter, but this is usually assumed to be zero for two-dimensional and three-dimensional subsystem. The edge effects were rigorously given by Vasil'ev [7]. Bogomolny and Hugues [8] and Bertelsen, Ellegaard and Hugues [9] give the expressions for the mode count of a rectangular plate under three standard boundary conditions: free, simple support and clamped. In their expressions, there is a perimeter term, which corresponds to Γ_{BC} given in reference [2]. However, for the plate under other combinations of boundary conditions, this perimeter term (or δ_{BC}) is still not available.

The determination of the mode count and modal density is essentially a mathematical problem. It involves the determination of the frequency equation for the structure under consideration from the appropriate equation of motion and then counting the resonance frequencies over all possible modes of vibration. This yields an expression for the mode count in terms of frequency. The differentiation of the expression of the mode count in terms of frequency will then yield the expression for the modal density in terms of frequency. For a simple vibrating rectangular plate, the asymptotic function of the distribution of eigenfrequencies has been analysed by Courant and

---

[†] That is the modal density of a structural system is proportional to the length for a one-dimensional structure, to the area for a two-dimensional one and to the volume for a three-dimensional one.

Hilbert [10] based on the *k*-space integration technique and by Bolotin [11] based on asymptotic methods. However, for a complicated structure, this mathematical solution is usually impractical.

For one-dimensional structural systems, the present authors [12] give a systematic study of the relationship between the mode count and boundary conditions. It was found that the modal density is largely independent of boundary conditions, although the mode count is dependent on boundary conditions. Any type of boundary condition has its own characteristic effect on the mode count of the system. This result is also applicable for a composite system, which contains a set of connected one-dimensional systems. A specific kind of boundary condition will result in the same effect on the mode count if it forms an intermediate boundary in a composite system as if it is at the edge of a single system. This discovery gives an appropriate approach to calculate the mode count and, further, the modal density of a composite system. It also reduces the error compared with simple summation over components of a system.

Based on these studies, the purpose of the present report is to investigate the approximate influence of line boundary conditions on the mode count and modal density of two-dimensional systems. In particular, the main work carried out concerns a rectangular plate in bending vibration with different boundary conditions. The effects of the intermediate constraints are also studied through the FEM to show that the intermediate constraint has the same effect as that applied on an edge. The conclusion drawn from the studies on the rectangular plate is then extended to an application to the modal density of extruded plates.

2 MODE COUNT OF RECTANGULAR PLATES

The mode count of a rectangular plate with a wavenumber less than a given value of k is given by Hart and Sbab [1] and Cremer, Heckl and Ungar [3] as

$$N = \frac{k^2 S}{4\pi} \tag{2.1}$$

where N is the mode count, k is structural wavenumber and S is the area of the plate under consideration.

Lyon and DeJong [2] give an equivalent expression

$$N \simeq \frac{k^2 S}{4\pi} + \Gamma_{BC} P k \tag{2.2}$$

where S is the area of system, P is the perimeter length and Γ_{BC} depends on the boundary conditions. The first term in equation (2.2) is same as equation (2.1).

The modal density of the system can be obtained by the differentiation of equation (2.2). This yields a term Γ'_{BC} in the expression for the modal density. It is suggested in [2] that the quantity Γ'_{BC} can often be determined for an isolated system and that it is best to assume it to be zero for connected systems because the effective boundary conditions change with frequency. Therefore the modal density of a plate is found to be a constant:

$$n(\omega)=\frac{\partial N}{\partial k}\cdot\frac{1}{c_g}=\frac{kS}{2\pi c_g}+\frac{\Gamma'_{BC}P}{c_g}\approx\frac{kS}{2\pi c_g}=\text{Constant} \tag{2.3}$$

as $k \propto \omega^{1/2}$ and $c_g \propto \omega^{1/2}$.

2.1 NATURAL MODES AND MODE COUNT

Before giving a detailed calculation of the mode count of a given rectangular plate, it is instructive to refer to the basic knowledge concerning the flexural vibration of a plate. The equation of motion for flexural waves in an infinite plate is

$$B\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial^2 x\partial^2 y}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{2.4}$$

where w is the out-of-plane displacement, B is the flexural rigidity, given by $B=\dfrac{Eh^3}{12\left(1-\mu^2\right)}$, ρ is the density, h is the thickness, E is Young's modulus and μ is the Poisson's ratio.

Harmonic plane wave solutions have the form

$$w(x,y,t)=e^{-jk_x x}e^{-jk_y y}e^{j\omega t} \tag{2.5}$$

where k_x and k_y are the trace wavenumbers in the x and y directions.

Substituting equation (2.5) into (2.4) shows that the free wave solution satisfies

$$B\left(k_x^2+k_y^2\right)^2=\rho h\omega^2 \tag{2.6}$$

The plate free wavenumber can be defined as

$$k=\sqrt{k_x^2+k_y^2}=\left(\frac{\rho h\omega^2}{B}\right)^{1/4} \tag{2.7}$$

For a finite plate, the natural modes will occur due to wave reflections at boundaries. Considering a rectangular plate of dimensions a and b as shown in Figure 2.1, as for the case of a beam, the phase-closure principle can be applied to it to find the natural modes. By ignoring the effect of near-field waves across the plate, the response of a rectangular plate may be assume to be composed of four free wave components. This is given by

$$w(x, y, t) = \left(A_1 e^{-jk_x x} e^{-jk_y y} + A_2 e^{jk_x x} e^{-jk_y y} + A_3 e^{jk_x x} e^{jk_y y} + A_4 e^{-jk_x x} e^{jk_y y}\right) e^{j\omega t} \tag{2.8}$$

where $k_x = k\cos\theta$ and $k_y = k\sin\theta$, with angle θ the heading of the wave, and k wavenumber determined by equation (2.7).

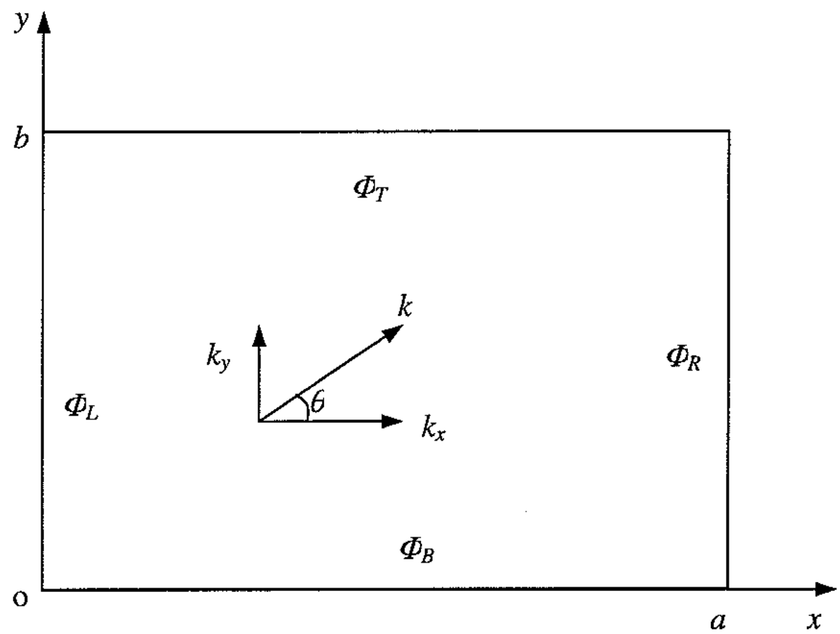

Figure 2.1 Illustration of the rectangular plate under consideration

The phase changes at the boundaries can be defined for each edge: ϕ_L, ϕ_R for reflection in the x direction at the left and right edges and ϕ_T, ϕ_B for reflection in the y direction at the top and bottom edges. The natural modes will occur when

$$2k_x a + \phi_L + \phi_R = 2m\pi \tag{2.9}$$

$$2k_y b + \phi_T + \phi_B = 2n\pi \tag{2.10}$$

for integer values of m and n. These results are arbitrary to within a multiple of 2π.

The natural frequency of the mode (m, n) can be found by substituting these values of k_x and k_y into equation (2.7)

$$\omega_{mn} = \left(\frac{B}{\rho h}\right)^{\frac{1}{2}} \left[k_x^2 + k_y^2\right] = \left(\frac{B}{\rho h}\right)^{\frac{1}{2}} \left[\left(\frac{m\pi}{a} - \frac{\phi_L + \phi_R}{2a}\right)^2 + \left(\frac{n\pi}{b} - \frac{\phi_T + \phi_B}{2b}\right)^2\right] \quad (2.11)$$

These natural modes can also be plotted in wavenumber k-space, as illustrated in Figure 2.2. Every point corresponds to one mode (m, n). The component wavenumbers in the two directions are given by

$$k_x = \frac{m\pi}{a} - \frac{\phi_L + \phi_R}{2a}, \quad k_y = \frac{n\pi}{b} - \frac{\phi_T + \phi_B}{2b} \quad (2.12)$$

It can be seen that the variation of the wavenumber in each direction from one mode to the next is constant, namely π/a and π/b. The boundary conditions effectively only influence the distance to the axes, not the separation between points. This characteristic is very useful allowing the use of k-space integration to calculate the mode count of the rectangular plate.

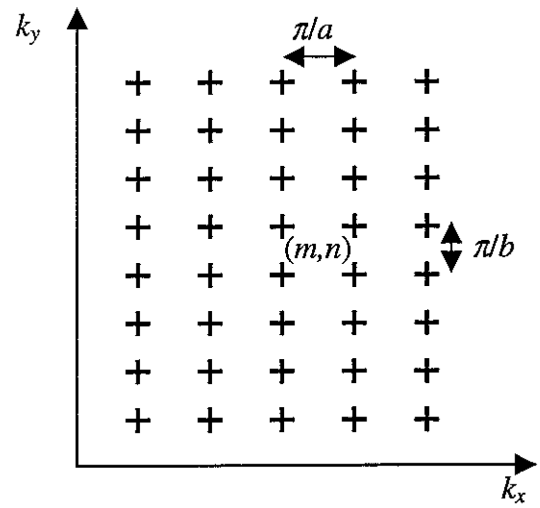

Figure 2.2 Illustration of the k-space of the natural modes of a rectangular plate.

2.1.1 SIMPLY SUPPORTED PLATE

If all four edges of the plate are simply supported, the phase change at each edge is π. The phase-closure principle gives

$$k_x a = m\pi$$
$$k_y b = n\pi \quad (2.13)$$

The natural frequencies are given by

$$\omega_{mn} = \left(\frac{B}{\rho h}\right)^{\frac{1}{2}} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \tag{2.14}$$

In this case the phase closure principle gives the exact result as no near-field waves are generated. The k-space plot of the modes is presented in Figure 2.3. The mode count below a particular frequency can be obtained by finding the number of modes located within the quarter circle of radius k. This is given by

$$N(k) = \frac{\int_S dk_x dk_y}{\Delta k_x \Delta k_y} \tag{2.15}$$

where S is the area of integration, Δk_x and Δk_y are the changes in component wavenumbers from one mode to the next, ie π/a and π/b.

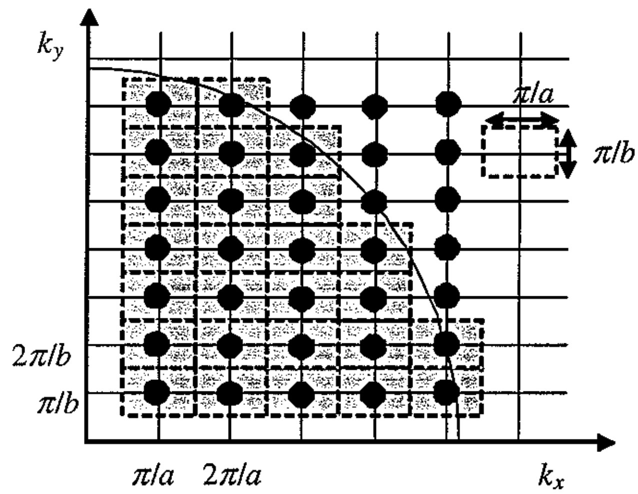

Figure 2.3 The modes of simply supported rectangular plate shown in k-space.

Equation (2.15) can be evaluated as

$$N(k) = \frac{\int_0^k \int_0^{\pi/2} k d\theta dk}{\Delta k_x \Delta k_y} = \frac{\frac{1}{4}\pi k^2}{\Delta k_x \Delta k_y} \tag{2.16}$$

In considering the average area occupied by each mode in k-space, it can be noted that each mode occupies an area of $\pi/a \times \pi/b$ and the area of two strips along the axes should not be taken into account by integration of equation (2.16). This is illustrated in Figure 2.3. Therefore equation (2.16) should be modified so that the average mode count below wavenumber k is given by

$$N(k)=\frac{\frac{1}{4}\pi k^2-k(\frac{\pi}{2a}+\frac{\pi}{2b})+\frac{\pi}{2a}\frac{\pi}{2b}}{\Delta k_x\Delta k_y} \tag{2.17}$$

where Δk_x and Δk_y are given by π/a and π/b.

The final expression for the mode count of a simply supported rectangular plate is

$$N(k)=\frac{k^2S}{4\pi}-\frac{1}{2}k\left(\frac{a+b}{\pi}\right)+\frac{1}{4} \tag{2.18}$$

or

$$N(k)=\frac{k^2S}{4\pi}-\frac{kP}{4\pi}+\frac{1}{4} \tag{2.19}$$

where P is the perimeter of the plate.

The first term in equation (2.19) is that is often used in the literature to estimate the mode count for a plate. It can be seen that the result for the mode count of a simply supported plate in equation (2.19) is less than that given by equation (2.1) due to the perimeter term.

Bogomolny and Hugues [8] give this perimeter term as

$$N_p=\beta\frac{L}{4\pi}k \tag{2.20}$$

where $\beta=-1$ for the simple supported plate. It can be seen that the equation (2.19) has the same results as that from equation (2.20) by [8].

To verify the result of equation (2.19), an aluminium plate of dimensions 0.4m×0.3m×0.002m is considered. The natural modes are calculated based on equations from Leissa [13] (see also Appendix A) and then the staircase function of the mode count is plotted. The average mode counts from equation (2.19) and (2.1) are both calculated and plotted for the purpose of comparison. This is presented in Figure 2.4. The differences between the mode count of equations (2.19), and (2.1) and the actual mode count are plotted in Figure 2.5. The values of the actual mode count are given by the top of each stair point minus 0.5 in each case (that is half way between the bottom and top of each vertical line in the staircase function. The average difference between the results from equation (2.19) and the actual average value is only –0.043. However, equation (2.1) has a systematic error increasing as frequency increases although the relative error decreases, which is shown in Figure 2.4.

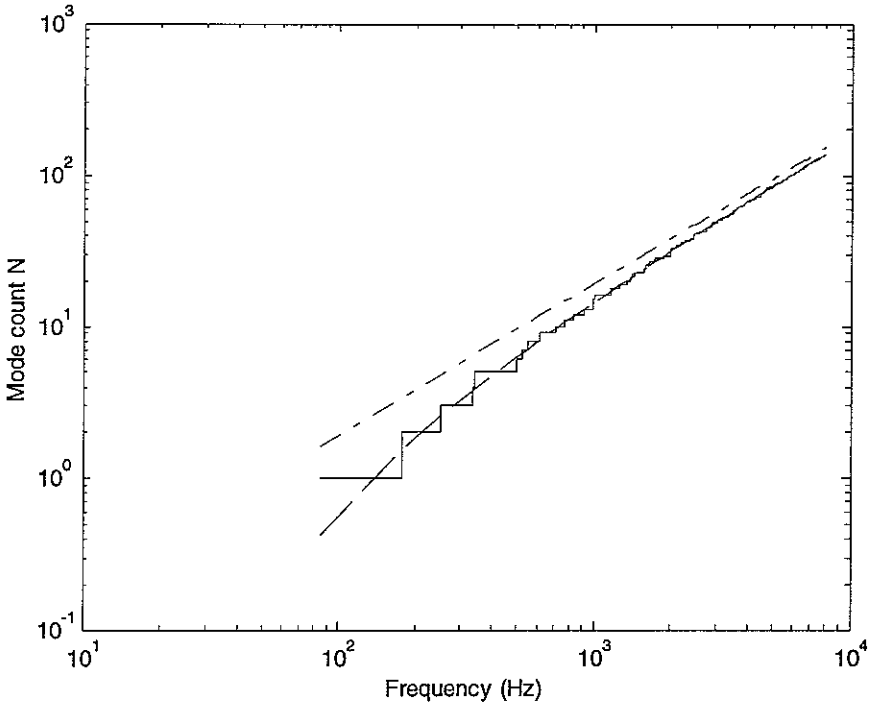

Figure 2.4 Comparison of the mode count for a simply supported aluminium plate 0.4m×0.3m×0.002m.

(—, staircase , ---, from (2.19), -·-·-, from (2.1))

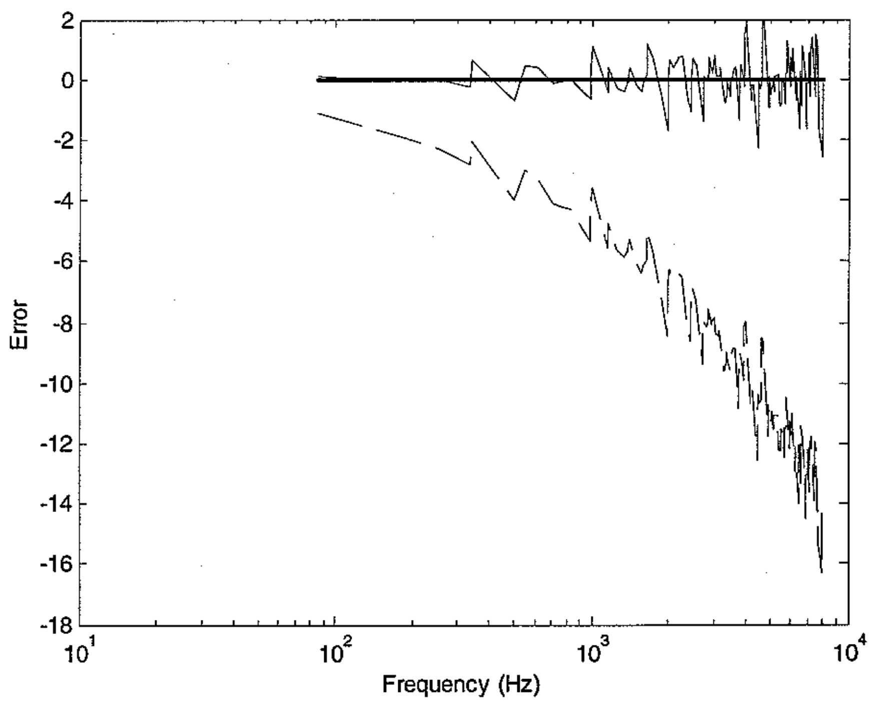

Figure 2.5 Difference between the estimated mode count and the actual result for a simply supported aluminium plate 0.4m×0.3m×0.002m.

(—, errors of (2.19), thick line, average of errors, -·-·-, error of (2.1),)

2.1.2 FULLY FIXED PLATE

Considering a rectangular plate with fully fixed edges, the natural modes occur approximately when the wavenumbers satisfy

$$k_x a = (m + \frac{1}{2})\pi, \quad k_y b = (n + \frac{1}{2})\pi \tag{2.21}$$

where $m, n = 1,2,3, \cdots$.

The k-space plot of the natural modes is presented in Figure 2.6. Based on the equation (2.16) and the concept of the average mode count, the expression of the mode count for a fully fixed rectangular plate can be given by

$$N(k) = \frac{\frac{1}{4}\pi k^2 - k(\frac{\pi}{a} + \frac{\pi}{b}) + \frac{\pi}{a}\frac{\pi}{b}}{\frac{\pi}{a}\frac{\pi}{b}}$$

$$N(k) = \frac{k^2 S}{4\pi} - k\left(\frac{a+b}{\pi}\right) + 1 \tag{2.22}$$

$$N(k) = \frac{k^2 S}{4\pi} - \frac{kP}{2\pi} + 1 \tag{2.23}$$

where P is the perimeter of the plate and S is the area of the plate. In terms of equation (2.20), Bogomolny and Hugues [8] give $\beta = -1.7627598$ for a fully fixed plate. From equation (2.23), β is approximately given as -2.

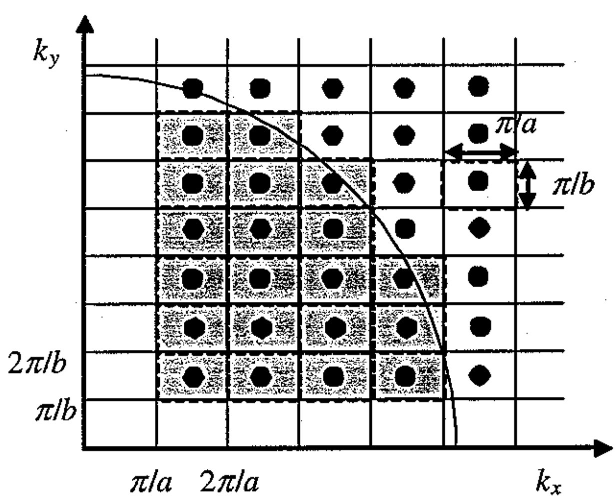

Figure 2.6 The modes of a fully fixed rectangular plate shown in k-space.

Having obtained equation (2.23), it is verified it by comparing it with the analytical solutions. As before an aluminium plate of dimensions 0.4m×0.3m×0.002m is considered as previously. The natural modes can be calculated based on the equation from Leissa [13] (see also Appendix A) and then the staircase function of the mode count can be obtained. Figure 2.7 presents the staircase function and the average mode count from equation (2.23). For comparison, the result from equation (2.1) is also plotted.

It can be seen that equation (2.23) has a much better agreement with the staircase function than equation (2.1), which result in considerable errors. Figure 2.8 shows the comparison of the difference from the actual result. The errors from equation (2.23) are caused by approximations made in applying the phase-closure principle from which it is derived. As the near-field waves are neglected in using the phase-closure principle to obtain the natural modes, the solutions for the first few modes and all modes with m=1 or n=1 tend to have larger values than those from the analytical solution. This eventually causes an underestimate of the mode count equation (2.23) by 2 or 3 at high frequencies. Some lower natural frequencies derived from the phase-closure principle and the analytical solution is listed in Table 1. These differences are larger than the corresponding difference for one-dimensional systems [12]. Compared with the result from equation (2.20) based on $\beta = -1.7627598$ given by Bogomolny and Hugues [8], equation (2.23) has larger error at high frequencies.

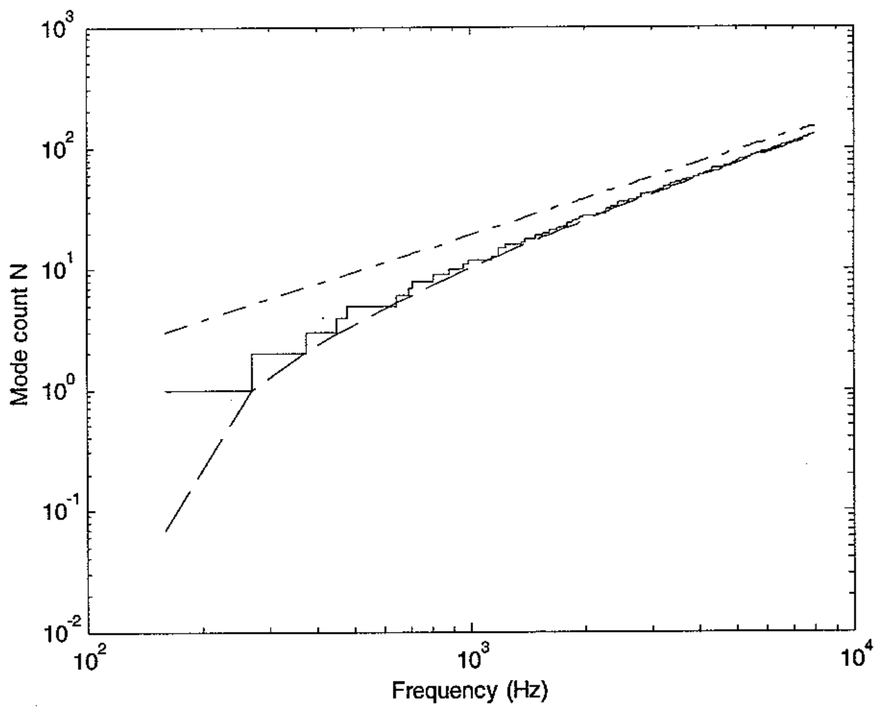

Figure 2.7 Comparison of the mode count for a fully fixed aluminium plate 0.4m×0.3m×0.002m.

(—, staircase , ---, from (2.23), -·-·-, from (2.1))

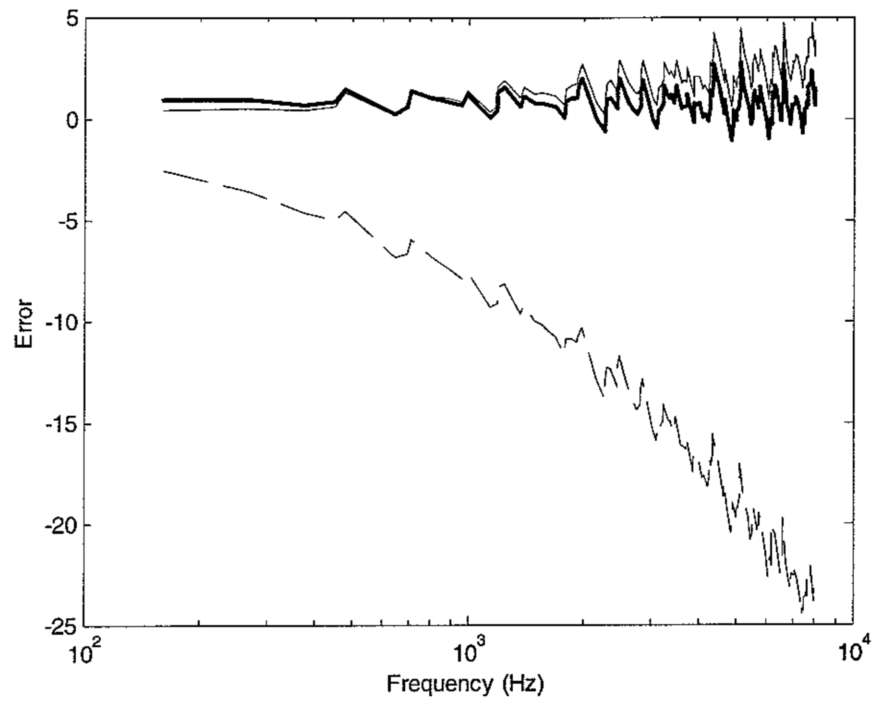

Figure 2.8 Difference between the estimated mode count and the actual result for a fully fixed aluminium plate 0.4m×0.3m×0.002m.

(—, errors of (2.23), thick line, error of (2.20), -·-·-, error of (2.1))

Table 1 Comparison of natural frequencies of fully fixed plate from phase-closure and analytical solution

Index of modes (m, n)	Phase-closure	Analytical	Difference (%)
1, 1	192.4	159.7	20.5
2, 1	315.6	269.3	17.2
1, 2	411.4	376.2	9.4
3, 1	500.4	448.5	11.6
2, 2	534.6	476.7	12.1
3, 2	719.3	645.5	11.4
4, 1	739.8	692.7	6.8
1, 3	746.7	704.8	5.9
2, 3	863.0	802.5	7.5
4, 2	965.6	882.5	9.4

2.1.3 FREE PLATE

For a plate with four free edges, three rigid body modes should be included. These consist of translation, rotation about the x direction and rotation about y direction. A fourth low frequency mode (not a bending mode) exists in which the plate flexes with opposite corners moving in phase. The natural modes in bending should satisfy approximately

$$k_x a = (m - \frac{3}{2})\pi, \quad k_y b = (n - \frac{3}{2})\pi \tag{2.24}$$

where $m, n = 3, 4, \cdots$.

Corresponding to each rigid mode in one direction (translation or rotation) a set of beam-like modes occur in the other direction. These beam-modes have a similar modal characteristic to a one-dimensional system except that the Poisson effect should be considered. The plate modes, therefore, should include two sets of beam-modes in each direction corresponding to the rigid modes in the other direction. This is shown in a k-space plot in Figure 2.9.

The average mode count hence can be given by

$$N(k) = \frac{\frac{1}{4}\pi k^2 + k(\frac{\pi}{a} + \frac{\pi}{b}) + \frac{\pi}{a}\frac{\pi}{b}}{\frac{\pi}{a}\frac{\pi}{b}} \tag{2.25}$$

$$N(k) = \frac{k^2 S}{4\pi} + k\left(\frac{a+b}{\pi}\right) + 1 \tag{2.26}$$

$$N(k) = \frac{k^2 S}{4\pi} + \frac{kP}{2\pi} + 1 \tag{2.27}$$

In terms of equation (2.20), Bertelsen, Ellegaard and Hugues [9] give $\beta = 1.7125908$ for a free plate. From equation (2.27), β is approximately given as 2.

Equation (2.27) is verified with the analytical solutions from Leissa [13] (see Appendix A). Calculations are performed for the same plate as in previous sections. Figure 2.10 shows the results and Figure 2.11 the differences. It can be seen that equation (2.27) gives a good agreement with the analytical solutions, although there is a small error that increases with frequency. This error is probably also caused by using the phase-closure principle to describe the natural modes of the free plate.

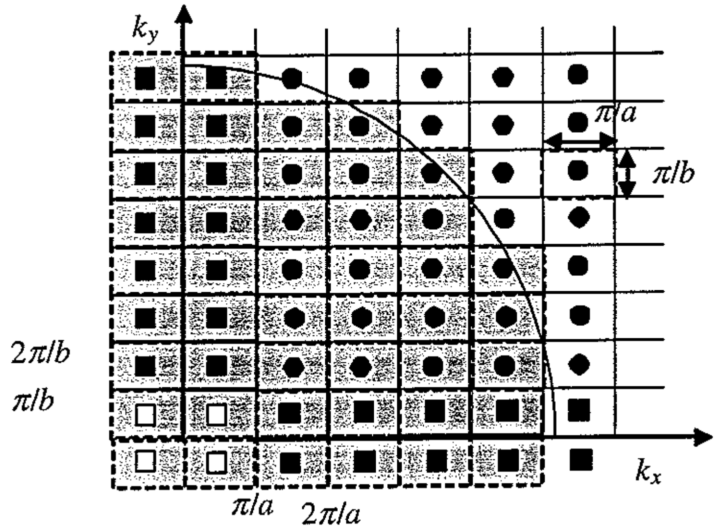

Figure 2.9 The modes of a free plate shown in k-space

(solid square: beam-like mode, square: rigid mode, solid circle: plate mode)

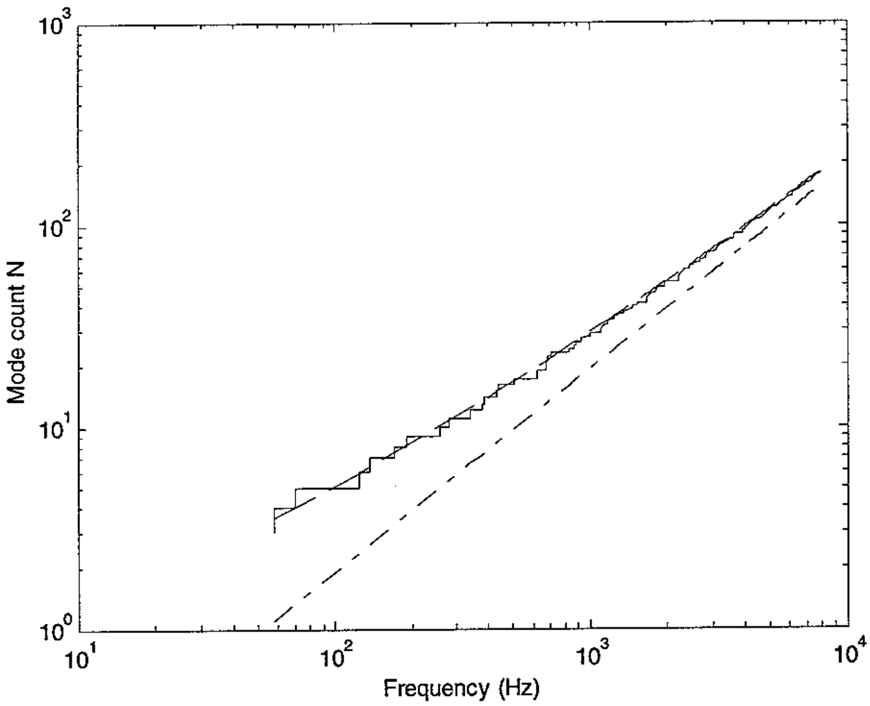

Figure 2.10 Comparison of the mode count for a free aluminium plate 0.4m×0.3m×0.002m

(—, staircase , ---, from (2.27), -·-·-, from (2.1))

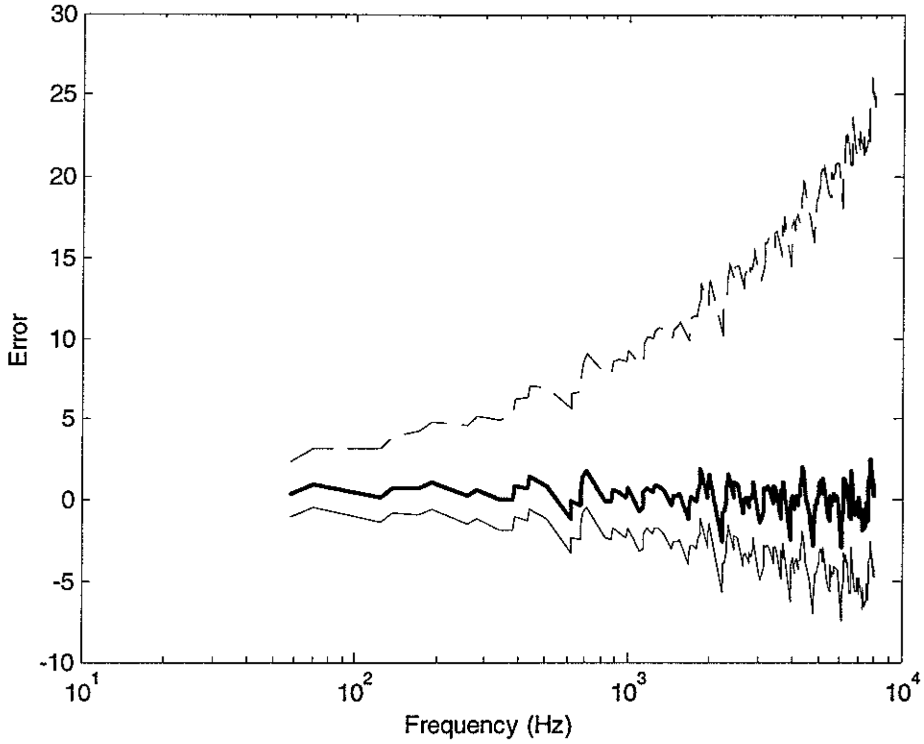

Figure 2.11 Difference between the estimated mode count and the actual result for a free aluminium plate 0.4m×0.3m×0.002m.

(—, errors of (2.27), thick line, errors of (2.20), -·-·-, error of (2.1))

2.1.4 PLATE WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED

Consider a plate having only two opposite edges in the x-direction with simple supports ($x = 0$, $x = a$), the other two edges ($y = 0$, $y = b$) being free. Two rigid modes will occur in the y-direction (considering a beam in this direction). These two rigid modes lead to two sets of simply supported beam-like modes in the x-direction of the plate. The natural modes in bending should satisfy approximately

$$k_x a = m\pi, \quad k_y b = (n - \frac{3}{2})\pi \tag{2.28}$$

where m and n are integers starting from 1 and 3, and for $n = 1, 2$, there are two sets of beam-like modes.

The modes for the plate are shown in a k-space plot in Figure 2.12. All the shaded area must be taken into account for calculating the mode count. The expression of the average mode count is given by

$$N(k)=\frac{\frac{1}{4}\pi k^2-k\frac{\pi}{2a}+k\frac{\pi}{b}-\frac{\pi}{2a}\frac{\pi}{b}}{\frac{\pi}{a}\frac{\pi}{b}}$$

thus

$$N(k)=\frac{k^2S}{4\pi}-\frac{1}{2}\frac{kb}{\pi}+\frac{ka}{\pi}-\frac{1}{2} \tag{2.29}$$

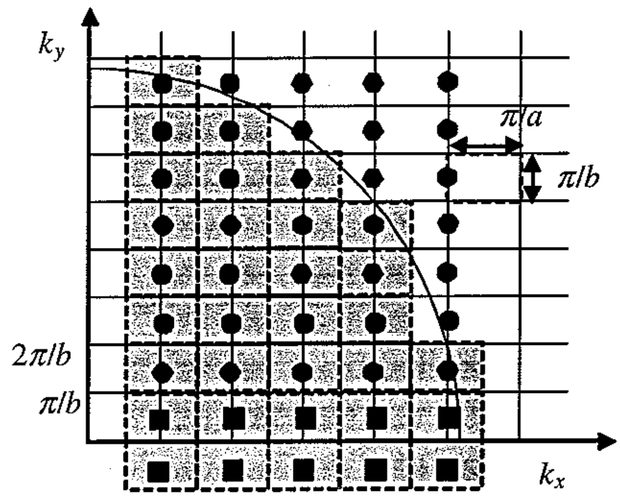

Figure 2.12 The modes of a plate with two opposite edges simply supported shown in *k*-space

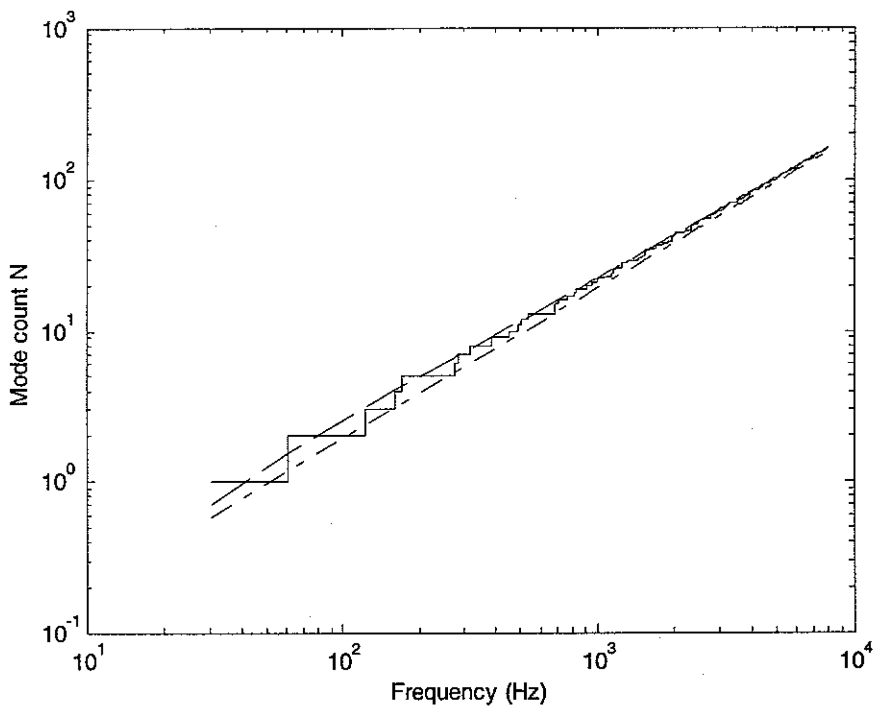

Figure 2.13 Comparison of the mode count for a aluminium plate 0.4m×0.3m×0.002m with two opposite edges free and others simply supported.

(—, staircase , ---, from (2.29), -·-·-, from (2.1))

The result of equation (2.29) is compared with analytical solutions. The same plate dimensions as in previous sections are used. The results are shown in Figure 2.13 and Figure 2.14. It can be seen that equation (2.29) give a good agreement although there is still a small error and equation (2.1) has a systematic error as frequency increases.

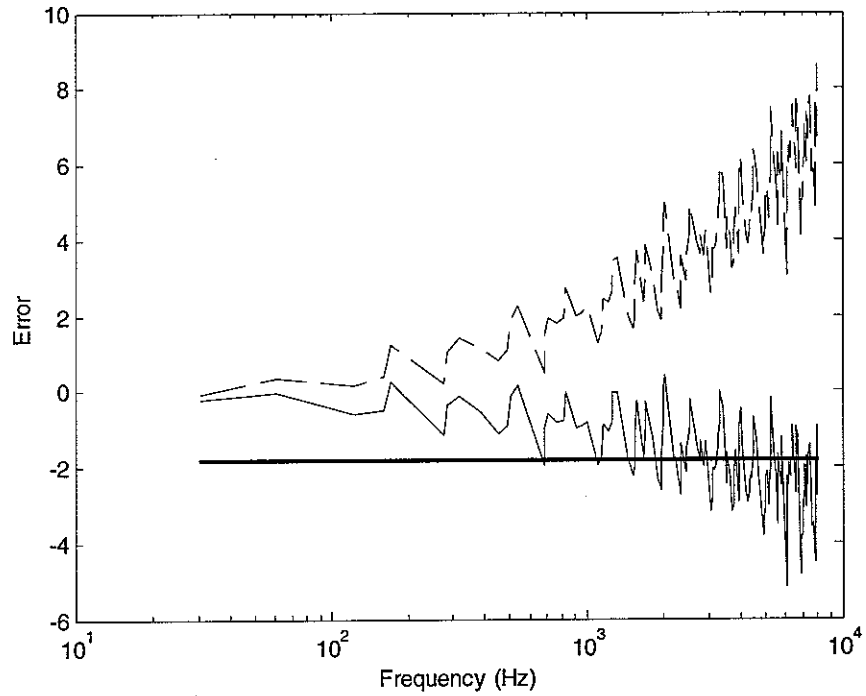

Figure 2.14 Difference between the estimated mode count and the actual result for an aluminium plate 0.4m×0.3m×0.002m with two opposite edges free and others simply supported.

(—, errors of (2.29), thick line, average of errors, -·-·-, error of (2.1))

2.2 RELATIONSHIP BETWEEN MODE COUNT AND BOUNDARY CONDITIONS

From sections 2.1.1 to 2.1.4, it has been seen that the mode count is not independent of the boundary conditions. Using equation (2.1) to evaluate the mode count of a plate will cause a considerable error. The formulae based on the phase-closure principle and *k*-space plot for four combinations of boundary conditions have presented a more correct result. It can also be concluded that in the *k*-space plot for the mode count of a rectangular plate, under any combination of boundary conditions, the mode lattice depends only on the material and dimensions but this is shifted depending on the boundary conditions. By observing equations

(2.18), (2.22), (2.26) and (2.29), the general expression for the mode count of a rectangular plate can be written as

$$N(k)=\frac{k^2S}{4\pi}+\delta_x\frac{k}{\Delta k_y}+\delta_y\frac{k}{\Delta k_x}+\Delta \qquad (2.30)$$

where δ_x and δ_y are the boundary effects along the x and y directions and Δ are is a constant term.

Actually δ_x and δ_y include the effects from two opposite edges in the corresponding directions. Because all examples used previously are symmetric with respect to their boundaries, it is reasonable to anticipate that half the value of δ_x or δ_y will be the boundary effect of one edge [12]. Considering the free plate as a base for comparison, the effect of each type of boundary can be obtained. The error Δ will be ignored because it is very small. It will be shown later that ignoring Δ is acceptable for calculation of the modal density. The mode count for a free rectangular plate is hence given by

$$N(k)=\frac{k^2S}{4\pi}+\frac{k}{\Delta k_y}+\frac{k}{\Delta k_x} \qquad (2.31)$$

Comparing equations (2.29) and (2.31), the effect due to the simple supports on the two edges in the x-direction is given by

$$\delta_x=-\frac{3}{2}\frac{k}{\Delta k_y} \qquad (2.32)$$

Therefore the effect due to a simple support on one edge in the x-direction is given by the half of this

$$\delta_{x-pinned}=-\frac{3}{4}\frac{k}{\Delta k_y} \qquad (2.33)$$

It is noted that the coefficient in the above expression (3/4) is equal to the constant effect of a simple support on the mode count of one-dimensional systems as discussed by Xie, Thompson and Jones [12]. By further comparing the case of a plate simply supported on four edges with the free plate, the effect due to a simple support on one edge in the y-direction can be given as

$$\delta_{y-pinned}=-\frac{3}{4}\frac{k}{\Delta k_x} \qquad (2.34)$$

Similarly, by comparing the case of a fully fixed plate with that of a free plate, the effect of a fixed condition on one edge can be given by

$$\delta_{x-fixed} = -\frac{k}{\Delta k_y} \tag{2.35}$$

$$\delta_{y-fixed} = -\frac{k}{\Delta k_x} \tag{2.36}$$

The constant coefficients in equations (2.35) and (2.36) also correspond with the result of a fixed condition for one-dimensional systems (1).

Based on the above derivations, it is straight forward and reasonable to conclude that the effect of an edge constraint on the mode count of a rectangular plate is equal to the product of the constant effect of the same type of constraint in a one-dimensional beam and a frequency dependent term, which depends on the dimensions and the dispersion relation of the plate. This can be represented by

$$\delta_{2-D} = \delta_{1-D} \frac{k}{\Delta k_{axis}} \tag{2.37}$$

where δ_{1-D} corresponds to the boundary effect in one-dimensional systems (see Table 2.1 in [12]) and Δk_{axis} corresponds to the direction parallel to the line constraint and equals $\frac{\pi}{L_{axis}}$ where L_{axis} is the length of the line constraint. (eg the edge constraint $x = a$ has length b).

Hence the mode count of a rectangular plate can be given by

$$N(k) = \frac{k^2 S}{4\pi} + \left(1 - \delta_{x-left} - \delta_{x-right}\right)\frac{k}{\Delta k_y} + \left(1 - \delta_{y-top} - \delta_{y-bottom}\right)\frac{k}{\Delta k_x} + \Delta \tag{2.38}$$

where δ_{x-left}, $\delta_{x-right}$, δ_{y-top} and $\delta_{y-bottom}$ are the one-dimensional boundary effects corresponding to the boundary conditions of the four edges, and Δ is a small constant.

2.3 FURTHER EXAMPLES

2.3.1 A PLATE WITH SIMPLE SUPPORTS AND CLAMPED EDGES IN TWO DIRECTIONS

Consider a plate with two adjacent edges simply supported and the other two edges clamped. Without considering the k-space plot for the natural modes, the mode count directly obtained from equation (2.38) can be given by

$$N(k)=\frac{k^2S}{4\pi}+\left(1-\delta_{x-pinned}-\delta_{x-clamped}\right)\frac{k}{\Delta k_y}+\left(1-\delta_{y-pinned}-\delta_{y-clamped}\right)\frac{k}{\Delta k_x} \tag{2.39}$$

$$N(k)=\frac{k^2S}{4\pi}+\left(1-\frac{3}{4}-1\right)\frac{k}{\Delta k_y}+\left(1-\frac{3}{4}-1\right)\frac{k}{\Delta k_x} \tag{2.40}$$

$$N(k)=\frac{k^2S}{4\pi}-\frac{3}{4}\frac{k}{\Delta k_y}-\frac{3}{4}\frac{k}{\Delta k_x} \tag{2.41}$$

$$N(k)=\frac{k^2S}{4\pi}-\frac{3}{4}k\frac{a}{\pi}-\frac{3}{4}k\frac{b}{\pi} \tag{2.42}$$

The result from equation (2.42) is compared with the analytical solutions from Leissa [13] in Figure 2.15. It can be seen that equation (2.42) give a good agreement although there is still a small error and equation (2.1) has a systematic error as frequency increases.

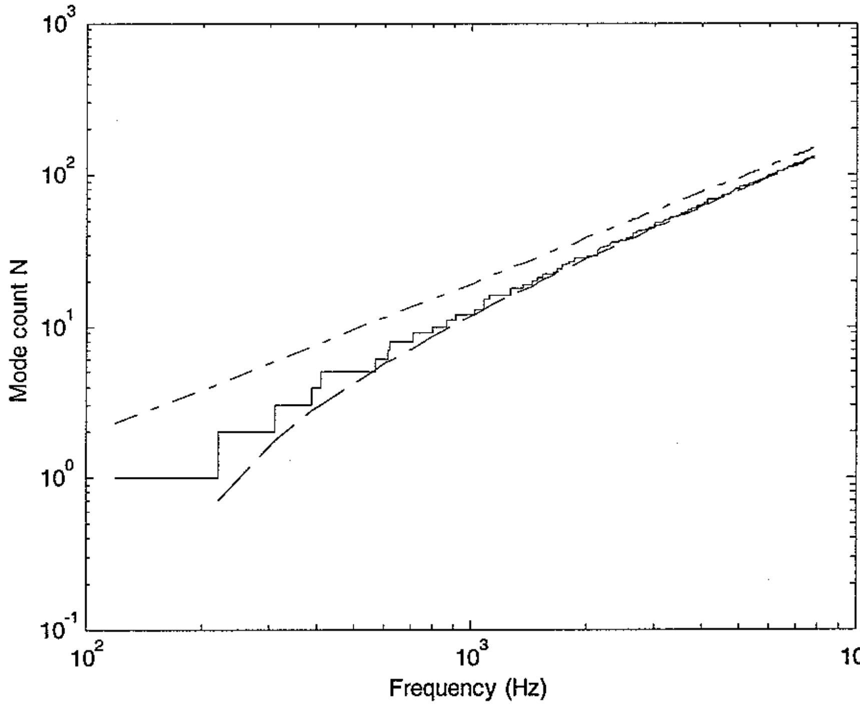

Figure 2.15 Comparison of the mode count for a pinned-pinned-fixed-fixed plate 0.4m×0.3m×0.002m. (—, analytical , ---, from (2.42), -·-·-, from (2.1))

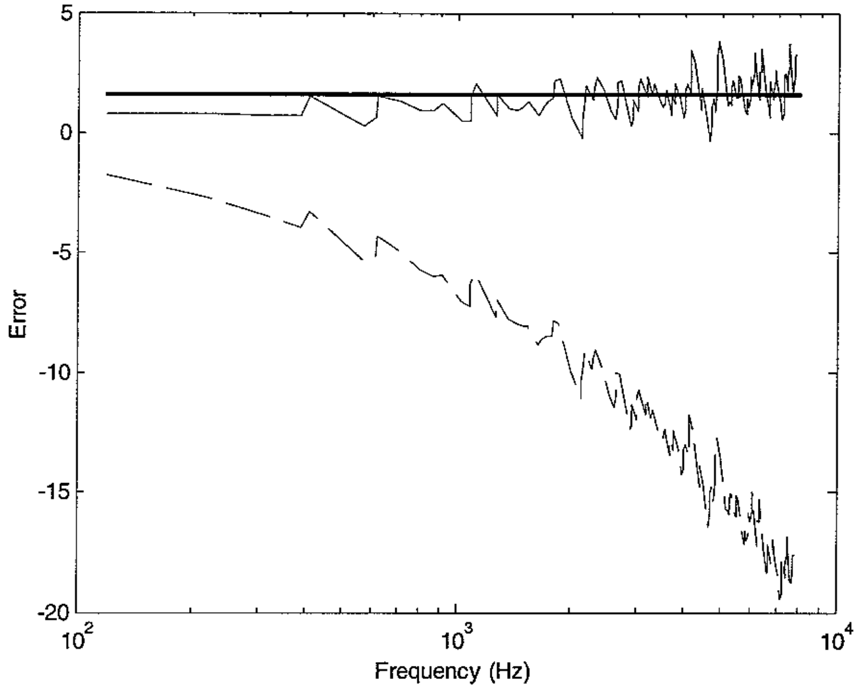

Figure 2.16 Difference between the estimated mode count and the actual result for a pinned-pinned-fixed-fixed aluminium plate 0.4m×0.3m×0.002m..
(—, errors of (2.42), thick line, average of errors, -·-·-, error of (2.1))

2.3.2 A PINNED-PINNED-PINNED-FREE PLATE

Consider a plate with one free edge in the *y*-direction ($y = 0$) and the others pinned. The mode count is given by

$$N(k) = \frac{k^2 S}{4\pi} + \left(1 - \delta_{x-pinned} - \delta_{x-pinned}\right)\frac{k}{\Delta k_y} + \left(1 - \delta_{y-pinned} - \delta_{y-free}\right)\frac{k}{\Delta k_x} \qquad (2.43)$$

$$N(k) = \frac{k^2 S}{4\pi} + \left(1 - \frac{3}{4} - \frac{3}{4}\right)\frac{k}{\Delta k_y} + \left(1 - \frac{3}{4} - 0\right)\frac{k}{\Delta k_x} \qquad (2.44)$$

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{2}\frac{k}{\Delta k_y} + \frac{1}{4}\frac{k}{\Delta k_x} \qquad (2.45)$$

$$N(k) = \frac{k^2 S}{4\pi} - \frac{1}{2}k\frac{b}{\pi} + \frac{1}{4}k\frac{a}{\pi} \qquad (2.46)$$

The comparison of the result from equation (2.45) with the analytical solution is shown in Figure 2.17. Equation (2.45) gives a good agreement with the analytical solution.

Further combinations of boundary conditions for a rectangular plate can be used to verify the applicability of equation (2.38) but these will not be discussed in the present report.

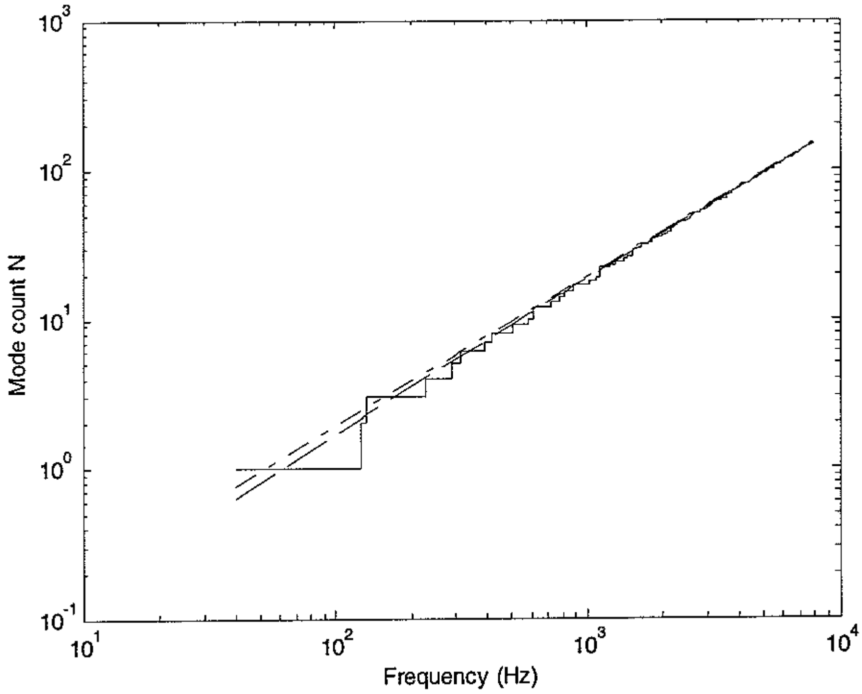

Figure 2.17 Comparison of the mode count for a pinned-pinned-pinned-free plate. (—, analytical , ---, from (2.45), -·-·-, from (2.1))

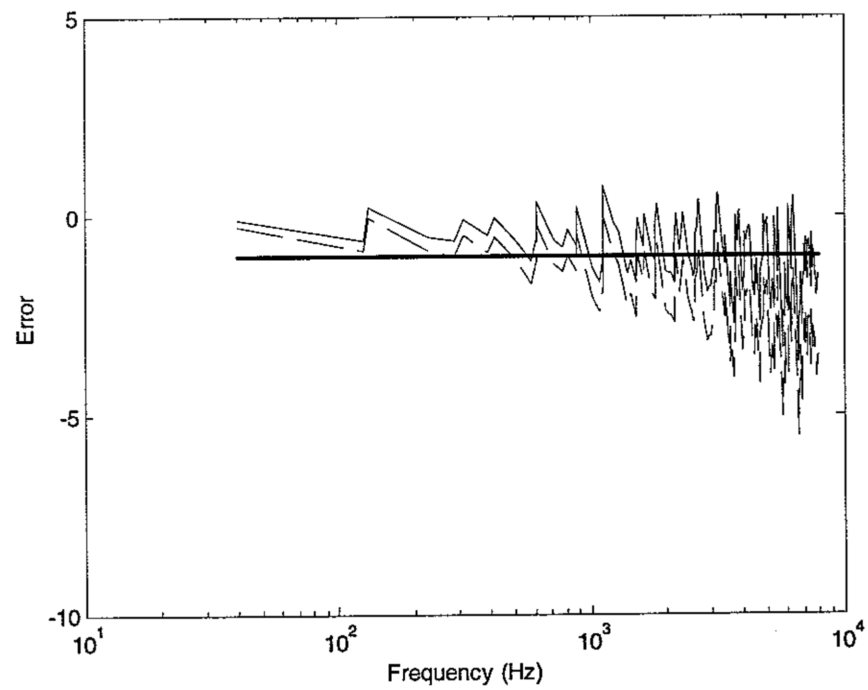

Figure 2.18 Difference between the estimated mode count and the actual result for a pinned-pinned-pinned-free aluminium plate 0.4m×0.3m×0.002m.. (—, errors of (2.46), thick line, average of errors, -·-·-, error of (2.1))

2.4 A PLATE WITH INTERMEDIATE LINE CONSTRAINTS

Following the conclusion from [12] that, for one-dimensional systems, an intermediate constraint has the same effect on the average mode count as the same type of constraint applied at an end, it can be anticipated that an intermediate line constraint will have the same effect as the same type of line constraint applied at an edge for two-dimensional systems. If a simply supported rectangular plate is considered with one or several intermediate line simple supports applied, the average mode count of such a two-dimensional system can be expected to be given by

$$N_{total}(k) = N(k) - m\delta_{BC} \tag{2.47}$$

where $N_{total}(k)$ is the average mode count of the whole system, $N(k)$ is the average mode count of the system without considering the intermediate constraints, given by equation (2.38), m is the number of applied intermediate constraints and δ_{BC} is the line boundary effect of a simple support on the mode count which is determined by equation (2.37).

To verify equation (2.47), two examples are studied. The first is a simply supported plate of dimension 0.4m×0.3m×0.002m with another line simple support applied at x_0 = 0.16m. The second is the same plate with two line simple supports applied at x_{01} = 0.16m and x_{02} = 0.27m. The position of the line constraint x_0 in both cases is randomly chosen. The average mode count of the first example is given by

$$N_{total}(k) = N(k) - \delta_{x-pinned} \tag{2.48}$$

where $N(k)$ is given by equation (2.18) and $\delta_{x-pinned} = -\frac{3}{4}\frac{kb}{\pi}$.

The average mode count of the second example is given by

$$N_{total}(k) = N(k) - 2\delta_{x-pinned} \tag{2.49}$$

Since the analytical solutions of natural frequencies for such systems are not available, an FEM modal analysis is used to obtain numerical results. Then the estimated results from equations (2.48) and (2.49) are compared with those from the FEM analysis, as shown in Figure 2.19 and Figure 2.20. It can be seen that the estimated values have a good agreement with the result from the FEM analysis.

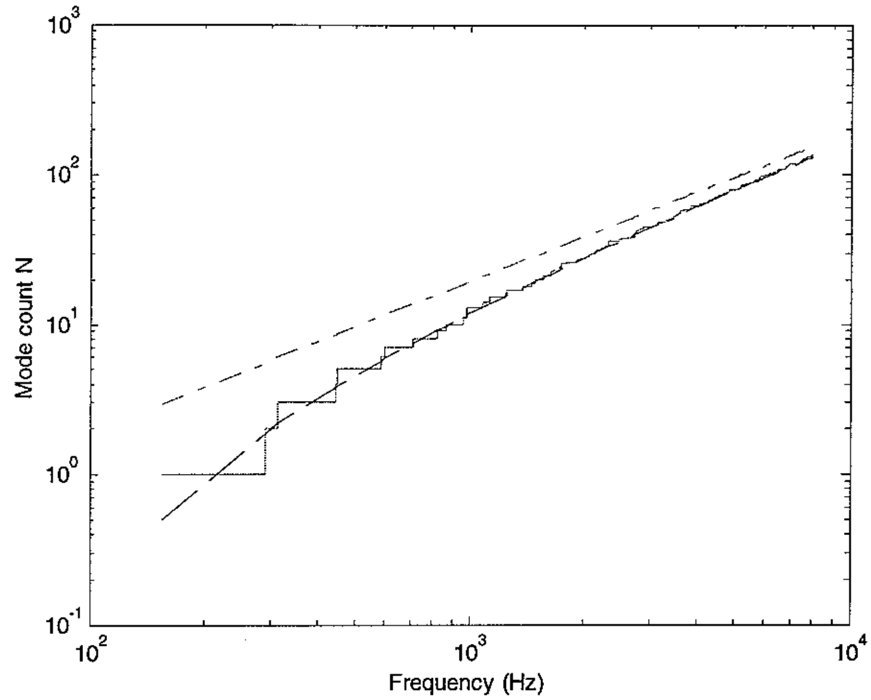

Figure 2.19 Comparison of the estimated mode count and FEM result for a plate 0.4m×0.3m×0.002m with a line intermediate simple support at x_0 = 0.16m.

(stair-case: FEM, ---: equation (2.48), -·-·-: equation (2.1))

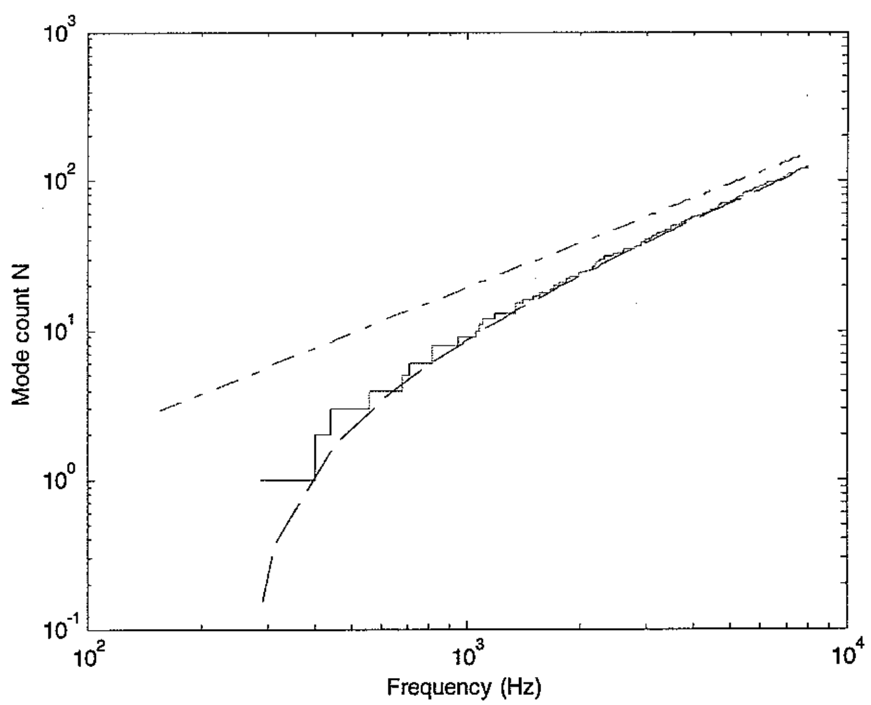

Figure 2.20 Comparison of the estimated mode count and FEM result for a plate 0.4m×0.3m×0.002m with two intermediate simple support at x_{01} = 0.16m and x_{02} = 0.27m.

(stair-case: FEM, --- : equation (2.49), -·-·-: equation (2.1))

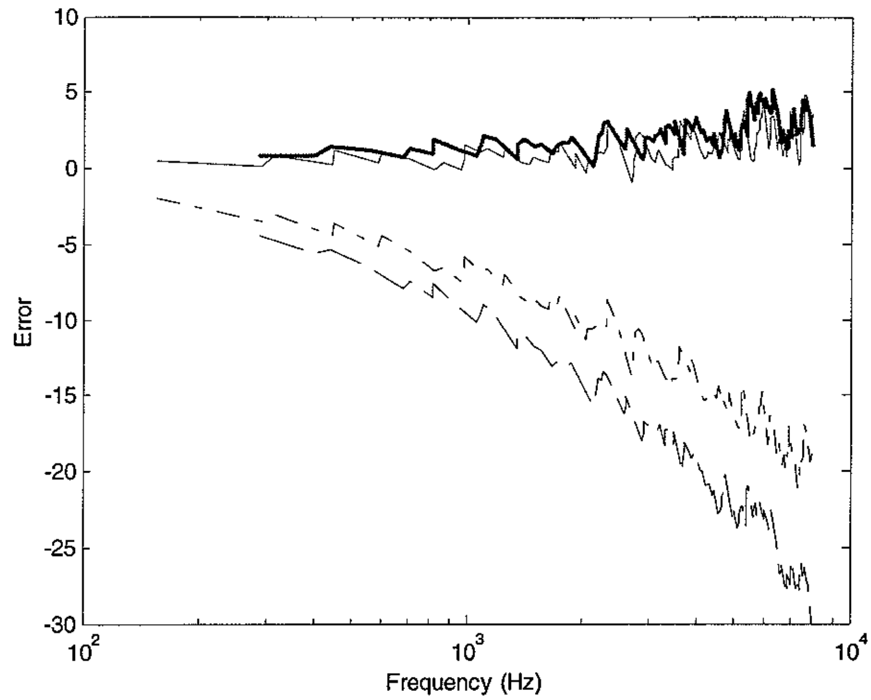

Figure 2.21 Difference between the estimated mode count and the FEM results. (solid line: errors of (2.48) for one intermediate constraint, thick solid line: errors of (2.49) for two intermediate constraints, -·-·-: error of (2.1) for one intermediate constraints, ----: error of (2.1) for two intermediate constraints)

2.5 CONCLUSIONS

Line boundary conditions have systematic effects on the mode count of a two-dimensional system. The effect depends on the type of the boundary condition and the geometric and material properties of the system but are independent of the position applied. The usual formula used to evaluate the mode count of a two-dimensional system contains a systematic error. For a composite two-dimensional system in a plane, the average mode count can be estimated by that of the system without intermediate constraints minus the product of the number of the constraints and the constraint effect δ_{BC}. Having included the effect of the boundary conditions, the estimated mode counts give a good agreement with actual values calculated using analytical and FEM results.

3 MODAL DENSITY OF RECTANGULAR PLATES

3.1 MODAL DENSITY OF RECTANGULAR PLATES

Having obtained the expression for the mode count of the rectangular plate with various boundary conditions, the modal density can be readily derived through the derivative of the mode count. In terms of frequency, the expression (2.38) of the mode count for a rectangular plate can be rewritten as

$$N(\omega)=\frac{S}{4\pi}\sqrt{\frac{m''}{B}}\omega+\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left[\frac{\left(1-\delta_{x-left}-\delta_{x-right}\right)b}{\pi}+\frac{\left(1-\delta_{y-top}-\delta_{y-bottom}\right)a}{\pi}\right]\omega^{\frac{1}{2}}+\Delta \quad (3.1)$$

It should be noted that there was a small constant term Δ in equation (3.1) which represents the systematic error from application of the phase-closure principle. The derivative of the mode count will eventually eliminate the effect of this error to the modal density. The modal density of a rectangular plate is hence given by

$$n(\omega)=\frac{\partial N(\omega)}{\partial\omega}=\frac{S}{4\pi}\sqrt{\frac{m''}{B}}+\frac{1}{2}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left[\frac{\left(1-\delta_{x-left}-\delta_{x-right}\right)b}{\pi}+\frac{\left(1-\delta_{y-top}-\delta_{y-bottom}\right)a}{\pi}\right]\omega^{-\frac{1}{2}} \quad (3.2)$$

It can be seen from equation (3.2) that the modal density of the plate is frequency-dependent. The first term in equation (3.2) is equivalent to equation (3.3) that is given in most literature (see [1], [3] and [2]).

$$n(\omega)=\frac{S}{4\pi}\sqrt{\frac{m''}{B}} \quad (3.3)$$

This first term is constant for a specific plate and depends only on the material, thickness and area of the plate under consideration. The second term in equation (3.2) is frequency dependent and contains information on the geometric characteristics of the plate. It becomes smaller and less important as frequency increases so that the modal density tends to a constant only at high frequency.

3.2 CASE STUDY OF MODAL DENSITY

The modal density of a simply supported plate can be given by

$$n(\omega)=\frac{S}{4\pi}\sqrt{\frac{m''}{B}}-\frac{1}{4}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left(\frac{a+b}{\pi}\right)\omega^{-\frac{1}{2}} \quad (3.4)$$

The modal density of a free plate can be given by

$$n(\omega)=\frac{S}{4\pi}\sqrt{\frac{m''}{B}}+\frac{1}{2}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left(\frac{a+b}{\pi}\right)\omega^{-\frac{1}{2}} \tag{3.5}$$

The modal density of a fully fixed plate can be given by

$$n(\omega)=\frac{S}{4\pi}\sqrt{\frac{m''}{B}}-\frac{1}{2}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left(\frac{a+b}{\pi}\right)\omega^{-\frac{1}{2}} \tag{3.6}$$

From the above three equations it can be noted that the modal density of the plate can be either larger or smaller than the constant value from equation (3.3). Taking the plate investigated in section 2 as an example, the result of the modal densities of a simply supported, a fully fixed and a free plate are shown in Figure 3.1 (NB results are plotted as $n(f) = 2\pi n(\omega)$). It can be seen that equation (3.3) can only represent the true modal density well at high frequencies. Although the first mode of the simply supported plate occur below 100Hz, even at 1000Hz the error can be as much as 13%. For the cases of the free and fully fixed plate, both errors are 26%.

The modal density is obtained directly by counting the number of modes in a band (it is actually impractical in applications). Figure 3.2 shows the results counted in one-third octave bands for a simply supported plate. It can be seen that the result from one-third octave bands agrees better with the curve from equation (3.4) and converges to the constant determined by equation (3.3) at high frequencies. However, for frequencies below 630Hz, the result counted in one-third octave bands has a quite large error. This error can be decreased by expanding the bandwidth of the frequency interval as every three adjacent one-third octave bands, as shown in Figure 3.3. Figure 3.4 and 3.5 present the results for the free plate. The same phenomenon as the simply supported plate can be observed.

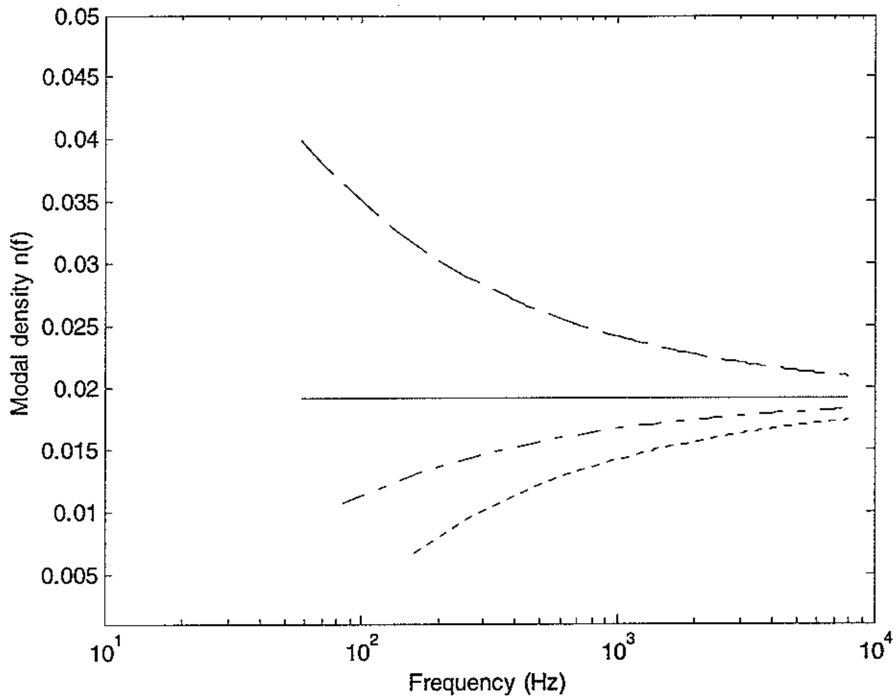

Figure 3.1 Modal density of the plate 0.4m×0.3m×0.002m

(solid: equation (3.3), -.-.: simple support, ----: free, ... : fixed)

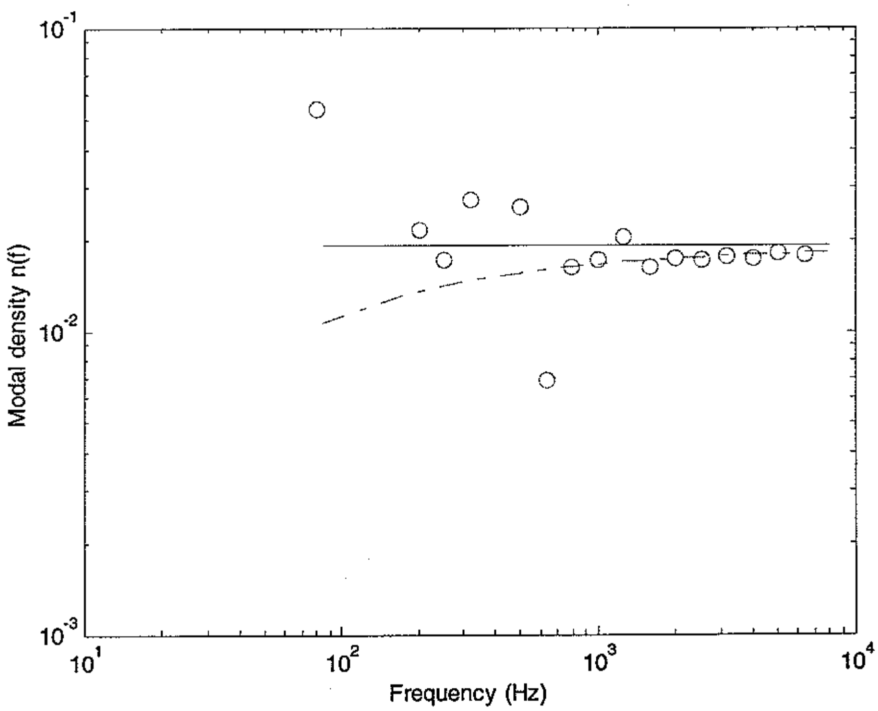

Figure 3.2 Modal density of the simply supported plate 0.4m×0.3m×0.002m

(solid: equation (3.3), -.-.: equation (3.4), circle : 1/3 octave bands)

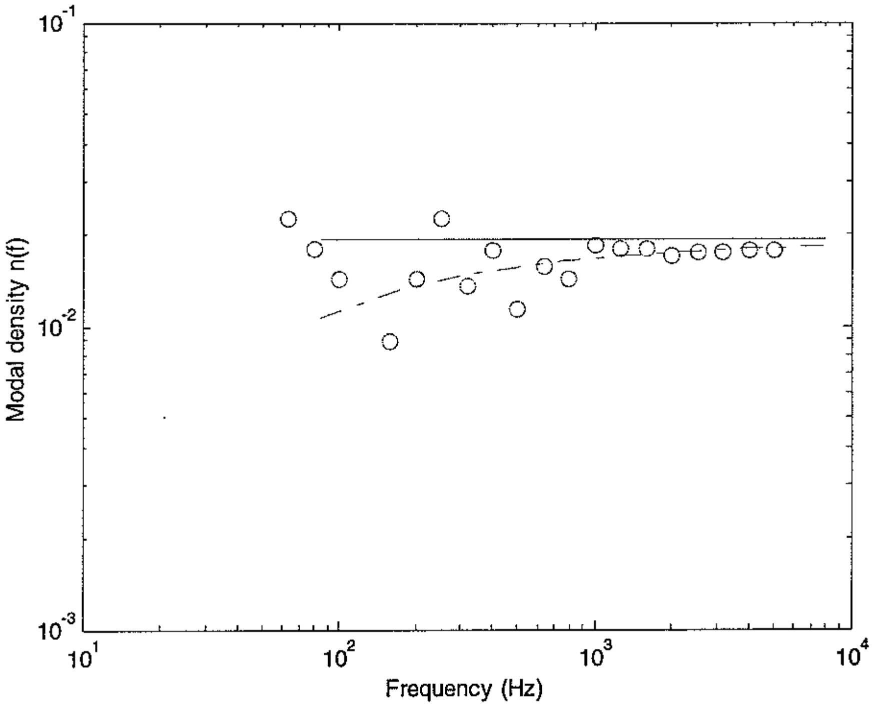

Figure 3.3 Modal density of the simply supported plate 0.4m×0.3m×0.002m (solid: equation (3.3), -.-.: equation (3.4), circle : counted by three adjacent 1/3 octave bands)

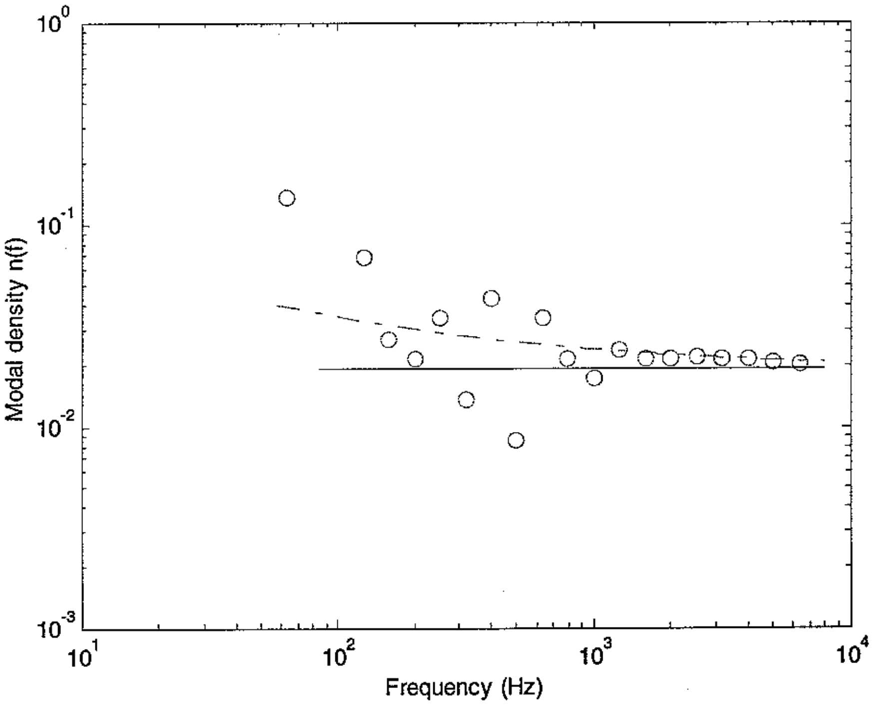

Figure 3.4 Modal density of the free plate 0.4m×0.3m×0.002m (solid: equation (3.3), -.-.: equation (3.5), circle : 1/3 octave bands)

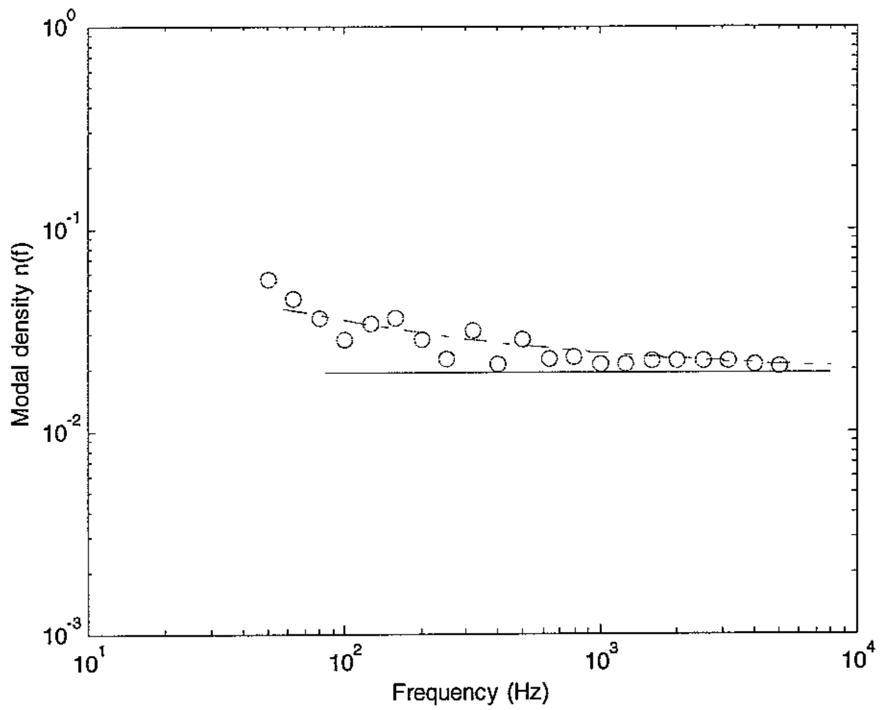

Figure 3.5 Modal density of the free plate 0.4m×0.3m×0.002m
(solid: equation (3.3), -.-.: equation (3.5), circle : counted by three adjacent 1/3 octave bands)

3.3 EXPERIMENTAL RESULTS

Another observation is the experimental determination of modal densities of flat plates. Clarkson and Pope [14] gave some results from experiments based on indirect methods for a flat rectangular plate. An indirect method to measure the modal density was proposed based on the relationship between the spatially average driving point mobility and modal density, rather than counting modes directly. The method should be used for a broad band random excitation having a flat spectral density in the frequency range of interest. Cremer, Heckl and Ungar [3] have shown that the modal density is given by

$$n(\omega)=\frac{2m}{\pi}\mathrm{Re}\{Y\} \tag{3.7}$$

where m is the mass of the plate and Y is the average driving point mobility.

Clarkson and Pope [14] did not indicate what kind of boundary condition was supposed in their experiment. However, the results based on the indirect method for the undamped case are rather low compared with theoretical value from equation (3.3). For the case of a damped plate, the experimental results match the theoretical value from equation (3.3) much better. The conclusion drawn from the experiment was that the indirect method can be used to obtain realistic estimates of the modal density of uniform structures. However, one point that should be noted is that, from the analysis in the present report, it is unlikely that the average experimental estimate for the modal density of a rectangular plate will be equal to the theoretical value from equation (3.3).

3.4 CONCLUSIONS

A theoretical expression for the modal density of a rectangular plate has been obtained. The effect of boundaries on the modal density has been considered. The modal density of the rectangular plate is frequency dependent, and depends on the geometry and dispersion relation of the plate under consideration. However, at high frequency, the modal density for any boundary condition tends to a constant value, which is determined only by the area of the plate, its mass and bending stiffness.

4 MODE COUNT AND MODAL DENSITY OF AN EXTRUDED PLATE

A longer term objective for the indirect experimental method to determine the modal density [14] was to develop the methods for application to structures for which there is no analytical result available. Based on the conclusions drawn in the present report, the modal density of some sort of composite structure can be estimated theoretically through including the effect of the boundary conditions and constraints applied. In practice, a flat panel may be supplied with internal supports or constraints to increase its stiffness. Consequently, the vibration level may decrease due to stiffening. This can be understood by considering a decrease of the modal density of the panel due to the stiffening.

An extruded plate of the type used in the construction of railway vehicles consists of a set of strips which form an upper plate, a lower plate and intermediate stiffening strips as shown in Figure 4.1. The vibration of the extruded plate involves global motion and local motion. At low frequencies the extruded plate behaves as a whole plate while the local vibration dominates at high frequencies [12]. The modal density of the extruded plate is a combination of the modal densities of global modes and local modes.

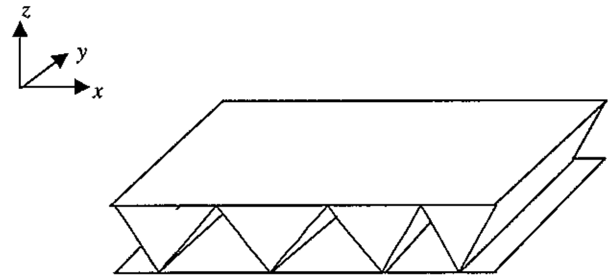

Figure 4.1 Example of an extruded plate

4.1 MODE COUNT AND MODAL DENSITY OF GLOBAL MODES

The global vibration of the extruded plate can be considered as an equivalent simple plate. The details of the model for global modes are described in reference [15].

It is assumed that the boundary conditions for the extruded plate are simple supports. The mode count for global modes of the extruded plate can be given by

$$N_g(\omega)=\frac{S}{4\pi}\sqrt{\frac{M''}{B_g}}\omega-\frac{1}{2}\left(\frac{M''}{B_g}\right)^{\frac{1}{4}}\left(\frac{L_x+L_y}{\pi}\right)\omega^{\frac{1}{2}} \tag{4.1}$$

where L_x and L_y are the dimensions of the extruded plate, S is the surface area of the extruded plate, M'' is the equivalent mass per unit area, and B_g is the equivalent bending stiffness for the global modes.

The modal density of the global modes is therefore given by the derivative of equation (4.1) as

$$n_g(\omega)=\frac{S}{4\pi}\sqrt{\frac{M''}{B_g}}-\frac{1}{4}\left(\frac{M''}{B_g}\right)^{\frac{1}{4}}\left(\frac{L_x+L_y}{\pi}\right)\omega^{-\frac{1}{2}} \tag{4.2}$$

4.1.1 MINDLIN THICK PLATE

Mindlin's thick plate theory should be considered for global modes at high frequencies [15]. The two-dimensional equation of motion for free vibration of a thick plate is given by [16]

$$\left(\nabla^2-\frac{\rho}{G'}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2-I\frac{\partial^2}{\partial t^2}\right)w+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{4.3}$$

where ρ is the density of plate, G' is the product of G and κ, G is the shear modulus, κ is the *Timoshenko* shear coefficient, I is the second moment of area, h is the thickness of the plate, D is the bending stiffness and ∇^2 is Laplace's two-dimensional operator.

The dispersion relation of the thick plate has a similar form to that for a thick beam [12]. This is given by

$$Dk_t^4 + \left(\frac{\rho D}{G'} + I\right)\omega^2 k_t^2 + \frac{\rho^2 I}{G'}\omega^4 - \rho h\omega^2 = 0 \tag{4.4}$$

where k_t is the wavenumber.

The solution of the first propagating wave from equation (4.4) is given by

$$k_t = \sqrt{\frac{\rho\left(I + \frac{D}{G'}\right)\omega^2 + \sqrt{\left[\rho\left(I + \frac{D}{G'}\right)\omega^2\right]^2 - 4D\left(-\rho h\omega^2 + \frac{\rho^2 I}{G'}\omega^4\right)}}{2D}} \tag{4.5}$$

The mode count of global modes of the thick plate can be calculated by

$$N_t = \frac{Sk_t^2}{4\pi} - \frac{1}{2}\left(\frac{L_x + L_y}{\pi}\right)k_t \tag{4.6}$$

The modal density of global modes of the thick plate can hence be given by

$$n_t(\omega) = \left[\frac{Sk_t}{2\pi} - \frac{1}{2}\left(\frac{L_x + L_y}{\pi}\right)\right]\frac{dk_t}{d\omega} \tag{4.7}$$

where no simple expression for $dk_t/d\omega$ can be found.

4.2 LOCAL MODES

4.2.1 MODE COUNT OF LOCAL MODES

The local modes occur above the first cut-on frequency of the widest strip. This behaviour was studied using a two-dimensional FEM model of the cross-section of an extruded plate by Xie, Thompson and Jones [12].

To consider the modal density of the local modes, the extruded plate can be treated as a large plate with a number of line constraints applied. The area of this large plate S_{total} is equivalent to the combined area of all the strips. These line constraints can be approximated as simple supports or a value between the simple support and clamped according to [12]. The mode count of local modes can be approximately given by

$$N_l(\omega) = N_{total}(\omega) - p\delta \frac{L_y}{\pi}\left(\frac{m''}{B}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}} \tag{4.8}$$

where N_{total} is the mode count of the plate having area S_{total}, p is the number of intermediate constraints, which is the number of strips minus 1, δ is the boundary effect of the intermediate constraints between strips, L_y is the length of the extruded plate in the y-direction, m'' is the mass per unit area and B is the bending stiffness of the strip.

$N_{total}(\omega)$ is given by

$$N_{total}(\omega) = \frac{S_{total}}{4\pi}\sqrt{\frac{m''}{B}}\omega - \frac{1}{2}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\left(\frac{L_y + \sum l_i}{\pi}\right)\omega^{\frac{1}{2}} \tag{4.9}$$

where S_{total} is the total area of all strips and l_i is the length of the shorter edge of each strip.

Theoretically, equation (4.8) is valid at frequencies which are sufficiently high that the first modes of all strips have occured. At low and mid frequency, the term containing the information relating to the constraints in equation (4.8) will become significant to the result. It may cause a negative value for the mode count at low and mid frequencies. This does not actually happen in reality, as the mode count must be positive. In practice, at low frequencies, the local modes occur only on those relatively wide strips. Therefore those relatively narrow strips and their corresponding constraints should not be taken into account for the application of equation (4.8). Therefore equation (4.8) can be rewritten as the sum of the local modes occurring on each strip

$$N_l(\omega) = \sum_{i=1}^{p+1}\left[\max\begin{pmatrix} N_i \\ 0 \end{pmatrix}\right] \tag{4.10}$$

where N_i is given by

$$N_i = \frac{l_i L_y}{4\pi}\sqrt{\frac{m''}{B}}\omega - \frac{1}{2}\frac{l_i}{\pi}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\omega^{\frac{1}{2}} - \frac{1}{p+1}\left(p\delta_{pinned} + \frac{1}{2}\right)\frac{L_y}{\pi}\left(\frac{m''}{B}\right)^{\frac{1}{4}}\omega^{\frac{1}{2}} \tag{4.11}$$

This is equivalent to treating each strip as simply supported parallel to the x-axis and with the assumed constraints defined by δ parallel to the y-axis. A negative N_i at one frequency is physically meaningless. For such a case, this strip only contribute zero to the local mode of the extruded plate. Note that $N_l = 0$ if the mode count of each strip is zero. It become non-zero as soon as local mode occur in the widest strip.

4.2.2 MODAL DENSITY OF LOCAL MODES

The modal density of the local modes of the extruded plate can hence given by

$$n_l(\omega)=\sum_i^{p+1} n_i \tag{4.12}$$

where n_i is given by

$$n_i=\begin{cases}\dfrac{l_i L_y}{4\pi}\sqrt{\dfrac{m''}{B}}-\dfrac{1}{4}\dfrac{l_i}{\pi}\left(\dfrac{m''}{B}\right)^{\frac{1}{4}}\omega^{-\frac{1}{2}}-\dfrac{1}{2(p+1)}\left(p\delta_{pinned}+\dfrac{1}{2}\right)\dfrac{L_y}{\pi}\left(\dfrac{m''}{B}\right)^{\frac{1}{4}}\omega^{-\frac{1}{2}} & \text{for } N_i>0\\ 0 & \text{for } N_i<0\end{cases} \tag{4.13}$$

4.3 MODE COUNT AND MODAL DENSITY OF EXTRUDED PLATES

The mode count and modal density of the extruded plate can be obtained by combining the expressions for global and local modes. The mode count of an extruded plate is given by

$$N(\omega)=N_g(\omega)+N_l(\omega) \tag{4.14}$$

where N_g can be obtained by equation (4.1) or (4.6) and N_l by equation (4.10).

The modal density of an extruded plate is given by

$$n(\omega)=n_g(\omega)+n_l(\omega) \tag{4.15}$$

where n_g is given by equation (4.2) or (4.7) and n_l is given by equation (4.13).

4.4 RESULTS

The example of the extruded plate used in the present report is of aluminium and of dimensions 2.016×1.0×0.07m. It is composed of 77 strips. The thickness of each strip is 0.003m. The widest strip is 0.16m wide while the narrowest strip is 0.046m wide.

An FEM model has been established using ANSYS for this example. A modal analysis is implemented to obtain the natural modes of the structure. The SHELL 93 element is chosen with dimension 0.03×0.03m^2. This allows valid analysis up to 3000Hz, based on six nodes per wavelength.

presents the results of the mode count from equation (4.14) and from the FEM model. Simple supports are assumed for the line constraints between strips in these calculations, ie δ = 3/4. The result by assuming the fixed constraints ($\delta = 1$) is also presented in Figure 4.2. The estimated result based on the assumption of simple supports constraints has a good agreement with that from the FEM model above about 700Hz. However it appears that the first local modes occur at a higher frequency in the FEM model (500Hz) than predicted using simple support (300Hz). This indicates the line constraints are various between the simple support and fixed conditions for frequencies below 700Hz, and tends to simple supports at high frequencies.

The global modes are estimated using thin plate theory as well as thick plate theory. The Timoshenko shear coefficient $\kappa = 0.2$ is used here for the thick plate. Figure 4.3 gives curves for the dispersion relation of the thin and thick plate theory used for the global modes of the extruded plate. It can be seen that the thick plate has a similar wavenumber to that of thin plate below 200Hz. The divergence above 200Hz causes the difference between the estimated global mode count of two models, shown in Figure 4.2.

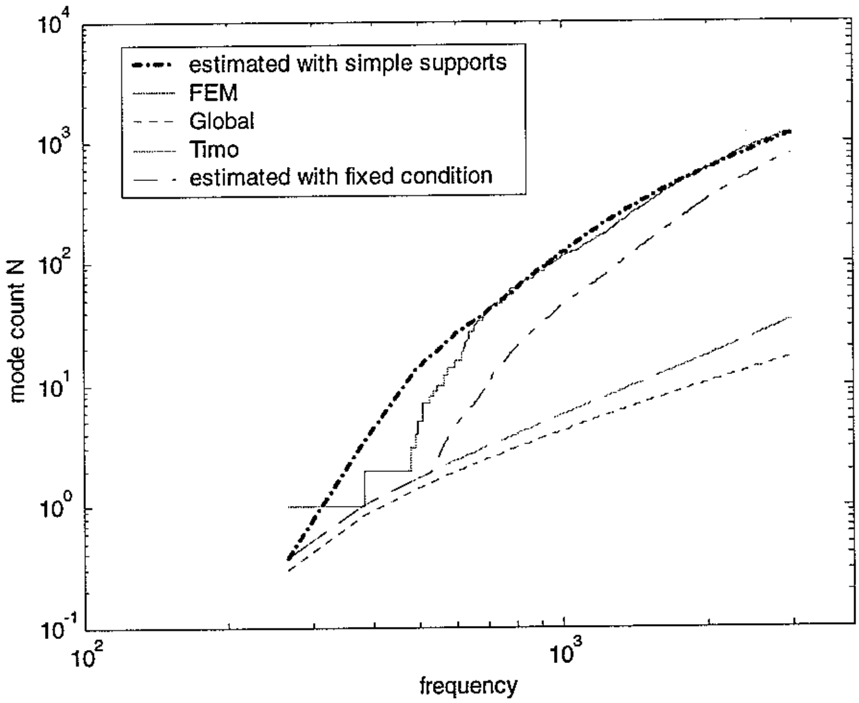

Figure 4.2 Mode count of the example extruded plate.

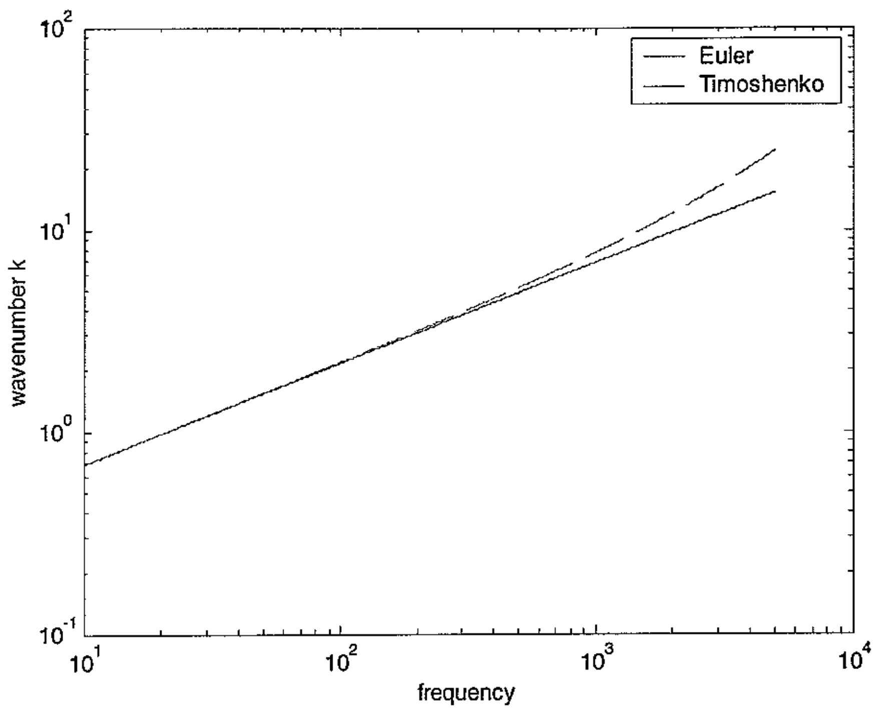

Figure 4.3 Dispersion relations of thin and thick plate theory used for global modes.

Figure 4.4 shows the estimated modal density of the extruded plate based on equation (4.15), compared with the result from the FEM modal analysis, expressed in one-third octave bands. The result generally agrees with that from the FEM model. It should be noted that the 315Hz band and all bands below 250Hz contain no modes in the FEM results. If the conjunctive three on-third octave bands used, the result has a better agreement.

The global modal density based on thin plate is calculated from equation (4.2). Equation (4.7), based on thick plate theory, is calculated numerically since the expression in terms of frequency is unavailable. However, it can be anticipated that it will have a negligible effect at high frequencies, where the local modes are dominant. The estimated result is also compared with that from the simple additive methods, which actually is equivalent to equation (2.3). The simple additive method is not able to consider the global behaviour and the effects due to intermediate constraints between strips. Therefore its result shows a quite large difference to that from the FEM, especially at low frequencies.

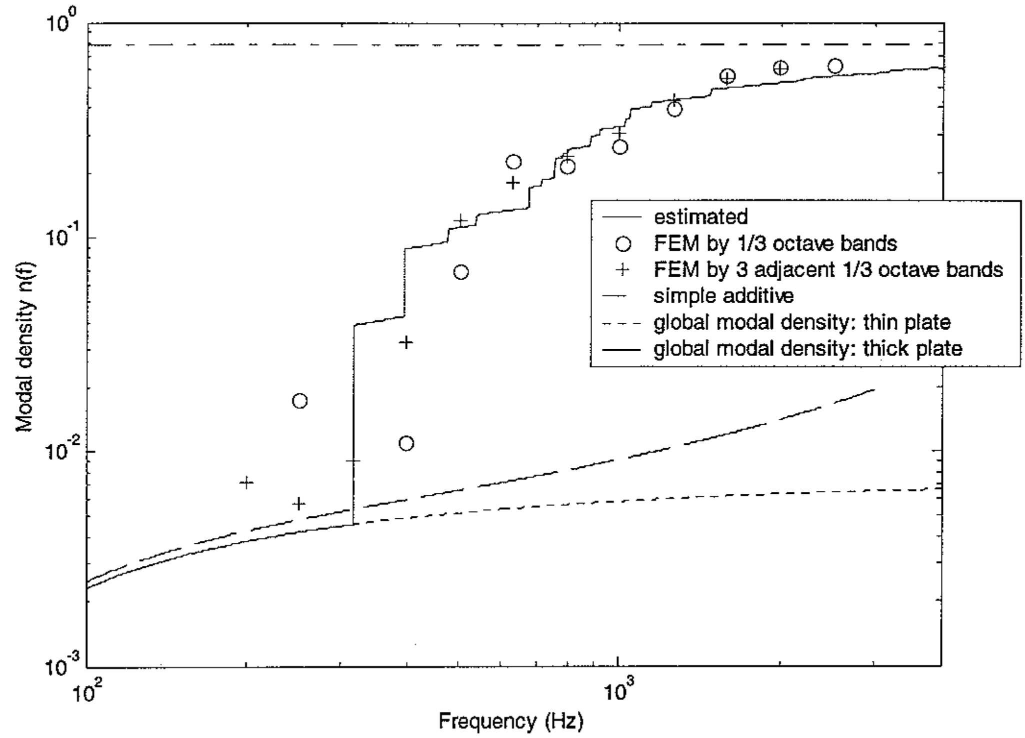

Figure 4.4 Modal density of the extruded plate.

5 CONCLUSION

The mode count and modal density of bending vibrations of two-dimensional systems have been investigated. A relation has been shown between the mode count and the boundary conditions in particular for rectangular plates. Line constraints have systematic effects on the mode count of a system. The effect depends on the type of the boundary condition, as well as the geometric and material properties of the system. The effect is an extension of that from the same type of boundary in one-dimensional system. Theoretical expressions used to estimate the mode count of a two-dimensional system have been obtained. The results from these estimated formulae have shown the limitation of the common formula used previously, which is based a form that neglects boundary effects. For a composite two-dimensional system, an intermediate line constraint has the same effect on the mode count as the equivalent constraint applied on an edge. The average mode count of such a composite system can be estimated by that of the system without intermediate constraints minus the product of the number of the constraints, the constraint effect δ_{BC} and the term kL/π, where k is the wavenumber and L is the length of the constrained edges.

Theoretical expressions for the modal density of a two-dimensional system have been obtained including boundary effects. The modal density of the rectangular plate is a frequency dependent parameter, which depends on the geometric information and dispersion relation of the plate under consideration. However, at high enough frequency, the modal density tends to a constant value, which is determined only by the area of the plate and dispersion relation and is independent of the boundary conditions.

The conclusion drawn from the multi-plate system leads to an approximation that can be used for the investigation of the extruded plate. The mode count of an extruded plate is predicted based on assuming simple supports between the strips. Although differences occur between the predicted results and those from the FEM analysis, the modal density shows a good agreement.

REFERENCES

1. F. D. Hart and K. C. Shah. 1971 *NASA Contractor Report.* CR-1773. Compendium of modal densities for structures.
2. R. H. Lyon and R. G. DeJong 1995 *Theory and Application of Statistical Energy Analysis.* London: Butterworth-Heinemann. See page136-137.
3. L. Cremer, M. Heckl and E. E. Ungar 1988 *Structure-borne Sound.* Berlin: Springer-Verlag; second edition.
4. P. M. Morse and R. H. Bolt 1944 *Reviews of Modern Physics*, **16**(2), 69-150. Sound waves in rooms.
5. D. Y. Maa 1939 *Journal of the Acoustical Society of America*, **10**, 235-238. Distribution of eigentones in a rectangular chamber at low frequency range.
6. G. M. Roe 1941 *Journal of the Acoustical Society of America* **13**, 1-7. Frequency distribution of normal modes.
7. D. G. Vasil'ev 1987 *Transactions of Moscow Mathematics Society.* **49**, 173-.
8. E. Bogomolny and E. Hugues 1998 *Physical Review*, **57(5)**, 5404-5424. Semiclassical theory of flexural vibrations of plates.
9. P. Bertelsen, C. Ellegaard and E. Hugues. 2000 *The European Physical Journal B*, **15**, 87-96. Distribution of eigenfrequencies for vibrating plates.
10. R. Courant and D. Hilbert 1953 *Methods of Mathematical Physics,* Vol. 1. Interscience Publishers, Inc., New York.
11. V. V. Bolotin, 1984 *Random Vibrations of Elastic Systems.* Martinus Nijhoff, The Hague.
12. G. Xie, D. J. Thompson and C. J. C. Jones 2002 *ISVR Technical Memorandum* **No: 882.** Investigation of the mode count of one-dimensional systems.
13. W. Leissa. 1969 *Vibration of Plates*, National Aeronautics and Space Administration, Washington, D.C.
14. B. L. Clarkson and R. J. Pope 1981 *Journal of Sound and Vibration* **77**(4), 535-549. Experimental determination of modal densities and loss factors of flat plates and cylinders.
15. G. Xie, D. J. Thompson and C. J. C. Jones 2002 *ISVR Technical Memorandum* **No: 895.** Investigation of the radiation efficiency of strips.
16. R. D. Mindlin. 1951 *Journal of Applied Mechanics* March, 31-38. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates.

APPENDIX A

Leissa [13] gives a comprehensive collection of solutions for natural frequencies of rectangular plates. The boundary conditions considered for a rectangular plate may be a combination of simple supports (SS), clamped (C) or free (F). There are six possible distinct sets of boundary conditions along either x- or y-direction. For a plate in the coordinate system illustrated in Figure 2.1, they are

a) simply supported at $x = 0$ and $x = a$;
b) clamped at $x = 0$ and $x = a$;
c) free at $x = 0$ and $x = a$;
d) clamped at $x = 0$ and simply supported at $x = a$;
e) free at $x = 0$ and simply supported at $x = a$;
f) clamped at $x = 0$ and free at $x = a$.

and similarly for constraints at $y = 0$ and $y = b$.

The natural frequencies are given by

$$\omega^2 = \frac{\pi^4 D}{a^4 m''}\left\{G_x^4 + G_y^4\left(\frac{a}{b}\right)^4 + 2\left(\frac{a}{b}\right)^2\left[\nu H_x H_y + (1-\nu)J_x J_y\right]\right\} \qquad \text{(A.1)}$$

where a and b are the dimensions of the rectangular plate, D is the flexural rigidity, m'' is the mass per unit area, G_x, H_x and J_x are functions determined from Table A.1. The quantities G_y, H_y and J_y are obtained from Table A.1 by replacing x by y and m by n.

The indicators m and n are seen to be the number of nodal lines parallel to the y- and x-axes, respectively, including the boundaries as nodal lines, except when the boundary is free.

Table A.1. Frequency coefficients in equation (A.1)

Boundary conditions at $x = 0$	Boundary conditions at $x = a$	m	G_x	H_x	J_x
SS	SS	2, 3, 4, …	$m-1$	$(m-1)^2$	$(m-1)^2$
C	C	2	1.506	1.248	1.248
		3, 4, 5, …	$m-\frac{1}{2}$	$\left(m-\frac{1}{2}\right)^2\left[1-\frac{2}{\left(m-\frac{1}{2}\right)\pi}\right]$	$\left(m-\frac{1}{2}\right)^2\left[1-\frac{2}{\left(m-\frac{1}{2}\right)\pi}\right]$
F	F	0	0	0	0
		1	0	0	$12/\pi^2$
		2	1.506	1.248	5.017
		3, 4, 5, …	$m-\frac{1}{2}$	$\left(m-\frac{1}{2}\right)^2\left[1-\frac{2}{\left(m-\frac{1}{2}\right)\pi}\right]$	$\left(m-\frac{1}{2}\right)^2\left[1+\frac{6}{\left(m-\frac{1}{2}\right)\pi}\right]$
C	SS	2, 3, 4, …	$m-\frac{3}{4}$	$\left(m-\frac{3}{4}\right)^2\left[1-\frac{1}{\left(m-\frac{3}{4}\right)\pi}\right]$	$\left(m-\frac{3}{4}\right)^2\left[1-\frac{1}{\left(m-\frac{3}{4}\right)\pi}\right]$
F	SS	1	0	0	$3/\pi^2$
		2, 3, 4, …	$m-\frac{3}{4}$	$\left(m-\frac{3}{4}\right)^2\left[1-\frac{1}{\left(m-\frac{3}{4}\right)\pi}\right]$	$\left(m-\frac{3}{4}\right)^2\left[1+\frac{3}{\left(m-\frac{3}{4}\right)\pi}\right]$
C	F	1	0.597	−0.0870	0.471
		2	1.494	1.347	3.284
		3, 4, 5, …	$m-\frac{1}{2}$	$\left(m-\frac{1}{2}\right)^2\left[1-\frac{2}{\left(m-\frac{1}{2}\right)\pi}\right]$	$\left(m-\frac{1}{2}\right)^2\left[1+\frac{2}{\left(m-\frac{1}{2}\right)\pi}\right]$