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The Analysis of Layered Systems Part V: Developer Notes and User Manual

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

The Analysis of Layered Systems
Part V: Developer Notes and User Manual

by

W.H. Cogill

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The analysis of layered systems.
Part V: Developer Notes and User Manual

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[1]Previous parts in this series on the analysis of layered systems were as follows. Part I: Single-Layered Inverse; ISVR Technical Memorandum No.833. January 1999. Part II:SH-waves; ISVR Technical Memorandum No.849. November 1999. Part III:Rayleigh waves in layered media; ISVR Technical Memorandum No.850. November 2000. Part IV: The analysis of layered systems. Surface deflections on layered media

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Chapter 1

Part 1: Hard over soft

1.1 Summary

This report contains notes for developers and a user manual.

Chapters 1 and 2 contain notes which were used in order to develop the programs described. The purpose of the notes is to obtain an approximate expression for the thickness of the surface layer of a single-layered system. The data required are the measurements of the reciprocal wavelength over a range of frequencies. The range of frequencies should encompass wavelengths of approximately a quarter to four times the thickness of the surface layer.[1]

Chapters 3 and 4 constitute a manual intended for users of the programs described. They present attempts to make theoretical matches of experimental results. The aim is to utilize theory to estimate the properties of pavement structures. We attempt to estimate both the thicknesses and the elastic moduli of the component layers of the structures.[2]

1.2 Statement of the problem

1.2.1 Introduction

The problem is to determine the structure of a pavement using as input the results of measurements of the wavelength of waves of the Rayleigh type and the corresponding frequencies.

[1]The notation employed throughout is that used by Ewing, Jardetzky and Press[2].

[2]For simplicity, we denote by

1. "hard" the medium having the higher stiffness, "soft" the medium having the lower stiffness
2. "Direct" and "Inverse" the systems in which the stiffnesses increase and decrease with the depth respectively
3. "Forward" and "Reverse" the calculations which lead to observed surface behaviour and the system properties - the stiffnesses and the thicknesses of the layers - respectiely

Experimental results are obtained in the form shown in Figure (1.1). This figure shows the reciprocal wavelength plotted against the frequency. The reciprocal slope of this plot yields the group velocities corresponding with the component materials.

The velocity of propagation of shear waves in the surface layer β_1 can be determined from the reciprocal of the slope of the figure at high frequencies. The quantity β_2 is the velocity of shear waves in the underlying semi-infinite medium. β_2 can be determined from the reciprocal of the slope of the figure at low frequencies. The point of intersection of the two sloping lines can be used to estimate the thickness of the surface layer.

The problem is to determine this thickness.

The problem is approached in two stages. In the first, we attempt to approximate the frequency response of a single-layered system, by expanding the transcendental functions in the frequency equation. The expansion is carried out to the first power of the argument of the transcendental functions. The work is performed with the aid of a Mathematica notebooks **APP_EQH.NB** and **APP_MH.NB**.

This stage is partly successful. It can only be used if the measured phase velocity exceeds the α-velocity in the underlying medium. Phase velocities lower than this value lead to negative square roots in the approximation to the frequency equation.

To overcome this limitation we attempt an approximation of a higher order.

1.2.2 The forward problem

We start with a pavement having a simplified cross section. The pavement is of the "hard-over-soft" type, i.e. a layer of stiff material overlying a subgrade composed of a material having a lower stiffness. We start with a calculated response of a layered structure, and not with the results of measurements. The result is shown in Figure (1.1). This figure shows the reciprocal wavelength of waves of the Rayleigh type plotted against the frequency.

The interpretation of the results is based on A.W.Lee's paper, a condensed version of which is given in **Rayleigh waves in a single-layered system, A_W_LEE.DOC**[3]. This version uses modern notation. The calculations for Figure (1.1) are based on imaginary arguments for the transcendental functions. The results shown in the figure were obtained with the aid of the program **SIN13.FOR**, which is based on A.W.Lee's paper.

1.2.3 The reverse problem

The output of the forward program yields the theoretically expected wave velocities as a function of the frequency.

The forward program **SIN13.FOR** appears to yield a valid solution, representing the zero of the determinant of the system.[3]

We hypothesize that an inverse must exist which connects the physical properties of

[3]The residual error oscillates, positive to negative. The amplitude of the oscillation is small.

the system to the measurements performed upon it. An attempt is made to find this inverse.

Mathematica notebooks have been written in order to provide numerical analysis of layered earthen structures. The object of this series of notebooks is to generate a Fortran program with which to interpret the field measurements made on waves of the Rayleigh type. This series of notebooks is based on **APP_MH.NB**[4]. The notebook **APP_MH.NB** attempts to inverse a layered structure consisting of a hard layer overlying a soft semi-infinite medium.

We attempt to interpret the results shown in Figure (1.1). We attempt to find the thickness of the surface layer. We use as data the measurements of reciprocal wave velocity and the corresponding frequency.

The steps toward doing this are as follows. The Mathematica notebook **APP_MH.NB** defines the frequency function in terms of η_1 , η_2 , ξ_1 , ξ_2 as first-order expansions [2].

1. **LEE_EQH.NB** uses the previous definitions of η_1, η_2, ξ_1, ξ_2; it generates the frequency equation; it solves the frequency equation for the wave number and writes the result in Fortran form. The wave number is expressed as a fraction of the thickness of the surface layer.

2. The output from **APP_MH.NB** is used as input to **LEE_EQH.NB** which writes its output in Fortran form. The corresponding Fortran program is **APP_H.FOR**.

3. The Fortran program **APP_H.FOR** is used in order to determine the depth of the interface. The data required are the reciprocal wave velocity and the corresponding frequency.

The logical sequence of these notebooks is: **APP_MH.NB, LEE_EQH.NB, LEE11.NB, LEE2.NB, LEE21.NB, REVERSE.NB**. The final output is **REVERSE.RES**, which is in Fortran form. **APP_MH.NB**

1.3 Notebook APP_MH.NB

The following definition is developed from A.W. Lee, "The effect of geological structure upon microseismic disturbance," Monthly Notices of the Royal Astronomical Society: Geophysical Supplement, Vol. 3, 1932, pp.83-105, equation (20). See A_W_LEE.DOC for a condensed version in modern notation[3]. It gives an approximation to the determinant for the case $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$, with the transcendental functions expanded to the first power of their arguments. The result is used to estimate the thickness of a surface layer from pairs of readings of reciprocal wavelength and the corresponding frequency. The output is indicative of equations (22) to (25) in **A_W_LEE.DOC**[3]. The output is used as input to **LEE_EQH.NB** which writes output in Fortran form. The resulting Fortran program is **APP_H.FOR**.

[4]The suffixes **.MA** and **.NB** are interchangable: both are capable of being executed by Mathematica version 4 and higher.

In[1]:=
$$\texttt{xi1} = \left(2 - \frac{kb1^2}{k^2}\right)(X + r2\ Y\ h) - \frac{2\ S\ \left(\frac{r2\ W\ S\ h}{k} + \frac{k\ Z}{S}\right)}{k};$$

$$\texttt{xi2} = \left(2 - \frac{kb1^2}{k^2}\right)\left(\frac{s2\ W}{k} + k\ Z\ h\right) - \frac{2\ S\ \left(X\ S\ h + \frac{s2\ Y}{S}\right)}{k};$$

$$\texttt{eta1} = \left(2 - \frac{kb1^2}{k^2}\right)\left(\frac{r2\ W}{k} + k\ Z\ h\right) - \frac{2\ R\ \left(X\ R\ h + \frac{r2\ Y}{R}\right)}{k};$$

$$\texttt{eta2} = \left(2 - \frac{kb1^2}{k^2}\right)(X + s2\ Y\ h) - \frac{2\ R\ \left(\frac{s2\ W\ R\ h}{k} + \frac{k\ Z}{R}\right)}{k};$$

In[2]:= `res = Numerator[Factor[Simplify[xi1 eta2 - xi2 eta1]]];`

res1 = res /. k -> k2^(1/2)

In[3]:= `xi11 = Simplify[xi1]`

Out[3]=
$$-\frac{2\ h\ r2\ S^2\ W}{k^2} + \left(2 - \frac{kb1^2}{k^2}\right)(X + h\ r2\ Y) - 2\ Z$$

In[4]:= `xi12 = Together[xi11]; xi13 = Apart[xi12];`

The result for xi1 follows

In[5]:= $\texttt{xi14 = Simplify}\left[\texttt{xi13/.S} \to \sqrt{k^2 - kb1^2}\right]$

Out[5]=
$$-2\ h\ r2\ W + \frac{2\ h\ kb1^2\ r2\ W}{k^2} + 2\ X - \frac{kb1^2\ X}{k^2} + 2\ h\ r2\ Y - \frac{h\ kb1^2\ r2\ Y}{k^2} - 2\ Z$$

In[6]:= `xi21 = Simplify[xi2]`

Out[6]=
$$-\frac{2\ (h\ S^2\ X + s2\ Y)}{k} + \left(2 - \frac{kb1^2}{k^2}\right)\left(\frac{s2\ W}{k} + h\ k\ Z\right)$$

In[7]:= `xi22 = Together[xi21]; xi23 = Apart[xi22];`

In[8]:= `xi24 = Simplify[xi23];`

In[9]:= $\texttt{Expand}\left[\texttt{xi24/.S} \to \sqrt{k^2 - kb1^2}\right]$

Out[9]=
$$\frac{2\ s2\ W}{k} - \frac{kb1^2\ s2\ W}{k^3} - 2\ h\ k\ X + \frac{2\ h\ kb1^2\ X}{k} - \frac{2\ s2\ Y}{k} + 2\ h\ k\ Z - \frac{h\ kb1^2\ Z}{k}$$

In[10]:= $\texttt{et11 = Simplify}\left[\texttt{eta1/.s1} \to \sqrt{kb1^2 - k^2}\right]$

Out[10]=
$$-\frac{2\ (h\ R^2\ X + r2\ Y)}{k} + \left(2 - \frac{kb1^2}{k^2}\right)\left(\frac{r2\ W}{k} + h\ k\ Z\right)$$

In[11]:= `et12 = Simplify[et11]`

Out[11]=
$$-\frac{2\ (h\ R^2\ X + r2\ Y)}{k} + \left(2 - \frac{kb1^2}{k^2}\right)\left(\frac{r2\ W}{k} + h\ k\ Z\right)$$

In[12]:= $\texttt{et13 = Expand}\left[\texttt{et12/.R}^2 \to k^2 - \frac{kb1^2}{3}\right]$

Out[12]=
$$\frac{2\ r2\ W}{k} - \frac{kb1^2\ r2\ W}{k^3} - 2\ h\ k\ X + \frac{2\ h\ kb1^2\ X}{3\ k} - \frac{2\ r2\ Y}{k} + 2\ h\ k\ Z - \frac{h\ kb1^2\ Z}{k}$$

In[13]:= $\texttt{et21 = Simplify}\left[\texttt{eta2/.s1} \to \sqrt{kb1^2 - k^2}\right];$

In[14]:= `et22 = Simplify[et21];`

In[15]:= $\texttt{et23 = Expand}\left[\texttt{et22/.R}^2 \to k^2 - \frac{kb1^2}{3}\right]$

Out[15]=
$$-2\ h\ s2\ W + \frac{2\ h\ kb1^2\ s2\ W}{3\ k^2} + 2\ X - \frac{kb1^2\ X}{k^2} + 2\ h\ s2\ Y - \frac{h\ kb1^2\ s2\ Y}{k^2} - 2\ Z$$

The Mathematica notebook **APP_MH.NB** yields an approximate expression for the determinant governing a single-layered system. The approximation is made by expanding the transcendental functions to the first power of their arguments.

1.4 The frequency equation: LEE_EQH.NB

The notebook which follows is **LEE_EQH.NB**. It yields a result in Fortran form, and the output is used as input to write the program **LEE_H.FOR**. **LEE_EQH.NB**

1.4.1 The notebook LEE_EQH.NB

A.W.Lee

See A.W. Lee "The effect of geological structure upon microseismic disturbance," Monthly Notices of the Royal Astronomical Society: Geophysical Supplement, Vol.3, 1932, pp. 83-105, eqn (20)[3]. The following notebook is an approximation to A.W.Lee's equation (20), the 'hard over soft' case. It generates a quadratic in the layer thickness h and is used to solve for h in terms of pairs of measurements of k and omega[3]. The following leads to a polynomial in h^2, h^1, h^0. Its output is used as input to write the program LEE_H.FOR. The logical sequence of these files is: **LEE_EQH.NB, LEE1.NB, LEE2.NB, LEE21.NB, REVERSE.NB.** The output of **REVERSE.NB** is **REVERSE.RES**, which is in Fortran form.

```
In[16]:= xi1 = 2 h r2 (1 + W) + (2 + X);

         xi2 = h k (2 X - 2 - Z) - s2 (2 + W)/k;

         eta1 = h k (2 X/3 - 2 - Z) - r2 (2 + W)/k;

         eta2 = h s2 (2 W/3 + 2 - Y) + (2 - X);

In[17]:= eqn = xi1 eta2 - xi2 eta1;

In[18]:= eqn2 = Expand[eqn];

In[19]:= aa = Coefficient[eqn2, h, 2];

In[20]:= bb = Coefficient[eqn2, h, 1];

In[21]:= Simplify[bb]

In[22]:= cc = Coefficient[eqn2, h, 0];

In[23]:= stmp = OpenWrite[a : tmp, FormatType → FortranForm, PageWidth → 70]

In[24]:= aaa = FortranForm[aa];

         bbb = FortranForm[bb];

         ccc = FortranForm[cc];
```

We solve next for the layer thickness, h, which makes the determinant xi1*eta2 - xi2*eta1 ==0

```
In[25]:= Write[stmp, aaa =, aaa, bbb =, bbb, ccc =, ccc];

         Write[stmp, Solve[aaa h^2 + bbb h + ccc == 0, h]];

In[26]:= Close[stmp]
```

1.5 The program *APP_H.FOR*

APP_H.FOR

The output of **LEE_EQN.NB** is the file **a:tmp**. This file can be used to write a Fortran program **APP_H.FOR** to perform the inverse computations. The data to the Fortran program **APP_H.FOR** consists of the values of the reciprocal wavelength and the corresponding frequency. Pairs of these data can be read from the plotted results of experimental measurements. In the case discussed here, the data are read from the results of calculations made with the aid of **SIN13.FOR**, and shown in Figure (1.1).

The output obtained from the program **APP_H.FOR** is shown in Figure (1.2). This figure shows the calculated thickness of the surface layer, plotted using successive data pairs obtained from Figure (1.1). The surface layer was one unit in thickness. The result obtained from the program **APP_H.FOR** is accurate to an order of magnitude only.

1.6 The program SIN13.FOR

Figure (1.1) shows the output from **SIN13.FOR**. The figure shows the dimensionless form of the results of measurements for a system composed of a layer of a stiff material overlying a subgrade composed of a less stiff material. The ratio of the shear moduli of the materials is $\frac{\mu_2}{\mu_1} = 0.01$. The dimensionless wavenumber is $\frac{\pi h}{\lambda}$, and the dimensionless frequency is $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$. The ratios $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

In order to achieve a plot which is comparable with field measurements, the axes must be scaled. The frequency axis must be multiplied by $\beta 1$, and divided by πh. The axis of the reciprocal wavelength must be divided by πh.

1.6.1 Operating SIN13.FOR

On execution, **SIN13.FOR** echoes "Enter a Starting value of V". It expects an initial wavelength, expressed as $V = \frac{\lambda}{\pi}$. It uses the value of **V** supplied by the user to start a search for a solution. If the velocity in the surface material is unity, a value of 0.4 appears to be a suitable starting point. This value of **V** corresponds with a wavelength of approximately the thickness of the surface layer.

However the termination condition, at which the solution is accepted for output, is not rigorous(see lines 126-130, SIN13.FOR). It does not necessitate a crossing of the zero of the determinant of the system. It requires only that the change in the value of the determinant at the final step should be small (about one hundredth or less of the initial trial value). This requirement leads to a more ready acceptance of the result at low velocities than at high velocities. At high velocities, the requirement is relatively stringent.

The effect is to produce different outputs depending on the starting value of **V**.

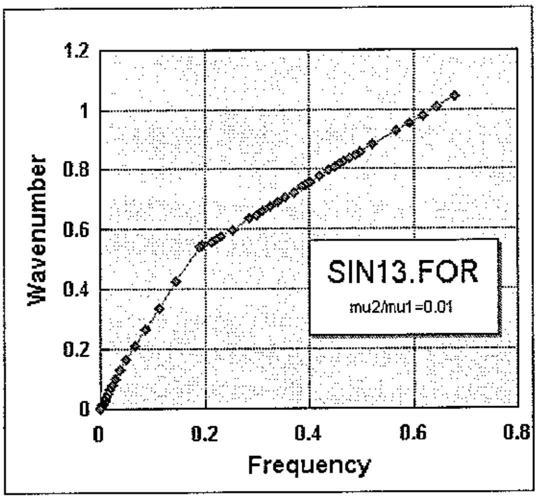

Figure 1.1: **Diagram showing the dimensionless form of the results of measurements for hard over soft; Output from SIN13.FOR. The wavenumber is** $\frac{\pi h}{\lambda}$**, and the frequency is** $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$**. The ratios** $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

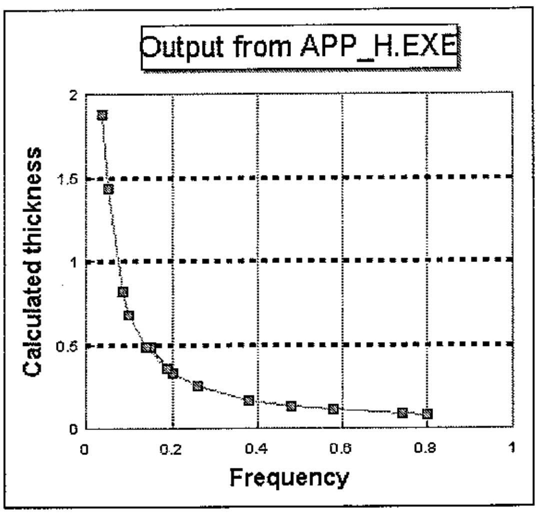

Figure 1.2: **Diagram showing the reverse of the results of measurements for hard over soft; Output from APP_H.FOR. The thickness** h **is unity, and the frequency is** $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$**.**

Chapter 2

Part 1: Soft over hard

2.1 Natural earthen systems

The following notebooks attempt to solve the problem (Section 1.2) for normal surface earthen systems. In these systems, the soft layer is at the surface. The underlying medium is a hard material. This is the normal progression in earthen systems.

2.1.1 The case of "soft over hard" - the normal geophysical progression

The case of "soft over hard" is the structure which is normally encountered in the field. Waves having a length shorter than the thickness of the surface layer yield information concerning the surface layer. Longer waves are generated by low frequencies. The results obtained indicate the properties of the underlying medium.

Two notebooks are utilized: **LEE1.NB** and **LEE2.NB**. The notebook **LEE1.NB** writes the series representing the determinant of the system in $h^n, n = 1..6$, where h is the thickness of the surface layer.[1] The output of **LEE1.NB** is **a:lee1tmp**, which is read as input by **LEE2.NB**. **LEE2.NB** writes the coefficients of the series for the determinant. The output from **LEE2.NB** is **LEE2.RES**. This file contains the coefficients of the series representing the determinant.

The coefficients are in a form which is suitable for reversion. The constant term is zero, and the coefficient of the term in h^1 is unity.

The Mathematica program **REVERSE.NB** yields an output **REVERSE.RES** in Fortran form. This output in turn yields a Fortran program **REVEW202.FOR**[2] and **REV9x.F90**. The reversion was carried out to terms in h. The series obtained is either divergent or so slowly convergent as to be unusable.

[1] Each of the expanded series of $\xi_1, \xi_2, \eta_1, \eta_2$ contains h^3; the product $\xi_1\ \eta_2 - \xi_2\ \eta_1$ contains h^6.
[2] Refer to Ewing, Jardetzky and Press[2] equation (202).

2.1.2 The program EW202.FOR

The example shown in Figure (2.2) is a forward calculation performed with the aid of **EW202.FOR**.

In Figure (2.2), the data used as input to the program **EW202.FOR** are as follows.

The values of $\beta_1/\alpha_1, \alpha_1/\alpha_2, \beta_2/\alpha_2, \mu_2/\mu_1$,which are coded as **B1A1, A1A2, B2A2, XMU2M1** , are the first line of the data input. The subsequent lines, eight values per line, are the values of c/α_1 for which calculations are required. The figures "999" in the final field specify a continuation line; a zero "0" specifies the final line of input.

```
0.4      0.1       0.4      100
1.1      1.3       1.5      1.7      1.9      2.1      2.3   999
2.6      2.9       3.1      3.3      3.5      3.53     3.57  999
3.6      3.63      3.66     3.69     3.72     3.74     3.77  0
```

The program **EW202.FOR** reads the velocity as a fraction of c/α_1. c must exceed α_1 and must be less than β_2. The corresponding wavelength is calculated within the program **EW202.FOR** as $V = \lambda/\pi$. The output of the reciprocal wavelength is π/λ, which is π times the actual reciprocal wavelength in the system, measured in reciprocal meters.

The frequency output is $c/\beta_1 * \pi h/\lambda$. It is thus $\pi h/\beta_1$ times the actual frequency in the system measured in Herz. The thickness h of the surface layer is unity.

2.1.3 Expansion of the determinant in h

The determinant, given by the notebook **LEE1.NB**, is expanded in powers of h. The coefficients are $[a(i), i = 1..power]$. The independent variable of the reversion is $z = \frac{-a[0]}{a[1]}$. The remaining coefficients $[a(i), i = 2..power]$ are all divided by $a[1]$.

A generalized series is then produced and reversed in notebook **REVERSE.NB**. The output is in **REVERSE.RES**. The output contains the coefficients of the reversion and the reversion itself, in terms of its coefficients.

The output of **LEE1.NB** is **LEE1TMP.DAT. LEE1TMP.DAT** was used as input to **LEE2.NB**. The output of **LEE2.NB** is **LEE2.RES** and is in Fortran form.

The program was tested using the data shown in Figure (2.2), a system composed of a layer having a thickness of unity overlying a semi-infinite medium.

2.1.3.1 Expansion to the power of three in h

Expansion was carried out to the power three in h. The result was that $a[7], a[8], a[9]$ are all equal to zero. The terms in the reversed series are large but defined. The program is **EWI83.F90** [3].

[3] The source language of this program **EWI83.F90** and of the programs **REV96.F90** and **REV99.F90** are not included as they are of small significance. They are generated from **LEE2.RES** which is in Fortran form

The series converges, at least for some values of frequency and reciprocal wavelength (the values 0.13 0.00073 lead to a value of 0.9 $for\ < h >$)

2.1.3.2 Expansion to the power of six in h

Expansion was carried out to the power of six in h . The program is **REV96.F90**. The result was that z, the independent variable of the reversion, was defined. All other terms, up to an including the sixth power of h, were defined. The series converges, at least for some values of frequency and reciprocal wavelength (the values 0.2 0.00238 lead to a value of 0.91 $for\ < h >$).

2.1.3.3 Expansion to the power of nine in h

Expansion was carried out to the power of nine in h . The program is **REV99.F90**. The series converges, at least for some values of frequency and reciprocal wavelength (the values 0.2 0.00238 lead to a value of 0.89 $for\ < h >$) . **LEE1.NB**

2.1.4 Mathematica notebook LEE1.NB

In[27]:=

```
(* LEE1.MA : The following is a definition of the period equation
(4-202) in Ewing, Jardetzky and Press, page 193  It leads to a
polynomial 'f' in h to the power of six.  Use Coefficient[f, h, n
], to extract the coefficients,  see page 791 of the Mathematica
manual, Version 4*)


x1 = ((2-kb1^2/k^2) (xx Cos[w r1 h] + r2/r1 yy Sin[w r1 h] )
         +2 s1/k (r2/k ww Sin[w s1 h] - k/s1 zz Cos[w s1 h] ));
x2 = ((2-kb1^2/k^2) (s2/k ww Cos[w r1 h] + k/r1 zz Sin[w r1 h] )
         +2 s1/k (xx Sin[w s1 h] - s2/s1 yy Cos[w s1 h] ));
```

In[28]:=

```
e1 = ((2-kb1^2/k^2) (r2/k ww Cos[w s1 h] + k/s1 zz Sin[w s1 h] )
          +2 r1/k ( xx Sin[ w r1 h] - r2/r1 yy Cos[ w r1 h] ));
e2 = ((2-kb1^2/k^2) (xx Cos[ w s1 h] +s2/s1 yy Sin[w s1 h])
          +2 r1/k (s2/k ww Sin[ w r1 h] - k/r1 zz Cos[ w r1 h] ));
```

In[29]:=

```
(*
k = w /c;
kb1 = w / b1;
kb2 = w / b2;

r1 =  ( - 1/c^2 + 1/a1^2) ^ (1/2);
s1 =  ( - 1/c^2 + 1/b1^2  ) ^(1/2);
r2 =  ( - 1/c^2 + 1/a2^2) ^ (1/2);
s2 =  ( - 1/c^2 + 1/b2^2  ) ^(1/2); *)

(* Expand each of the elements in a series, and simplify *)

xi1 = Factor[Normal[Series[x1,{h,0,3}] ] ];
xi2 = Factor[Normal[Series[x2,{h,0,3}] ] ];
```

In[30]:=

```
eta1 = Factor[Normal[Series[e1,{h,0,3}] ] ];
eta2 = Factor[Normal[Series[e2,{h,0,3}] ] ];
```

In[31]:=

```
(* Now write the determinant, to be solved for the thickness, h *)

f = xi1 eta2 - xi2 eta1; (* EJP's equation 4-202 *)
y = Simplify[Factor[ f ] ];(* Prepare the polynomial to extract *)
  (* the coefficients; 'y' is now a polynomial, and the  coefficients
   of h^n  can be extracted by a Coefficient[y,h,n] command *)
stmp = OpenWrite["a:lee1tmp"];    (* Open a file to receive
intermediate output*)

Write[stmp,y];  (* Write 'y' to A:LEE1TMP as input to LEE2.M *)
Close[stmp];    (* Close the file *)

(* Definition of the ancilliary variables follows *)

(*

ww = 2 ( mu2/mu1 -1 );
xx = mu2/mu1 kb2^2/k^2 - ww;

yy =kb1^2/k^2+ww;
zz = mu2/mu1 kb2^2/k^2 - kb1^2 /k^2 - ww; *)
```

2.1.5 Generation of the coefficients for the reversion: Mathematica notebook LEE2.NB

```
In[32]:=
        (* File LEE2.NB: Continuation of LEE1.NB the expansion
             of Ewing Jardetzky and Press eqn (4-202) *)
        y = <<a:leeltmp;    (* Read in the file LEE1TMP, y1 from LEE1.NB *)
        z = -Coefficient[y,h,0]/Coefficient[y,h,1];(*Independent variable
        *)

        y1 = y - Coefficient[y,h,0]; (* Subtract constant term from
        series, and   *)

        y2 = y1/Coefficient[y1,h,1]; (* arrange that h^1 has a coefficient
        of unity.*)

        a[1] = Simplify[Coefficient[y2,h,1]] (* Check that h^1 has a
        coefficient of unity.*)

        a[0] = Coefficient[y2,h,0]      (* Check that the constant term is
        zero.*)

        a[2] = Simplify[Coefficient[y2,h,2] ]; (* Prepare coefficients *)
        a[3] = Simplify[Coefficient[y2,h,3] ]; (*       for            *)
        a[4] = Simplify[Coefficient[y2,h,4] ]; (*     reversion *)

        a[5] = Simplify[Coefficient[y2,h,5] ]; (*        of        *)

        a[6] = Simplify[Coefficient[y2,h,6] ]; (*       series *)

        stmp = OpenWrite["a:lee2.res"];

        Write[stmp, " z = ", FortranForm[z],
                "a[0] =",FortranForm[a[0]],
                "a[1] =",FortranForm[a[1]],     (*  Write the results    *)
                "a[2] =",FortranForm[a[2]],     (*  of the coefficients  *)
                "a[3] =",FortranForm[a[3]],     (*    in the series      *)
                "a[4] =",FortranForm[a[4]],     (*   to be reversed      *)
                "a[5] =",FortranForm[a[5]],     (*  in Fortran form      *)
                "a[6] =",FortranForm[a[6]] ];
        Close[stmp]

Out[32]= 1
Out[32]= 0
Out[32]= a : lee2.res
```

LEE21.NB

2.1.6 Notebook LEE21.NB

The output of the coefficients of the series to be reversed can be obtained with the aid of Notebook **LEE21.NB**.

2.2 LEE21.NB

Output of the coefficients of the reversed series: Mahematica notebook LEE21.NB

```
In[33]:=
        stmp = OpenWrite["a:lee.dat"];
        Write[stmp, " z = ",z,
                "a[0] =",a[0],
                "a[1] =",a[1],
                "a[2] =",a[2],    (*     Write           *)
                "a[3] =",a[3],    (*    results          *)
                "a[4] =",a[4],
                "a[5] =",a[5],
                "a[6] =",a[6] ];
        Close[stmp]
```

2.3 Series for the thickness h in terms of c and ω.

REVERSE.NB

The notebook **REVERSE.NB** generates a series, then finds the reverse of the series. The $a(i), i = 1..6$ are the coefficients of the forward series. This file includes an independent variable of the reversion, p. Then the coefficients of the reversed series $b(i), i = 1..6$ are written in terms of $a(i), i = 1..6$. The results are written to **REVERSE.RES**.

The coefficients $a(i), i = 1..6$ can be read from **LEE2.RES**. **REVERSE.NB** performs the same operation on its input series as that performed on the output series in **LEE2.NB** .

2.4 Series reversion: Notebook REVERSE.NB

In[34]:=

```
(* REVERSE.NB Mathematica program to generate the reversed series
of a polynomial having defined coefficients.  The program reverses
a series having coefficients a[i]. The coefficients of the
reversion are b[i]. The coefficients of the reversion and the
reversion itself are written to a file A:REVERSE.RES *)

y3 = Sum[{ a[i] x^i},{i,0,7}];      (* Write a series with defined
coeffts *)

y2 = Expand[Simplify[(y3[[1]]-a[0])/a[1]] ]; (* Set zero and unity
coeffts *)

z = - a[0]/a[1];    (* Independent variable of the reversion *)

y1= Series[y2,{x,0,7}]; (*Set the series form for the reversion *)

y = Normal[InverseSeries[y1,p] ]; (* Carry out the reversion *)

b[1] = Simplify[Coefficient[y ,p,1] ]; (* Extract the coefficients *)
b[2] = Simplify[Coefficient[y ,p,2] ]; (*                          *)
b[3] = Simplify[Coefficient[y ,p,3] ]; (*        of the            *)
b[4] = Simplify[Coefficient[y ,p,4] ]; (*       reversed           *)
b[5] = Simplify[Coefficient[y ,p,5] ]; (*                          *)
b[6] = Simplify[Coefficient[y ,p,6] ]; (*        series            *)

w = Sum[b[i] p^i, {i,1,6}];               (*  Form the output series
*) stmp = OpenWrite["a:reverse.res"];
    Write[stmp,"p = ", z ,                (*      Define  p        *)
        "b[1] =",FortranForm[b[1]],       (*        Write          *)
        "b[2] =",FortranForm[b[2]],       (*         the           *)
        "b[3] =",FortranForm[b[3]],       (*       results         *)
        "b[4] =",FortranForm[b[4]],       (*   to a Fortran file   *)
        "b[5] =",FortranForm[b[5]],       (*         and           *)
        "b[6] =",FortranForm[b[6]] ];     (*     close the file    *)
    Write[stmp, " w = ", FortranForm[w] ]; (* Write the output to a
file *)
Close[stmp]
```

figures (1.1) and (1.2) indicate the results obtained from a single-layered system, of the "hard-over-soft" type. Figure (1.1) is the output from **SIN13.FOR**. It shows the reciprocal of the wavelength plotted against the frequency, for a system consisting of a hard layer of unit thickness overlying a soft semi-infinite medium.

Figure (1.2) shows the results obtained with the aid of the Fortran program **APP_H.FOR**. The input to **APP_H.FOR** consists of the data pairs read from Figure (1.1). The result is an approximation to the thickness of the surface layer.

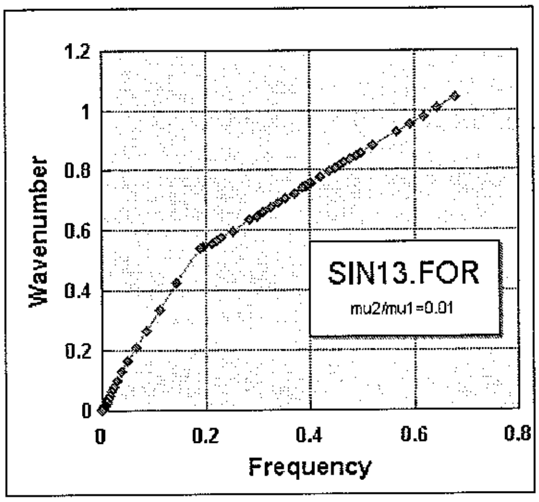

Figure 2.1: **Diagram showing the dimensionless form of the results of measurements for hard over soft; Output from SIN13.FOR. The wavenumber is** $\frac{\pi h}{\lambda}$**, and the frequency is** $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$**. The ratios** $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

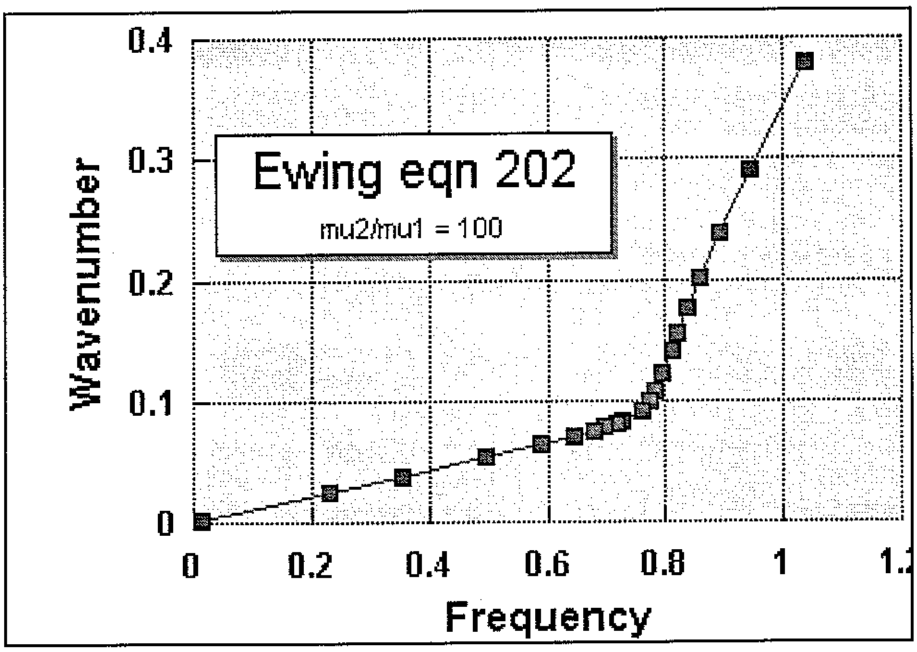

Figure 2.2: **Diagram showing the dimensionless form of the results of measurements for soft over hard; Output from EW202.FOR. The wavenumber is** $\frac{\pi h}{\lambda}$**, and the frequency is** $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$**. The ratios** $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

Laboratory manual Part II

Part 2 : User manual

Chapter 3

Part 2: Theoretical Match with Experimental Results

3.1 Maple worksheets hardk.mws and softk.mws

The following is a series of outputs obtained with the aim of matching experimental results to theoretical predictions.

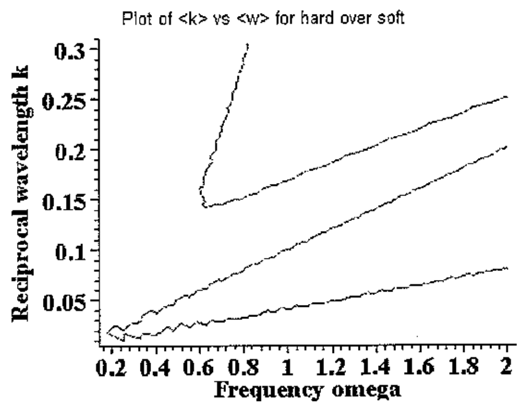

Figure 3.1: **Implicit Plot from hardk.mws: a stiff layer overlying a less stiff semi-infinite medium.**

The two figures (3.1) and (3.2) were plotted with the aid of Maple version 7. The figures show the reciprocal wavelength k plotted against the angular frequency ω. The programs were **hardk.mws** (see section 5.8 on page 83) and **softk.mws** (see section 5.9 on page 85). The programs **hardk.mws** and **softk.mws** were developed from A.W.Lee, and are based on Ewing, Jardetzky and Press equation (4-202)[2].

The system consists of a single layer overlying a semi-infinite medium. In Figure (3.1), the surface layer has the higher modulus. In Figure (3.2), the surface layer has the lower modulus.

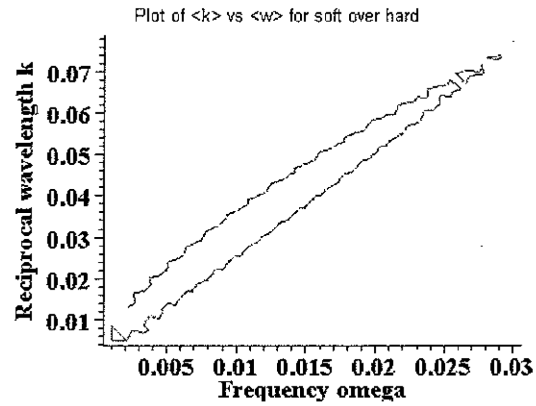

Figure 3.2: **Implicit Plot from softk.mws:a less stiff layer overlying a more stiff semi-infinite medium.**

The scale of the reciprocal wavelength in figures (3.1) and (3.2) is $\frac{2\pi}{\lambda}$ where λ is in units of the thickness of the layer. The scale of frequency $\omega = 2\pi f$ where f is the frequency in Herz.

The shear-moduli μ_1 and μ_2 refer to the materials in the surface layer and the underlying medium respectively. The values of the moduli are 1.0 and 100.0, and Poisson's ratio is equal to 0.4. The thickness of the surface layer is 0.1 units in both cases.

3.2 Hard over soft: reverse calculation

3.2.1 Scaling the output from SIN13.FOR

The results shown in Figure (3.3) were obtained with the aid of **SIN13.FOR** (see section 5.1 on page 53). [1]

The output variables are the reciprocal wavelength **V** = **LAMBDA/PI** and the frequency, in the form **CB1/V**.

The velocity of shear waves in the material composing the surface layer β_1 is 1300 m/s. The ratio of the shear modulus in the underlying medium to that in the surface layer $\frac{\mu_2}{\mu_1}$ is 0.01. The thickness of the surface layer is 0.05 m. The ratio of velocities of transverse waves, β, to the velocities of dilatational waves, α, is $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$.

[1]In the output from **SIN13.FOR**: The wavenumber is $\frac{\pi h}{\lambda}$, and the frequency is $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$. The ratios $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

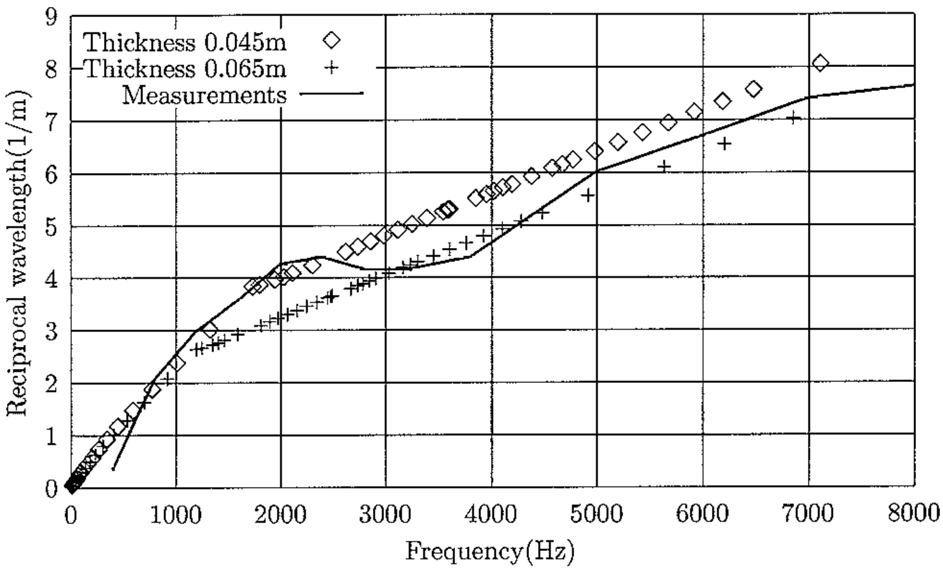

Figure 3.3: **Experimental results and theoretical match. Southampton 240a2.wk4**

3.2.2 Routine for finding the theoretical match

The routine for finding a theoretical match with the experimental results shown in Figure (3.3) was as follows.

1. The ratio of the elastic modulus in the surface course to that in the underlying course was estimated from the plot of reciprocal wavelength against the frequency. This ratio is taken as the square of the ratio of the group velocities.[2]
2. Two sets of theoretical points are shown. The two trial sets of points shown have first-order discontinuities at (1000, 2.5) assuming a thickness of 0.045m, and (1600, 3.9) assuming a thickness of 0.065m. The first-order discontinuity in the plot of theoretical results was used to scale the theoretical results to correspond with the experimental results.
3. The program **SIN13.FOR** was used to obtain a table of values of reciprocal wavelength against the corresponding frequency. The velocity in the surface material was assumed to be unity.

   The output from **SIN13.FOR** is shown in Figure (2.1). The discontinuity in the curve of the theoretical results is at (0.19, 0.55). This curve is calculated for a thickness of unity for the surface layer.
4. In the output from **SIN13.FOR**: The dimensionless reciprocal wavelength is $\frac{\pi h}{\lambda}$, and the dimensionless frequency is $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$. The ratios $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2} = 0.4$

[2]In a system composed of two media, the ratio of the elastic moduli $\frac{\mu_1}{\mu_2}$ is greater than the ratio of the group velocities.

The match shown is obtained as follows.

1. The **WAVENUMBER** output from **SIN13.FOR** is multiplied by $\frac{1}{0.045\ \pi}$ and by $\frac{1}{0.065\ \pi}$ to obtain matches for thicknesses of 0.045 meters and of 0.065 meters respectively.

2. The **FREQUENCY** output from **SIN13.FOR** is multiplied by $\frac{1300}{0.045\ \pi}$ and $\frac{1300}{0.065\ \pi}$ to obtain matches for a structure having a surface-layer thickness of 0.045 meters and of 0.065 meters respectively.

3.2.3 Problems in attaining a theoretical match - Hard over soft: inverse system

3.2.3.1 Hurn Airport hurn547.tex

We attempt to improve on the present match between the measurements and the expected theoretical results.

The experimental results labelled **hurn547** were obtained on a parking bay at Hurn Airport at a spacing of 547 millimeters. The results are shown in Figure (3.4). They indicate that three media are present in the system. The velocities in the media are 1300 m/s, 450 m/s and 160 m/s. We attempt first to match the results with those expected from a system consisting of two layers.

The theoretical match was reached as follows.

1. Read the velocity in the surface layer from the experimental results. It is the reciprocal of the slope of the plot of reciprocal wavelength versus frequency, measured at high frequencies.

2. Choose a value of h which most nearly matches the remainder of the experimental results.

3. Increase the contrast in the matching data, between the upper and lower layers. This is done by increasing the ratio $\frac{\alpha_1}{\alpha_2}$ and the corresponding $\frac{\mu_2}{\mu_2}$ in the data file. See **SIN13.FOR** (Section 5.1page 53) for the format of the data file.

The match was obtained using **SIN13.FOR** and the following data file.

```
0.4 100.0 0.4 1.0E-4 \\
0.64 0.63 0.62 0.61 0.60 0.59 0.585 999.\\
0.58 0.57 0.56 0.555 0.55 0.546 0.54 999.\\
 0.524 0.523 0.52 0.51 0.50 0.49 0.48 999.\\
 0.47 0.46 0.45 0.42 0.40 0.39 0.38 999.\\
0.36 0.35 0.34 0.33 0.32 0.31 0.30 999. \\
0.290 0.285 0.280 0.275 0.270 0.265 0.250 999. \\
0.283 0.282 0.281 0.280 0.279 0.278 0.277 999. \\
```

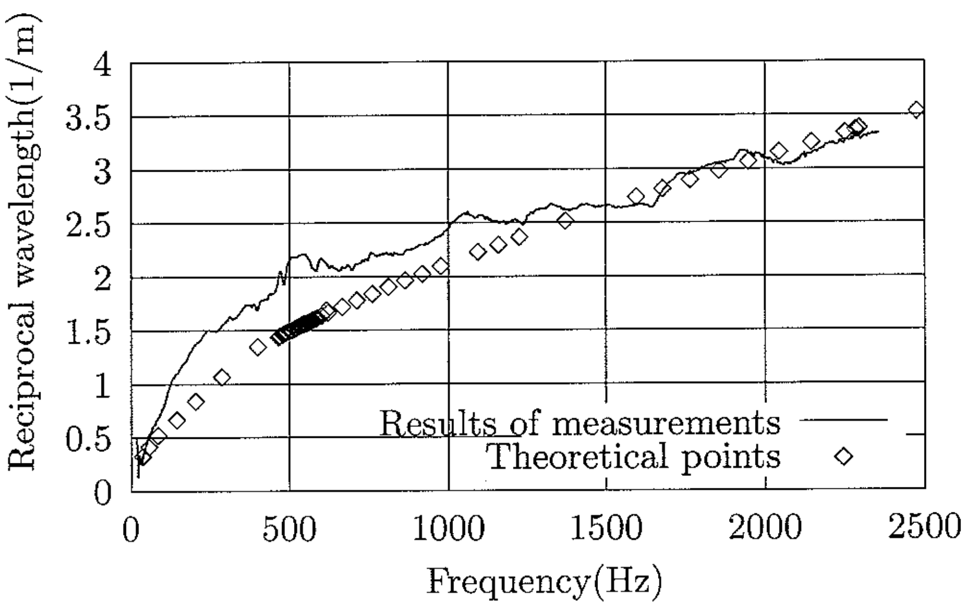

Figure 3.4: **Experimental results and theoretical match assuming that the system consists of two media. Hurn airport Hurn547** $\beta_1 = 1300ms^{-1}, h = 0.060m$. *hurn547.dat* is result of measurements; *hurn5471.dat* is the theoretical match.

```
0.276 0.275 0.274 0.273 0.272 0.271 0.270 999. \\
0.269 0.268 0.266 0.264 0.262 0.260 0.258 999. \\
 0.257 0.256 0.255 0.254 0.252 0.251 0.250 999.\\
 0.23 0.21  0.19   0.17 0.13   0.11 0.09 \\
 0.0 0.0 0.0 0.0 0.0
```

The thickness is calculated from the height of the first-order discontinuity in the curve, in both the experimental results and in the results of the calculation. The match was achieved by

1. multiplying the theoretical values of the wavenumber by $\frac{1}{\pi\, 0.060}$, indicating that the thickness is 0.060m.

2. multiplying the theoretical values of the frequency by $1300\frac{1}{\pi\, 0.060}$, indicating that the velocity of Rayleigh type waves in the surface medium is $1300m/s$.

The best match achieved is shown in Figure (3.4). The match achieved for the underlying layers and for the semi-infinite medium is poor.

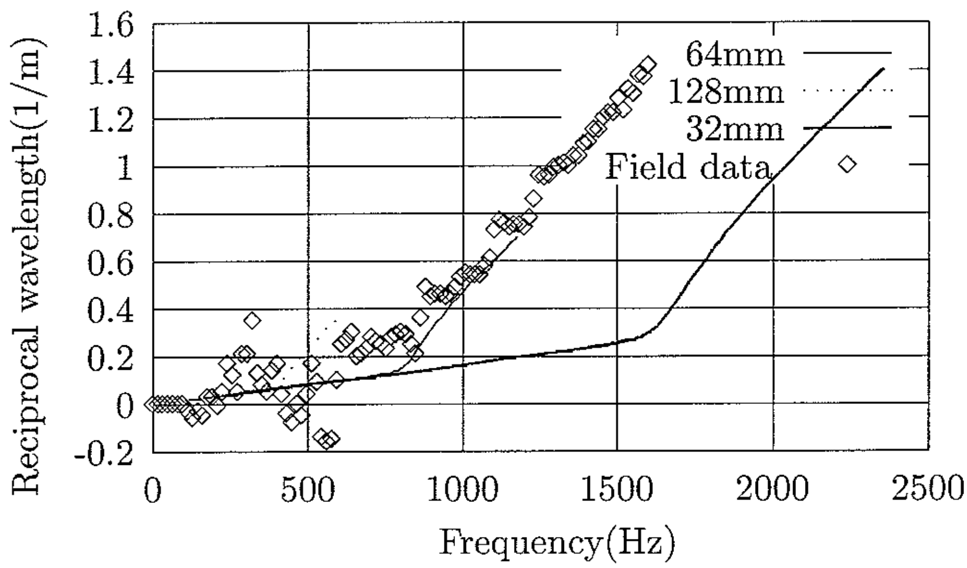

Figure 3.5: **Experimental results and theoretical match. Hurn airport Hurn1p**

3.3 Soft over hard: reverse calculation

The results shown in Figure (3.5) were obtained on a parking bay at Hurn Airport. The figure shows also a theoretical match with the experimental results. The match indicates that the surface layer is approximately 64 millimeters in thickness.

The calculations for the theoretical match were performed with the aid of the program **SON.FOR**. The datafile **SON.DAT** was as follows.

```
0.4 0.101 0.4 100. \\
2.51 2.52 2.53 2.54 2.55 2.56 2.57 999.\\
2.58 2.6 2.75 2.8 2.9 3.1 3.3 999.
3.45 3.8 3.9 3.95 4.0 4.05 4.08 999.\\
4.2 4.4 5.1 5.5 6.7 7.5 7.9 999.\\
8.0 8.1 8.2 8.3 8.4 8.5 8.6 999.\\
8.7 8.8 8.9 9.0 9.1 9.15 9.18 999.\\
9.22 9.26  9.28 9.29 9.295 9.297 9.299\\
0.0 0.0 0.0 0.0 0.0
```

The match was achieved by placing $\beta_1 = 670 ms^{-1}$ and $h = 0.64/\pi$, $h = 0.32/\pi$,and $h = 1.28/\pi$,.

3.3.1 Problems in attaining a theoretical match - Soft over hard: Direct system

The steps taken were as follows.

1. Read the ratio of the highest to the lowest velocity from the plot of the reciprocal wavelength against the frequency.
2. This ratio, squared, gives the ratio of the shear moduli $\frac{\mu_2}{\mu_1}$
3. Enter the value of the ratio of the shear moduli in **SON.DAT** Section 5.2 on page 56 or in **EW202.DAT** Section 5.4.1 on page pagerefprg:EW202.DAT.
4. Enter the values of the Poisson's ratios in the surface layer and in the underlying medium in **SON.DAT**or in **EW202.DAT**
5. Enter the value of $\frac{\alpha_1}{\alpha_2}$. This value can be arbitrarily small: it needs only to ensure that $\alpha_1 < c < \beta_2$.
6. Enter the values of the velocity ratios to be calculated, as fractions $\frac{c}{\alpha_1}$. The velocity must exceed α_1 and be less than β_2.

There are infinitely many solutions to Ewing's equation (4-202). The solution produced by **SON.FOR** or by **EW202.FOR** depends upon the initial value of $V = \frac{\lambda}{\pi}$.

The numerical solution is probably easiest for $\frac{\mu_2}{\mu_2}$ large, 60 or more. For $\frac{\mu_2}{\mu_2}$ small, equal to 4 or less, the response depends upon the initial value of V. For $\frac{\alpha_1}{\alpha_2} = 0.1$, the first-order discontinuity in the curve is sharp for V=1.0, and is gradual for V=4.0

3.3.2 Scaling the output from SON.FOR or EW202.FOR

The output from **SON.FOR** and from **EW202.FOR** is in dimensionless form. The reciprocal of the wavelength is in dimensionless units of $\frac{\pi h}{\lambda}$. Divide the output of dimensionless reciprocal wavelength by πh in order to make it dimensional.

The frequency output from **SON.FOR** and from **EW202.FOR**is in dimensionless units of $\frac{c}{\beta_1} * \frac{\pi h}{\lambda}$. The output from **SON.FOR** and from **EW202.FOR** must be divided by $\frac{\pi h}{\beta_1}$, in order to make it dimensional.

3.3.3 Routine to provide a theoretical match with experimental data

3.3.3.1 Hurn Airport hurn12.eps

The output obtained theoretically requires scaling in order to match it with experimental results. The following procedure was followed in order to produce Figure (3.6).

1. Using the first-order discontinuity of the curve of the reciprocal wavelength against the frequency,[3] adjust the calculated reciprocal wavelength so that it matches with the experimental measurements.

2. Calculate and supply the value of β_1 which yields the measured frequency at this point.

3. The supplied numerical values then represent the field quantities.

Figure (3.6) shows the experimental results obtained for Hurn airport, site 1. The theoretical match shown was obtained with the aid of EW202.FOR. In the surface layer, $\beta_1 = 800m/s$, and the layer thickness was $h = 0.4m$. The value of Poisson's ratio was equal to 0.4 in both the layer and in the underlying semi-infinite medium.

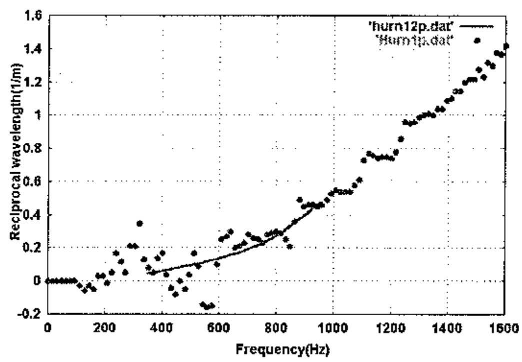

Figure 3.6: **Experimental results and theoretical match. Hurn Airport hurnl2p** $\beta_1 = 800ms^{-1}, h = 0.4m$

The datafile **EW202.DAT** used in the program **EW202.FOR** was as follows.

```
0.4      0.1      0.4      9
   1.005   1.01          1.013    1.014    1.015    1.017    1.019 999
   1.02    1.04         1.05     1.06     1.07     1.08      1.09 999
   1.1      1.2         1.3      1.4      1.5      1.6       1.7  999
   1.8      1.82        1.84     1.86     1.88      1.93       2.0    999
   2.0      2.1         2.15    2.2     2.28     2.3        2.32  999
   2.33       2.34      2.35    2.36     2.37      2.38      2.40    999
   2.6     2.8          3.0     3.4      3.6       3.7       3.8   0
```

The starting point for the iteration was $V = 1$. The ratio of the moduli was obtained as the square of the ratio of the extreme velocities, $[\frac{2010}{670}]^2 = 9$. Note that the datafile contains the unrealistic value for $\frac{\alpha_1}{\alpha_2} = 0.1$. A realistic value would be $\frac{1}{3}$.

This indicates an error in the assumption concerning the ratio of the velocities $\frac{\alpha_1}{\alpha_2}$: this ratio is not equal to the ratio of the group velocities.

[3]The first-order discontinuity is not sharp in this case. It is sharper as the ratio $\frac{\mu_2}{\mu_1}$ is decreased to about 0.1. See the next example.

3.3.3.2 Hurn Airport hurn13.tex

The attempt to match the experimental results with theoretical calculations was continued. Figure (3.7) shows the result obtained using the program **EW202.FOR** with the following data file

```
0.4       0.1          0.4        100
     1.1       1.3          1.5        1.7       1.9       2.1       2.3  999
     2.6       2.9          3.1        3.3       3.5      3.53      3.57  999
     3.6      3.63         3.66       3.69      3.72      3.74      3.77  0
```

The theoretical match of the experimental results shown in Figure (3.7) was obtained by scaling the theoretical results shown in Figure (2.2).

3.3.3.3 Details of the scaling process

The velocity in metres per second at high frequencies is 570m/s. It was obtained from the plot of the experimental results, showing the wavenumber plotted against the frequency, Figure (3.7). It is the reciprocal of the slope of this plot, at high frequencies.

The starting point for the iteration in the program **EW202.FOR** was V=1. The output from **EW202.FOR** was scaled as follows. The frequency was multiplied by $\frac{570 * 0.15}{0.08}$ and the wavenumber was multiplied by $\frac{0.15}{0.08}$. The values 0.15 and 0.08 are the wavenumber ordinates of the first-order discontinuity in the curve, for the experimental results and for the theoretical match obtained from **EW202.FOR** shown in Figure (2.2) respectively.

3.4 Conclusion

The aim of this chapter is to provide details of the process of scaling theoretical results. The operations are

1. Estimate the group velocities in the surface and the underlying media, with the aid of a plot of the experimental measurements of the reciprocal wavelength against the frequency.
2. Use the co-ordinates of the first-order discontinuity in the plot of the experimental measurements to estimate the depth of the interface between the surface layer and the underlying semi-infinite medium.

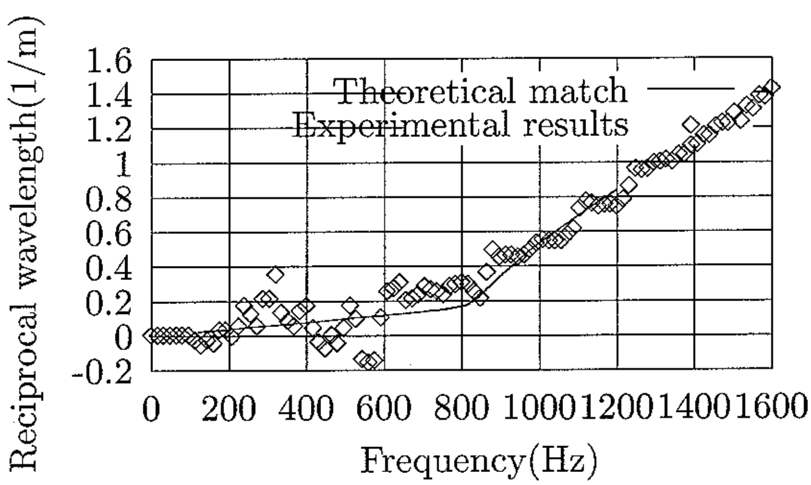

Figure 3.7: **Experimental results and theoretical match. Hurn airport Hurn13p** $\beta_1 = 570ms^{-1}, h = 0.6m$

Chapter 4

Part 2: Field Results

4.1 Pavement structure

The general pavement structure is more complex than those considered so far.

The stiffnesses of the materials within the layers can be in any order. The material having the highest elastic modulus may not be that in the top layer.

The results of the field measurements are matched with those expected theoretically from waves of the Rayleigh type. The solution is performed with the aid of the theory of Rayleigh waves [5]. The solution is not unique, as the component matrices contain trigonometric functions. In finding a theoretical match for experimental results, it is attempted to obtain the solution corresponding with the first mode. This is the mode having the lowest frequencies, and is probably the mode which is generated in the field.

In conclusion, suggestions are made on how to avoid mode-skipping during the calculations, and to concentrate on output relating to the first or any other selected mode.

4.2 Results of measurements

The results of experimental measurements made on two structures follow. The structures were of the general type, having stiffnesses in any order of depth. We make attempts to find matching theoretical structures.

4.2.1 LCPC results

Measurements were performed on a major road by the Laboratoire Central des Ponts et Chaussees (LCPC) [1]. The results of the measurements are shown in Table 4.1.

The measurements of the phase velocity made over a range of frequencies were plotted. An attempt was made to match the results of the measurements with theoretical predictions. See Figure (4.2.1). This figure shows the phase velocity plotted against the wavelength. It shows also the theoretical prediction obtained with the aid of the

Velocity(m/s)	Wavelength(m)	Frequency(Hz)	Recip w'length(1/m)
1320	0.2	6600	5
1350	0.25	5400.00	4.00
1370	0.4	3425.00	2.50
1300	0.65	2000.00	1.54
1085	0.75	1446.67	1.33
1120	1.1	1018.18	0.91
1040	1.25	832.00	0.80
900	1.8	500.00	0.56
850	2.15	395.35	0.47
840	2.8	300.00	0.36
790	2.6	303.85	0.38
700	3.05	229.51	0.33
560	2.8	200.00	0.36
500	2.95	169.49	0.34
460	3.25	141.54	0.31
410	4.6	89.13	0.22

Table 4.1: **Table giving the results of measurements from a pavement structure on Route A4 Auxerre-Lyon**

programs **PREP.FOR** and **RAY.FOR.**

4.2.1.1 Theoretical prediction

The theoretical prediction was made using the data obtained concerning the properties of the structure. The elastic moduli used were those measured by static means. The values of the moduli may not be correct in a situation which involves wave propagation.

The data file used in the program **PREP.FOR** was as follows.

```
LCPC DATA
E1=11200,E2=28500,E3=200;H1=0.1,H2=0.4,NU1=NU3=0.25,NU2=0.4 3
11200. .25  28500. .4  200. .3  0.0
      2.   2.   2.
 0.1 0.4
(5, 0.0)  (0.05, 0.00)  (0.1, 0.01)  0.05 0.1
```

The parameters used differ from those given by Bonitzer and Leger[1]. The values used by Bonitzer and Leger are shown in Table 4.2. Table 4.2 shows also the values used in Figure (4.2.1) and Figure (4.1).

The scatter of the experimental points in Figure (4.2.1) may be explained by the results shown in Figure (3.3). The results shown in Figure (3.3) were obtained automatically, by means of a programmed routine. The results shown in Figure (4.2.1) were obtained manually and depend upon the judgement of the operator.

The results showing the reciprocal of the wavelength plotted against the frequency are

Bonitzer and Leger	Figure 4.2.1 and Figure 4.1
$\beta_1 = 1320ms$	$\beta_1 = 1020ms$
$\beta_2 = 1415ms$	$\beta_2 = 1300ms$
$\beta_3 = 400ms$	$\beta_3 = 400ms$
$h_1 = 0.2m$	$h_1 = 0.2m$
$h_2 = 0.15m$	$h_2 = 0.4m$

Table 4.2: **Table showing velocities and thicknesses used in order to match the experimental results with theoretical calculations .**

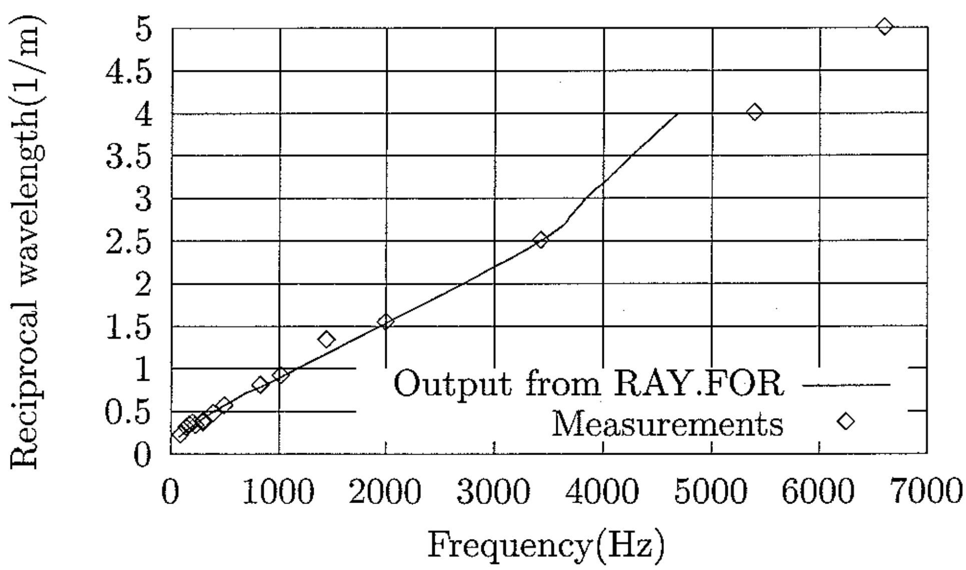

Figure 4.1: **Results of measurements and theoretical curve: reciprocal wavelength plotted against the frequency: LCPC structure Route A4 Auxerre-Lyon**

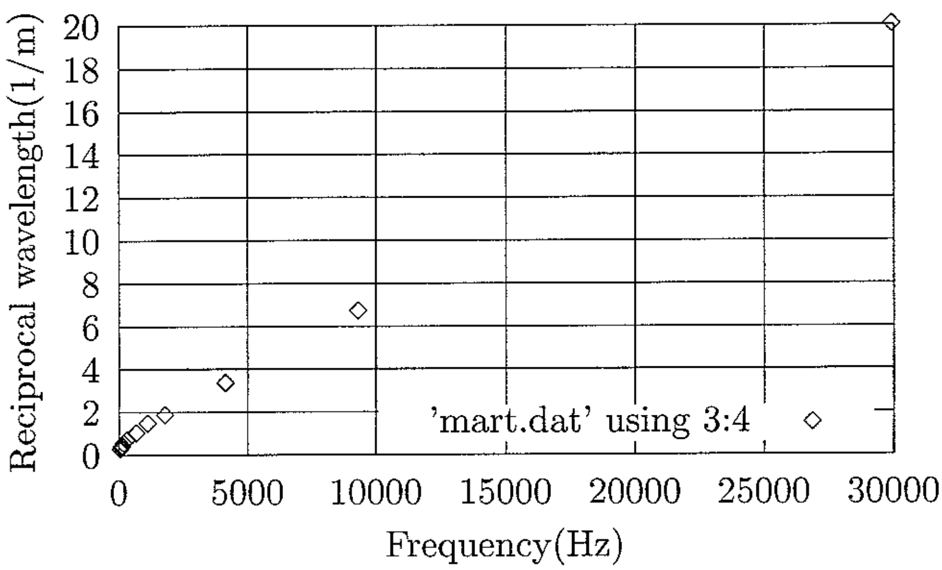

Figure 4.2: **Figure giving results of measurements from Martinzec[4] Figure 1.55**

Asphalt cement	5cm
Coated sand/gravel	8cm
Cement stabilized gravel	25cm
Lime stabilized gravel	10cm

Table 4.3: **Table giving the composition of the pavement from Martinzec[4] Figure 55**

shown in Figure (4.1). This figure shows both the results of measurements and the calculated output from **RAY.FOR**.

4.2.2 Martinzec, experimental results

Measurements were made by Martinzec [4]. The results of some of the measurements are shown in Figure (4.2) and in Table 4.4 . The velocities of shear waves read from Figure (4.2) are $\beta_1 = 1520ms^{-1}, \beta_2 = 1300ms^{-1}, \beta_3 = 240ms^{-1}$. Assuming that the density is $2000kg/m^3$,the values of the shear moduli are 4.5, 3.38 and 0.11 GPa. The corresponding values of $E = (1 + \sigma)\beta^2$ for $\sigma = 0.25$ are 11.1, 8.4 and 0.28 GPa.

The composition of the pavement is shown in Table 4.3.

Figure (4.3) shows the velocity of waves of the Rayleigh type plotted against the wavelength. The figure shows experimental measurements, and also the expected theoretical values. The pavement is treated as a structure composed of three media. All media extend laterally to an infinite extent. The subgrade extends to an infinite depth.

The theoretical curve shown in Figure (4.3) was obtained with the aid of the programs **PREP.FOR** and **RAY.FOR**.

MARTINZEC.TEX

Velocity(m/s)	Wavelength(m)	Frequency(Hz)	Recip w'length(1/m)
1496	0.05	29920	20
1400	0.15	9333	6.67
1250	0.3	4167	3.33
1000	0.55	1818	1.82
800	0.7	1143	1.43
700	1	700	1
500	1.55	323	0.65
400	2	200	0.5
350	2.5	140	0.4
300	3	100	0.33
250	4	63	0.25

Table 4.4: **Table giving results of measurements from Martinzec[4] Figure 1.55**

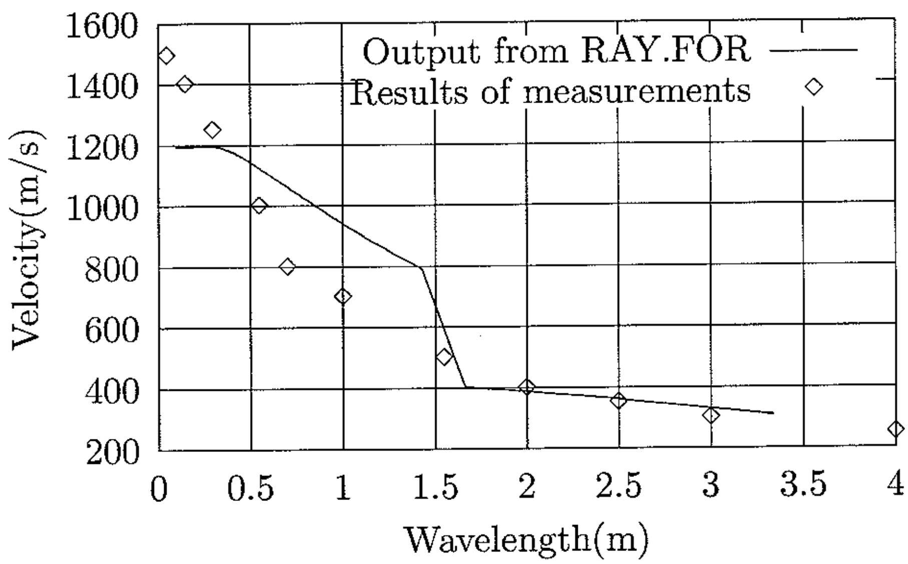

Figure 4.3: **Results of measurements of phase velocity and corresponding wavelength, showing theoretical curve: Martinzec Figure 55 [4]**

The following data were used for the program **PREP.FOR**.

```
MARTINZEC55E1=11100,E2=8400,E3=280;H1=0.13,H2=0.25,NU1=NU2=NU3=0.25
3 11100. .25  8400. .25  280. .25  0.0
      2.   2.   2.
0.13  0.25
(5, 0.0)  (0.2, 0.00)  (0.01, 0.01)  0.02 0.1
```

The figure shows a discrepancy between the measured and predicted values of wave velocity at the shorter wavelengths. This indicates that the properties of the surface layers, their thicknesses and elastic moduli, are not accurately defined. We attempt to improve the theoretical match by treating the system as composed of five media. The corresponding data file for **PREP.FOR** follows.

```
MARTINZEC55E1=11100,E2=84000,E5=280;H1=0.13,H2=0.25,NU1=NU2=NU3=0.25
5 11100. .25 84000.  .25  20000.  .25 40000.  .25  280. .25  0.0
      2.   2.   2.  2.   2.
0.05 0.08 0.25 0.10

(5, 0.0)  (0.4, 0.00)  (0.05, 0.01)  0.0001 0.1
```

This data file leads to a continuous dispersion curve of wave velocity against the wavelength, as shown in Figure (4.3). However the measured velocities at wavelengths less
mart3 than 1.5 meters are higher than the predicted values.

In order to correct the discrepancy, a five-medium system was analyzed. The wavelength output from the program **RAY.FOR** was multiplied by 0.3 and the velocity of waves of the Rayleigh type was multiplied by 1020. This suggests that the actual layer thickness are 0.3 times those assumed in the data used for the program **RAY.FOR**,
mart31 shown in Table 4.3.

4.2.2.1 Theoretical prediction

The program **RAY.FOR** operates using trigonometric functions. Therefore infinitely many solutions exist, providing phase velocity of waves of the Rayleigh type at specified wavelengths. The experimental results match with only one of these solutions. This solution may correspond with the mode having the lowest possible frequency.

Mode selection is carried out using the wave phase velocity as the criterion: the solution is sought for a given phase velocity. A better criterion may be to use the frequency. Use of the frequency may possibly decrease the tendency to skip modes.

The theoretical curve shown in Figure (4.3) shows adequate agreement with measurements at large wavelengths. At short wavelengths there is a discrepancy between the measurements and the theoretical curve.

This discrepancy is corrected in Figures (4.4) and (4.5) as follows. The wavelengths output from **RAY.FOR** have been multiplied by 0.3; the velocity at the surface has

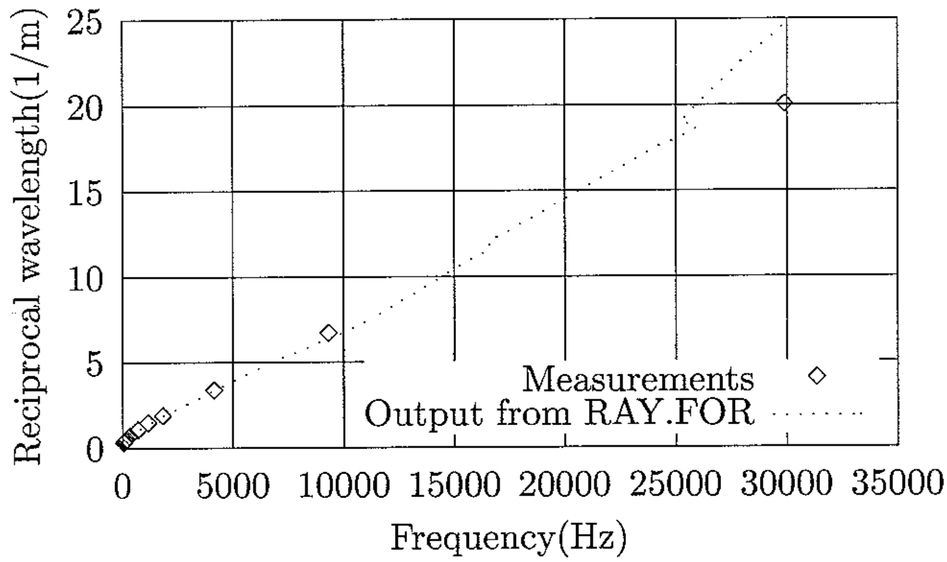

Figure 4.4: **Results of measurements and theoretical curve: the reciprocal of the wavelength plotted against the frequency of waves of the Rayleigh type, Martinzec Figure 55. Theoretical values of the wavelengths have been multiplied by 0.3;** $\beta_1 = 1020ms^{-1}$

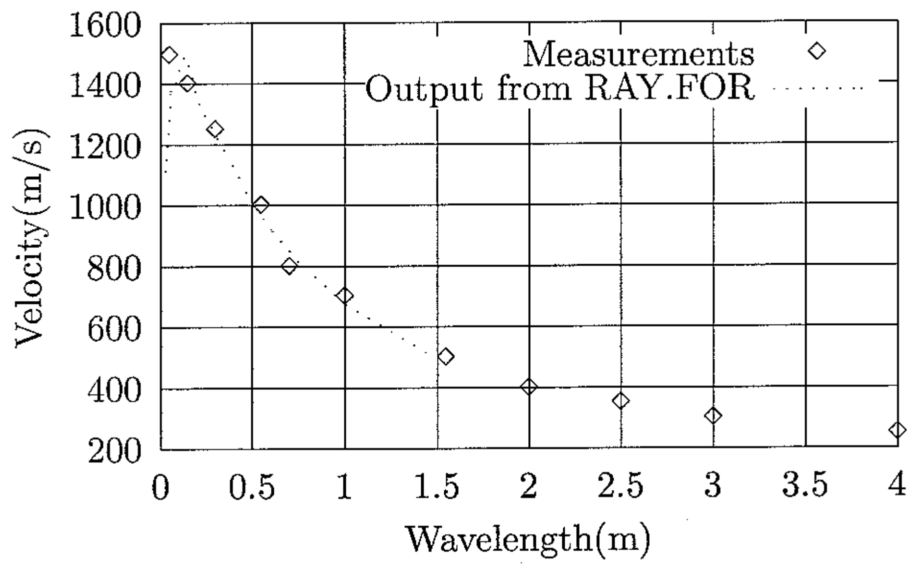

Figure 4.5: **Results of measurements and theoretical curve: phase velocity of waves of the Rayleigh type plotted against the wavelength, Martinzec Figure 55. Wavelengths have been multiplied by 0.3;** $\beta_1 = 1020ms^{-1}$

been made equal to $1020ms^{-1}$. The match between measured and theoretical values at short wavelengths is improved, as shown in Figures (4.4) and (4.5).

It may be deduced that the system is 0.3 times as thick as the data indicates in Table 4.3.

4.3 Routine for obtaining a theoretical match of the general structure

The results shown suggest a routine to follow in order to obtain a theoretical match with a series of experimental measurements.

1. Estimate the velocity of propagation of shear waves in the components of the structure. The estimate is obtained from a plot of the reciprocal wavelength against the frequency. The velocities are successively the reciprocals of the slopes of the plot. The largest values of the reciprocal wavelength correspond with the layers closest to the surface.

2. Estimate the thickness of the surface layer. The thickness is approximately the wavelength at which the plot of velocity against the wavelength passes through a maximum.

3. Estimate the thickness of the underlying layer. Increase the thickness of the underlying layer, if the drop in the phase velocities occurs at too small a wavelength.

4. Execute the program **RAY.FOR**. Plot the following.

   (a) Reciprocal wavelength against the frequency. The frequency output of **RAY.FOR** must be multiplied by the selected value of β_1, in order to yield actual frequencies in Herz.

   (b) The velocity of propagation of waves of the Rayleigh type against the wavelength. The output of **RAY.FOR** gives the velocity as a fraction $\frac{c}{\beta_1}$.

   (c) Return to (1).

5. If the maximum occurs at too large a wavelength, decrease the thickness of the surface layer.

6. Continue until a satisfactory match is reached.

4.4 Conclusion

The object of the present work is to determine the properties of a pavement structure. The velocities of waves of the Rayleigh type are measured at the surface of a layered structure, such as a road surface. The results of the measurements are used to estimate the thicknesses and the elastic moduli of the media of which the structure is composed.

1. Measurements of wave velocities on pavements are able to lead to information concerning the properties of the pavement.

2. A simplified interpretive method exists. It is applicable to a structure consisting of a single layer overlying a semi-infinite medium. It involves the use of the programs **SIN13.FOR** and **SON.FOR**

3. The general structure can be analysed with the aid of the program **RAY.FOR.** The program **RAY.FOR** is able to analyse the results obtained from the surface of a multi-layered structure. The underlying medium is treated as being semi-infinite.

4. The use of Maple worksheets provides an indication of the valid solutions of the determinant given by equation (4-202) in Ewing, Jardetzky and Press[2].

Bibliography

[1] Bonitzer, J. and Ph. Leger(1967) "LCPC studies in pavement design," *Proceedings*, Second International Conference on the Structural Design of Asphalt Pavements, The University of Michigan, pp.781-788

[2] Ewing,W.M., W.S. Jardetzky and F.Press, "Elastic waves in layered media". New York, McGraw-Hill. 1957.

[3] Lee, A.W. extract "Rayleigh waves in a single-layered system", in W.H. Cogill, *"Single-layered inverse: First-order approximation"*, ISVR Technical Memorandum No. 833, January 1999

[4] Martinzec, G., Dynamics of pavement structures, E. and F.N. Spon Chapman and Hall 1994 ISBN 0 419 18100 8, Figure 55, page 65.

[5] Thrower, E.N. (1965) "The computation of the dispersion of elastic waves in layered media," *Journal of Sound and Vibration*, Vol. 2, pp.210-226

Chapter 5

Appendix

5.1 Listing of *SIN*13.*FOR*

The following is a listing of the program *SIN*13.*FOR*. It leads to a table of reciprocal wavelength and the corresponding frequency, for a system consisting of a hard layer overlying a soft semi-infinite medium.

The quantities r_1, r_2, s_1, s_2 are imaginary.

The reciprocal wavelength is calculated as $\frac{\pi h}{\lambda}$. The output must be divided by π to obtain reciprocal wavelength, with the thickness h of the layer as unity.

The frequency is output as $CB1/V$. That is $\frac{c}{\lambda} * \frac{\pi * h}{\beta_1}$, or $\frac{\pi * h}{\beta_1}$ times the frequency $\frac{c}{\lambda}$. *SIN*13.*FOR*

5.1.1 Program listing

```
C      003G4        001         025 TTI BASEMENT OET BUILDING WHC33F
C           THERE ARE SEVEN TEN COLUMN DATA FIELDS ON THE VELOCITY CARD.
C           ANY FIELD MAY CONTAIN  999.,  WHICH CAUSES
C           THE PROGRAM TO READ A FURTHER VELOCITY CARD.
C         Version for hard over soft see Lee,A.R. "The effect of structure
C  upon microseismic disturbance" page 91 eqn (20)
C*********************************************************************
C*********************************************************************
C                                                                  10
C  RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.   JANUARY   1968   WHC33F
C
C*********************************************************************
C*********************************************************************
c  Trial data follows, works on starting value of 0.4
C        0.4 10.0 0.4 0.01
C        0.58 0.53 0.529 0.528 0.527 0.526 0.525 999.
```

```
C        0.524 0.523 0.52 0.51 0.50 0.49 0.48 999.
C        0.47 0.46 0.45 0.42 0.40 0.39 0.38  999.
C        0.36 0.35 0.34 0.33 0.32 0.31 0.30 999.
C        0.290 0.285 0.280 0.275 0.270 0.265 0.250 999.
C        0.283 0.282 0.281 0.280 0.279 0.278 0.277 999.
C        0.276 0.275 0.274 0.273 0.272 0.271 0.270 999.
C        0.269 0.268 0.266 0.264 0.262 0.260 0.258 0.0
C        0.0 0.0 0.0 0.0
C
      PROGRAM WHC33F
      DIMENSION CB(8)
      REAL*8 R2K,S1K,S2K,
     1R1H,R2H,S1H,S2H,XKR1,XKS1,R2R1,S2S1,XI1,XI2,ETA1,ETA2,XITEM1,
     2XITEM2
  101 FORMAT (4F10.0)
  102 FORMAT(13X,'RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.'
     15X'EWING EQN (4 - 202).'//7X'POISSON''S RATIO IN LAYER = ',
     2F5.2,',      IN HALF SPACE = ', F5.2,',     MU2/MU1 = ',F7.2,////)
  103 FORMAT(8F10.0)
  104 FORMAT(6X,'VELOCITY      WAVELENGTH         FREQUENCY
     1 WAVE NUMBER     COUNTER REG.'///)
  105 FORMAT(5X,G12.4,5X,G12.4,7X,G12.4,4X,G12.4,7X,I3)
  106 FORMAT(5X,G12.4,5X,G12.4,7X,G12.4,4X, G12.4,7X,I3,3X,
     1  'DELODD EQUALS ', E18.6)
  107 FORMAT(1H1)
        OPEN(1,FILE='SIN13.DAT',STATUS='OLD',FORM='FORMATTED')
        OPEN(3,FILE='SIN13.PRN',STATUS='UNKNOWN',FORM='FORMATTED')
  201 READ(1, * ) B1A1,A1A2,B2A2,XMU2M1
      IF(B1A1.EQ. 0.0) GO TO 999
      SIGMA1 = (1.-2.*B1A1*B1A1)/(2.-2.*B1A1*B1A1)
      SIGMA2 = (1.-2.*B2A2*B2A2)/(2.-2.*B2A2*B2A2)
      WRITE(3,102) SIGMA1,SIGMA2,XMU2M1
      WRITE(3,104)
C "2/V" is 2*pi/lamda = k; so V = lamda / pi
      WRITE(*,*)  'ENTER A STARTING VALUE OF V'
      READ(*,*) V
  202 READ(1, * ) (CB(J),J=1,8)
C      REF=10.V
      I=0
C*********************************************************************
C**********STATEMENT 203 IS RE-ENTRY POINT FOR NEW VALUE OF CB1*********
C*********************************************************************
  203 CONTINUE
C       V = VST  ! RESET V TO THE STARTING VALUE
      I=I+1
      IF(I.GT.8) GO TO 201
      KOUNT = 0
      CB1=CB(I)
      IF(CB1.EQ.0.) GO TO 201
      IF(CB1.EQ.999.) GO TO 202
```

```
C**********************************************************************
      CA1=CB1*B1A1
      CA2=CB1*B1A1*A1A2
      B1B2=B1A1*A1A2/B2A2
      CB2=CB1*B1B2
C       WRITE(*,*) CA1,CB1,CA2,CB2
C**********************************************************************
      R1K=SQRT(1.0-CA1*CA1)
      R2K=SQRT(-1.0+CA2*CA2)
      S1K=SQRT(1.0-CB1*CB1)
      S2K=SQRT(-1.0+CB2*CB2)
      W = 2.*(XMU2M1-1.)
      X = XMU2M1*CB1*CB1*B1B2*B1B2-W
      Y = CB1*CB1+W
      Z = X-CB1*CB1
C**********************************************************************
      XKR1=1./R1K
      XKS1=1./S1K
      R2R1=R2K/R1K
      S2S1=S2K/S1K
C**********************************************************************
C**************START OF ITERATION LOOP***************
C**********************************************************************
  250 CONTINUE
      R1H = (2./V)*R1K
      KOUNT = KOUNT + 1
      R2H=(2./V)*R2K
      S1H=(2./V)*S1K
      S2H=(2./V)*S2K
C**********************************************************************
      XI1=(2.-CB1*CB1)*(X*DCOSH(R1H)+R2R1*Y*DSINH(R1H))
     1      -2.*S1K*(R2K*W*DSINH(S1H)+XKS1*Z*DCOSH(S1H))
      XI2=(2.-CB1*CB1)*(S2K*W*DCOSH(R1H)+XKR1*Z*DSINH(R1H))
     1      -2.*S1K*(X*DSINH(S1H)+S2S1*Y*DCOSH(S1H))
C**********************************************************************
      ETA1=(2.-CB1*CB1)*(R2K*W*DCOSH(S1H)+XKS1*Z*DSINH(S1H))
     1     -2.*R1K*(X*DSINH(R1H)+R2R1*Y*DCOSH(R1H))
      ETA2=(2.-CB1*CB1)*(X*DCOSH(S1H)+S2S1*Y*DSINH(S1H))
     1     -2.*R1K*(S2K*W*DSINH(R1H)+XKR1*Z*DCOSH(R1H))
C**********************************************************************
      XITEM1=XI1*ETA2
      XITEM2=XI2*ETA1
C**********************************************************************
      DEL = REAL(XITEM1 - XITEM2)
      IF (KOUNT - 1) 301, 301, 302
  301 VEVEN = V
      V = 1.27 * V
      VODD = V
      DELEVEN = DEL
 3011 GO TO 250
```

```
sin13.for                                                    5903 Bytes
2.4.101                                                            10:0
C***********************************************************************
  302 IF (KOUNT - 2*(KOUNT/2)) 305, 305, 304
  304 DELEVEN = DEL
      VEVEN = V
 3041 GO TO 306
  305 DELODD = DEL
      VODD = V
C***********************************************************************
  306 IF(ABS(DELODD-DELEVEN).LE. 0.01)    GO TO 599 ! Failed output
      DIFF =  ABS((DELODD-DELEVEN)/XITEM1)
      IF(DIFF .LE. .000001*DELODD)  GO TO 600   ! Successful output
      V = VEVEN -0.5*DELEVEN*(VODD - VEVEN)/(DELODD- DELEVEN) !Convergence D
 3061 GO TO 250
C
C***********************************************************************
C*************END OF ITERATION LOOP**************
C***********************************************************************
  599 FREQ=CB1/V
      VI = 1./V
      WRITE(3,106) CB1, V, FREQ,VI, KOUNT,DELODD
      GO TO 203
  600 FREQ=CB1/V
      VI=1./V
      WRITE(3,105) CB1,V,FREQ,VI,KOUNT
      GO TO 203
  999 WRITE(3,107)
      STOP
      END
```

The result shown in Figure (1.1) was obtained using the following data for *SIN*13.*FOR*
.

```
0.4 10.0 0.4 0.01
0.66 0.65 0.64 0.63 0.62 0.61  0.59 999.
0.58 0.575 0.57 0.565 0.56 0.555 0.55 999.
0.54 0.53 0.529 0.528 0.527 0.526 0.525 999.
0.524 0.523 0.52 0.51 0.50 0.49 0.48 999.
0.47 0.46 0.45 0.42 0.40 0.39 0.38  999.
0.36 0.35 0.34 0.33 0.32 0.31 0.30 999.
0.290 0.285 0.280 0.275 0.270 0.265 0.250 999.
0.283 0.282 0.281 0.280 0.279 0.278 0.277 999.
0.276 0.275 0.274 0.273 0.272 0.271 0.270 999.
0.269 0.268 0.266 0.264 0.262 0.260 0.258 0.0
0.0 0.0 0.0 0.0
```

5.2 Listing of *SON.FOR*

The following is a listing of the program *SON.FOR*. It leads to a table of reciprocal wavelength and the corresponding frequency, for a system consisting of a soft layer overlying a hard semi-infinite medium. *SON.FOR*

5.2.1 Program listing

```
C     003G4        001         025 TTI BASEMENT OET BUILDING WHC33F
C          THERE ARE SEVEN TEN COLUMN DATA FIELDS ON THE VELOCITY CARD.
C          ANY FIELD MAY CONTAIN  999.,  WHICH CAUSES
C          THE PROGRAM TO READ A FURTHER VELOCITY CARD.
C     SON.FOR IS FOR SOFT OVER HARD
C     TEST DATA FOLLOWS: (STARTING POINT V=3.0)
C        0.4 0.101 0.4 100.
C        2.51 2.52 2.53 2.54 2.55 2.56 2.57 999.
C        2.58 2.6 2.75 2.8 2.9 3.1 3.3 999.
C        3.45 3.8 3.9 3.95 4.0 4.05 4.08 999.
C        4.2 4.4 5.1 5.5 6.7 7.5 7.9 999.
C        8.0 8.1 8.2 8.3 8.4 8.5 8.6  999.
C        8.7 8.8 8.9 9.0 9.1 9.15 9.18 999.
C        9.22 9.26  9.28 9.29 9.295 9.297 9.299 0.0
C        0.0 0.0 0.0 0.0
C
C**********************************************************************
C**********************************************************************
C
C  RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.   JANUARY   1968   WHC33F
C
C**********************************************************************
C**********************************************************************
      PROGRAM WHC33F
      DIMENSION CB(8)
c     COMPLEX*16 R1K,R2K,S1K,S2K,
c    1R1H,R2H,S1H,S2H,XKR1,XKS1,R2R1,S2S1,XI1,XI2,ETA1,ETA2,XITEM1,
c    2XITEM2
  101 FORMAT (4F10.0)
  102 FORMAT(1H1,33X,'RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.'
     15X'EWING EQN (4 - 202).'//27X'POISSON''S RATIO IN LAYER = ',
     2F5.2,',       IN HALF SPACE = ', F5.2,',    MU2/MU1 = ',F7.2,////)
  103 FORMAT(8F10.0)
  104 FORMAT(1H0,26X,'VELOCITY      WAVELENGTH      FREQUENCY      WAVE
     1 NUMBER      COUNTER REG.'///)
  105 FORMAT(27X,G10.5,7X,G10.5, 9X,G10.5,6X,G10.5, 9X,I3)
  106 FORMAT(27X,G10.5,7X,G10.5, 9X,G10.5,6X, G10.5,  9X, I3, 3X,
     1  'Y EQUALS ', E18.6)
  107 FORMAT(1H1)
        OPEN(1,FILE='SON.DAT',STATUS='OLD',FORM='FORMATTED')
        OPEN(3,FILE='SON.PRN',STATUS='UNKNOWN',FORM='FORMATTED')
  201 READ(1, * ) B1A1,A1A2,B2A2,XMU2M1
      IF(B1A1.EQ. 0.0) GO TO 999
      SIGMA1 = (1.-2.*B1A1*B1A1)/(2.-2.*B1A1*B1A1)
      SIGMA2 = (1.-2.*B2A2*B2A2)/(2.-2.*B2A2*B2A2)
```

```
      WRITE(3,102) SIGMA1,SIGMA2,XMU2M1
      WRITE(3,104)
      WRITE(*,*)  'ENTER A STARTING VALUE OF V'
C "2/V" is 2*pi/lamda = k; so V = lamda / pi
      READ(*,*) V   ! V IS LAMDA / PI
  202 READ(1, * ) (CB(J),J=1,8)
C     REF=10.V
      I=0
C**********************************************************************
C*********STATEMENT 203 IS RE-ENTRY POINT FOR NEW VALUE OF CB1*********
C**********************************************************************
  203 I=I+1
      IF(I.GT.8) GO TO 201
      KOUNT = 0
      CB1=CB(I)
      IF(CB1.EQ.0.) GO TO 201
      IF(CB1.EQ.999.) GO TO 202
C**********************************************************************
      CA1=CB1*B1A1
      CA2=CB1*B1A1*A1A2
      B1B2=B1A1*A1A2/B2A2
      CB2=CB1*B1B2
      write(*,*)ca1,ca2,cb1,cb2
C**********************************************************************
      R1K=SQRT(-1.0+CA1*CA1)
      R2K=SQRT(1.-CA2*CA2)
      S1K=SQRT(-1.+CB1*CB1)
      S2K=SQRT(1.-CB2*CB2)
      W = 2.*(XMU2M1-1.)
      X = XMU2M1*CB1*CB1*B1B2*B1B2-W
      Y = CB1*CB1+W
      Z = X-CB1*CB1
C**********************************************************************
      XKR1=1./R1K
      XKS1=1./S1K
      R2R1=R2K/R1K
      S2S1=S2K/S1K
C**********************************************************************
C*************START OF ITERATION LOOP**************
C**********************************************************************
  250 R1H = (2./V)*R1K
      KOUNT = KOUNT + 1
      R2H=(2./V)*R2K
      S1H=(2./V)*S1K
      S2H=(2./V)*S2K
C**********************************************************************
      XI1=(2.-CB1*CB1)*(X*COS(R1H)+R2R1*Y*SIN(R1H))
     1       +2.*S1K*(R2K*W*SIN(S1H)-XKS1*Z*COS(S1H))
      XI2=(2.-CB1*CB1)*(S2K*W*COS(R1H)+XKR1*Z*SIN(R1H))
     1       +2.*S1K*(X*SIN(S1H)-S2S1*Y*COS(S1H))
```

```
C***********************************************************************
      ETA1=(2.-CB1*CB1)*(R2K*W*COS(S1H)+XKS1*Z*SIN(S1H))
     1      +2.*R1K*(X*SIN(R1H)-R2R1*Y*COS(R1H))
      ETA2=(2.-CB1*CB1)*(X*COS(S1H)+S2S1*Y*SIN(S1H))
     1      +2.*R1K*(S2K*W*SIN(R1H)-XKR1*Z*COS(R1H))
C***********************************************************************
      XITEM1=XI1*ETA2
      XITEM2=XI2*ETA1
C***********************************************************************
      DEL = REAL(XITEM1 - XITEM2)
      IF (KOUNT - 1) 301, 301, 302
  301 XEVEN = V
      V = 1.01 * V
      XODD = V
      YEVEN = DEL
 3011 GO TO 250
C***********************************************************************
  302 IF (KOUNT - 2*(KOUNT/2)) 305, 305, 304
  304 YEVEN = DEL
      XEVEN = V
 3041 GO TO 306
  305 YODD = DEL
      XODD = V
C***********************************************************************
  306 IF(ABS(YODD-YEVEN).LE.. 000001)   GO TO 599
      IF(ABS(XODD - XEVEN) .LE. .001)  GO TO 600
      V = XEVEN - YEVEN*(XODD - XEVEN)/(YODD- YEVEN)
 3061 GO TO 250
C
C***********************************************************************
C*************END OF ITERATION LOOP*************
C***********************************************************************
  599 FREQ=CB1/V
      VI = 1./V
      WRITE(3,106) CB1, V, FREQ,VI, KOUNT,YODD
      GO TO 203
  600 FREQ=CB1/V
      VI=1./V
      WRITE(3,105) CB1,V,FREQ,VI,KOUNT
      GO TO 203
  999 WRITE(3,107)
      STOP
      END
```

5.3 Listing of *APP_H.FOR*

The following is a listing of the program *APP_H.FOR*. The program yields an approximate value of the thickness of the surface layer. The data are pairs consisting of the

APP_H.FOR

reciprocal wavelength and the corresponding frequency.

5.3.1 Program listing

```
c The following file is an approximation to Lee's determinant for the
c "hard over soft" case, using a first order expansion of the transcendental
c functions. It is based on equation (21) in the condensed version of
c A.W.Lee's paper, A_W_LEE.DOC. It solves for <h> in terms of the
c measured quantities, c the reciprocal wavelength and the corresponding
c frequency. The coefficients "aaa", "bbb", "ccc" are derived in the
c Mathematica notebook LEE_EQH.NB, where they are expressed in Fortran
c form.
c
        implicit real(i-n)
 201    format(3g16.3)
 10     write(*,*) "  Enter betal and beta2 "
        read(*,*) betal, beta2
        alphal=1.732*betal
        alpha2=1.732*beta2
 20     write(*,*) "Enter reciprocal wavelength, frequency"
        read(*,*) rlngth,freq
        if(rlngth.eq.0) goto 10    !backtrack if required
        write(*,201) rlngth,freq
        mu1=1.0
        mu2=mu1*(beta2/betal)**2
        c=freq/rlngth
        omega= 6.28*freq
        k=6.28*rlngth
        ka1=omega/alphal
        kb1=omega/betal
        ka2=omega/alpha2
        kb2=omega/beta2
c        write(*,*) "to herel"
        write(*,*) "omega=",omega,"   c=",c
        Rarg= -(omega**2/alphal**2) + omega**2/c**2
        if(Rarg.lt.0) write(*,*)"Rarg.lt.0"
        R=Sqrt(-(omega**2/alphal**2) + omega**2/c**2)
        write(*,*) "R=", R
c       write(*,*) "betal = ",betal,"beta2 = ",beta2
        Sarg= -(omega**2/betal**2) + omega**2/c**2
        S=Sqrt(-(omega**2/betal**2) + omega**2/c**2)
        write(*,*) "S=", S
        write(*,*) "  alphal=",alphal,"  alpha2=",alpha2
        r2arg= omega**2/alpha2**2 - omega**2/c**2
        if(r2arg) 21,21,22
 21     write(*,*) "r2arg .lt. 0; c must exceed alpha2"
        goto 20
 22     r2=Sqrt(omega**2/alpha2**2 - omega**2/c**2)
c       write(*,*) " r2=",r2
```

```
      s2arg= omega**2/beta2**2 - omega**2/c**2
      s2=Sqrt(omega**2/beta2**2 - omega**2/c**2)
c     write(*,*) " s2=",s2
c  now define the Love parameters
c      write(*,*) "to here"

      X=c**2*mu2/(beta2**2*mu1) - 2*(-1 + mu2/mu1)
      Y=c**2/beta1**2 + 2*(-1 + mu2/mu1)
      Z=-(c**2/beta1**2) + c**2*mu2/(beta2**2*mu1) - 2*(-1 + mu2/mu1)
      W=2*(-1 + mu2/mu1)
c now define the value of the determinant (=zero) aa*h^2+bb*h+cc=0
      aa=-4*k**2 + 4*r2*s2 + 16*r2*s2*W/3 + 4*r2*s2*W**2/3 +
     -   16*k**2*X/3 - 4*k**2*X**2/3 - 2*r2*s2*Y - 2*r2*s2*W*Y -
     -   4*k**2*Z + 8*k**2*X*Z/3 - k**2*Z**2
      bb=
     - 2*r2*W - 2*s2*W/3 + 2*r2*X + 10*s2*X/3 +
     -   4*s2*W*X/3 - 2*s2*Y - s2*X*Y - 2*r2*Z - 2*s2*Z - r2*W*Z -
     -   s2*W*Z
      cc=4 - 4*r2*s2/k**2 -
     -   4*r2*s2*W/k**2 - r2*s2*W**2/k**2 - X**2

      write(*,*) "X*Y+W*Z =",X*Y+W*Z ! a factor of <bb>
c     write(*,*) "cc=",cc
      write(*,*) "            aa            bb              cc    "
      write(*,201) aa,bb,cc
      h=(-bb+sqrt(bb**2-4*aa*cc))/(2*aa)
      h1= (-bb-sqrt(bb**2-4*aa*cc))/(2*aa)
      write(*,*) "   Thickness(1)     Thickness(2)        S*h  "
      write(*,201) h,h1,S*h
      goto 20
      end
```

5.4 Listing of *EW202.FOR*

The following is a listing of the program *EW202.FOR*. It leads to a table of reciprocal wavelength and the corresponding frequency, for a system consisting of a soft layer overlying a hard semi-infinite medium. The quantities r_1, r_2, s_1, s_2 are all real.

The reciprocal wavelength is calculated as $\frac{\pi h}{\lambda}$. The output must be divided by π to obtain reciprocal wavelength, with the thickness h of the layer as unity.

The frequency is output as *CB1/V*. That is $\frac{c}{\lambda} * \frac{\pi * h}{\beta_1}$, or $\frac{\pi * h}{\beta_1}$ times the frequency $\frac{c}{\lambda}$. *EW202.FOR*

5.4.1 Program listing

```
C     003G4     001     025 TTI BASEMENT OET BUILDING WHC33F
```

```
C         THERE ARE SEVEN TEN COLUMN DATA FIELDS ON THE VELOCITY CARD.
C         ANY FIELD MAY CONTAIN  999.,  WHICH CAUSES
C         THE PROGRAM TO READ A FURTHER VELOCITY CARD.
C  This is the real version of Ewing Jardetzky and Press equation (4-202).
C It is for a soft layer overlying a hard semi-infinite medium.
C  See CTAC93 for output obtained from it.
C
C*********************************************************************
C*********************************************************************
C
C  RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.   JANUARY   1968   WHC33F
C
C*********************************************************************
C*********************************************************************
      PROGRAM WHC33F
      DIMENSION CA(8)
C     THE INPUT VELOCITIES ARE CA(J): CB2 > C > CA1
  101 FORMAT (4F10.0)
  102 FORMAT(6X,'RAYLEIGH WAVES IN A SINGLE-LAYERED SYSTEM.'
     +5X'EWING EQN (4 - 202).'/25X' REAL VERSION. CB2 > C > CA1  '
     +/1X'POISSON''S RATIO IN LAYER = ',
     +F5.2,', IN HALF SPACE = ', F5.2,', MU2/MU1 = ',F7.2,/)
  103 FORMAT(8F10.0)
  104 FORMAT('    VELOCITY   WAVELENGTH  WAVE NUMBER ',
     1'  FREQUENCY  COUNTER REG.'/)
  105 FORMAT(5X,G8.3,3X,G8.3,4X,G8.3,7X,G8.3,3X,I3)
  106 FORMAT(5X,G8.3,3X,G8.3,4X,G8.3,7X, G8.3, 3X, I3, 3X,
     1  'Y EQUALS ', E8.2)
  107 FORMAT(1H1)
        OPEN(1,FILE='EW202.DAT',STATUS='OLD',FORM='FORMATTED')
        OPEN(3,FILE='EW202.PRN',FORM='FORMATTED')
  201 READ(1, *,END=999) B1A1,A1A2,B2A2,XMU2M1
      IF(B1A1.EQ. 0.0) GO TO 999
      SIGMA1 = (1.-2.*B1A1*B1A1)/(2.-2.*B1A1*B1A1)
      SIGMA2 = (1.-2.*B2A2*B2A2)/(2.-2.*B2A2*B2A2)
      WRITE(*,102) SIGMA1,SIGMA2,XMU2M1
      WRITE(*,104)
C       V =   LAMBDA / PI
      V=1.0
      write (*,*) ' enter V : '
      read (*,*) V
  202 READ(1, * ) (CA(J),J=1,8)
C     REF=10.V
      I=0
C*********************************************************************
C**********STATEMENT 203 IS RE-ENTRY POINT FOR NEW VALUE OF CB1*********
C*********************************************************************
  203 CONTINUE
      I=I+1
      IF(I.GT.8) GO TO 201
```

```
      KOUNT = 0
      CA1=CA(I)
      IF(CA1.EQ.0.) GO TO 201
      IF(CA1.EQ.999.) GO TO 202
C*********************************************************************
      CB1=CA1/B1A1
      CA2=CB1*B1A1*A1A2
      B1B2=B1A1*A1A2/B2A2
      CB2=CB1*B1B2
C     write(*,*) 'cb1 = ',cb1,' ca1 = ',ca1,' cb2 = ',cb2, ' ca2 = ',ca2
C*********************************************************************
      R1K=SQRT(-1.0+CA1*CA1) !C > ALPHA1 AND C < BETA2
      R2K=SQRT(+1.-CA2*CA2)
      S1K=SQRT(-1.0+CB1*CB1)
      S2K=SQRT(+1.-CB2*CB2)
      W = 2.*(XMU2M1-1.)
      X = XMU2M1*CB1*CB1*B1B2*B1B2-W
      Y = CB1*CB1+W
      Z = X-CB1*CB1
C*********************************************************************
      XKR1=1./R1K
      XKS1=1./S1K
      R2R1=R2K/R1K
      S2S1=S2K/S1K
C*********************************************************************
C**************START OF ITERATION LOOP***************
C*********************************************************************
  250 CONTINUE
      R1H = (2./V)*R1K
      KOUNT = KOUNT + 1
      R2H=(2./V)*R2K
      S1H=(2./V)*S1K
      S2H=(2./V)*S2K
C*********************************************************************
      XI1=(2.-CB1*CB1)*(X*COS(R1H)+R2R1*Y*SIN(R1H))
     1        +2.*S1K*(R2K*W*SIN(S1H)-XKS1*Z*COS(S1H))
      XI2=(2.-CB1*CB1)*(S2K*W*COS(R1H)+XKR1*Z*SIN(R1H))
     1        +2.*S1K*(X*SIN(S1H)-S2S1*Y*COS(S1H))
C*********************************************************************
      ETA1=(2.-CB1*CB1)*(R2K*W*COS(S1H)+XKS1*Z*SIN(S1H))
     1      +2.*R1K*(X*SIN(R1H)-R2R1*Y*COS(R1H))
      ETA2=(2.-CB1*CB1)*(X*COS(S1H)+S2S1*Y*SIN(S1H))
     1      +2.*R1K*(S2K*W*SIN(R1H)-XKR1*Z*COS(R1H))
C*********************************************************************
      XITEM1=XI1*ETA2
      XITEM2=XI2*ETA1
C*********************************************************************
      DEL = XITEM1 - XITEM2
c     write(*,*) 'del = ', DEL , '    V = ',V
      IF (KOUNT - 1) 301, 301, 302
```

```
 301 XEVEN = V
     V = 1.01 * V          !SET INCREMENT OF THE SEARCH
     XODD = V
     YEVEN = DEL
3011 GO TO 250
C************************************************************************
 302 IF (KOUNT - 2*(KOUNT/2)) 305, 305, 304
 304 YEVEN = DEL
     XEVEN = V
3041 GO TO 306
 305 YODD = DEL
     XODD = V
C************************************************************************
 306 IF(ABS(YODD-YEVEN).LE.. 00000000001)   GO TO 599
     IF(ABS(XODD - XEVEN) .LE. .00000001)  GO TO 600
     V = XEVEN - YEVEN*(XODD - XEVEN)/(YODD- YEVEN)
3061 GO TO 250
C
C************************************************************************
C**************END OF ITERATION LOOP**************
C************************************************************************
 599 FREQ=CB1/V
     VI = 1./V
     WRITE(*,106) CA1, V,VI, FREQ, KOUNT,YODD
     WRITE(3,106) CA1, V,VI, FREQ, KOUNT,YODD
     GO TO 203
 600 FREQ=CB1/V
     VI=1./V
     WRITE(*,105) CA1,V,VI,FREQ,KOUNT
     WRITE(3,105) CA1,V,VI,FREQ,KOUNT
     GO TO 203
 999 WRITE(*,107)
     STOP
     END
```

A sample data file follows.

```
0.4      0.1        0.4      100
   1.1      1.3        1.5      1.7      1.9      2.1      2.3  999
   2.6      2.9        3.1      3.3      3.5      3.53     3.57 999
   3.6      3.63       3.66     3.69     3.72     3.74     3.77 0
```

5.5 Listing of *REVEW202.FOR*

The following is a listing of the program *REVEW202.FOR*. It attempts to reverse the output of the program *EW202.FOR*. It aims to yield the thickness of the surface layer for a medium consisting of a soft surface layer overlying a hard semi-infinite medium.

The series on which the program is based is divergent, so the results are accurate to an order of magnitude only. *REVEW202.FOR*

5.5.1 Program listing

```
c Program REVEW202.FOR
c This program attempts to inverse Ewing Jardetzky and Press equation (202)
c It does so by using the solution for the determinant of the system, expanded in terms of
c the thickness h of the surface layer.
c Expansion to the power of five leads to a polynomial which is reversible, using
c REVERSE.NB  However the resulting reversion is not convergent.
c The first part of this program is the output of LEE2.NB, called LEE2.DAT
c The second part of this program is the output of REVERSE.NB, called REVERSE.DAT

        real mu1,mu2,kb1,kb2,k
        dimension a(8)
  201   format(8g14.3)
  202   format( '   z = ',g14.3, '   h= ',g14.3)

 100    continue
        write(*,*) "Enter  frequency, reciprocal w'lngth . . "
        read(*,*) freq, xk
        k =  6.28 * xk
        w =  6.28 * freq
        c = freq / xk


      b1 = 0.1
      b2 = 1.0
      a1 = 2.5*b1
      a2 = 2.5*b2
      mu1 = 1.0
      mu2 = 100.0

      k = w /c
      kb1 = w / b1
      kb2 = w / b2

      r1 =  ( - 1/c**2 + 1/a1**2) ** (1/2)
      s1 =  ( - 1/c**2 + 1/b1**2  ) **(1/2)
      r2 =  ( - 1/c**2 + 1/a2**2) ** (1/2)
      s2 =  ( - 1/c**2 + 1/b2**2  ) **(1/2)

       ww = 2 * ( mu2/mu1 -1 )
       xx = mu2/mu1 * kb2**2/k**2 - ww
       yy = kb1**2/k**2 + ww
       zz = mu2/mu1 * kb2**2/k**2 - kb1**2 /k**2 - ww
       z0 = -(-144*k**4*r2*s2*ww**2 + 144*k**2*kb1**2*r2*s2*ww**2 -
     # 36*kb1**4*r2*s2*ww**2 + 144*k**6*xx**2 - 144*k**4*kb1**2*xx**2 +
```

```
     # 36*k**2*kb1**4*xx**2 + 288*k**4*r2*s2*ww*yy -
     # 144*k**2*kb1**2*r2*s2*ww*yy - 144*k**4*r2*s2*yy**2 -
     # 288*k**6*xx*zz + 144*k**4*kb1**2*xx*zz + 144*k**6*zz**2)
      z1=(144*k**6*r2*w*xx*yy - 144*k**4*kb1**2*r2*w*xx*yy +
     # 36*k**2*kb1**4*r2*w*xx*yy + 144*k**4*r2*s1**2*w*xx*yy +
     # 144*k**6*s2*w*xx*yy - 144*k**4*kb1**2*s2*w*xx*yy +
     # 36*k**2*kb1**4*s2*w*xx*yy + 144*k**4*r1**2*s2*w*xx*yy -
     # 144*k**6*r2*w*ww*zz + 144*k**4*kb1**2*r2*w*ww*zz -
     # 36*k**2*kb1**4*r2*w*ww*zz - 144*k**4*r2*s1**2*w*ww*zz -
     # 144*k**6*s2*w*ww*zz + 144*k**4*kb1**2*s2*w*ww*zz -
     # 36*k**2*kb1**4*s2*w*ww*zz - 144*k**4*r1**2*s2*w*ww*zz)

       z = z0 / z1
c       a(0) =0
       a(1) =1
       a(2) =-(w*(-(kb1**4*r2*(r1**2 + s1**2)*s2*ww**2) +
     #  k**2*(-8*r1**2*r2*s1**2*s2*ww**2 +
     #     4*kb1**2*r2*(r1**2 + s1**2)*s2*ww**2 +
     #     kb1**4*(r1**2*xx**2 + s1**2*xx**2 - 2*r2*s2*yy**2)) +
     #  8*k**8*zz**2 - 2*k**4*(2*kb1**2*s1**2*xx**2 +
     #     2*r2*s2*(-2*kb1**2*yy**2 + s1**2*(ww**2 + yy**2)) +
     #     2*r1**2*((kb1**2 - 2*s1**2)*xx**2+r2*s2*(ww**2 + yy**2)) -
     #     kb1**4*zz**2) + 4*k**6*(r1**2*(xx**2 + zz**2)
     #  + s1**2*(xx**2 + zz**2) - 2*(r2*s2*yy**2 + kb1**2*zz**2))))/
     #  (2.*k**2*(4*k**4*(r2 + s2) + kb1**4*(r2 + s2) -
     #  4*k**2*(-(r2*s1**2)-r1**2*s2+kb1**2*(r2 + s2)))*(xx*yy-ww*zz))

       a(3) =-((4*k**4*(s1**2*(3*r2 + s2) + r1**2*(r2 + 3*s2)) +
     #  kb1**4*(s1**2*(3*r2 + s2) + r1**2*(r2 + 3*s2)) -
     #  4*k**2*(-(r2*s1**4) - r1**4*s2 - 3*r1**2*s1**2*(r2 + s2) +
     #     kb1**2*(s1**2*(3*r2 + s2) + r1**2*(r2 + 3*s2))))*w**2)/
     #   (6.*(4*k**4*(r2 + s2) + kb1**4*(r2 + s2) -
     # 4*k**2*(-(r2*s1**2) - r1**2*s2 + kb1**2*(r2 + s2))))
c  a(4) = a(7) / a(8) to facilitate shortening
       a(7)=(w**3*(-3*kb1**4*r1**2*r2*s1**2*s2*ww**2+k**2*(-8*r1**2*
     #r2*s1**2*(r1**2 + s1**2)*s2*ww**2 +2*kb1**2*r2*s2*ww*(6*r1**2
     #*s1**2*ww+r1**4*yy+s1**4*yy)+kb1**4*(-2*r2*s1**2*s2*yy**2+r1**2*
     #(3*s1**2*xx**2-2*r2*s2*yy**2)))+8*k**8*(r1**2+s1**2)*zz**2-
     #2*k**4*(r1**4*(-4*s1**2*xx**2+2*r2*s2*ww*yy+kb1**2*xx*zz)+r1**2
     #*(6*kb1**2*s1**2*xx**2-4*s1**4*xx**2+2*r2*s2*(-2*kb1**2*yy**2
     #+ 3*s1**2*(ww**2 + yy**2))-kb1**4*zz**2)+ s1**2*(2*r2*s2*yy*
     #(s1**2*ww - 2*kb1**2*yy) +kb1**2*zz*(s1**2*xx - kb1**2*zz))) +
     #4*k**6*(r1**4*xx*zz + s1**2*(-2*r2*s2*yy**2 + zz*(s1**2*xx -
     #2*kb1**2*zz)) + r1**2*(3*s1**2*(xx**2 + zz**2) -
     # 2*(r2*s2*yy**2 + kb1**2*zz**2)))))

       a(8)= (12.*k**2*(4*k**4*(r2 + s2) + kb1**4*(r2 + s2) -
     # 4*k**2*(-(r2*s1**2)-r1**2*s2+kb1**2*(r2+s2)))*(xx*yy - ww*zz))
       a(4) = a(7) / a(8) ! to facilitate shortening
       a(5) =(r1**2*s1**2*w**4)/12.
```

```
      a(6) =(w**5*(4*r1**4*s1**4*(r2*s2*ww**2 - k**2*xx**2) -
     # 2*(2*k**2 - kb1**2)*r1**6*(-(r2*s2*ww*yy) + k**2*xx*zz) -
     # 2*(2*k**2 - kb1**2)*s1**6*(-(r2*s2*ww*yy) + k**2*xx*zz) -
     #(-2*k**2+kb1**2)**2*r1**2*s1**2*(-(r2*s2*yy**2)+k**2*zz**2)))/
     #(36.*(4*k**4*(r2 + s2) + kb1**4*(r2 + s2) -
     #4*k**2*(-(r2*s1**2)-r1**2*s2+kb1**2*(r2+s2)))*(xx*yy-ww*zz))

c        write(*,201) (a(i),i=1,8)

       bb1 =1
               bb2 =-(a(2)/a(1))
       bb3 =(2*a(2)**2 - a(1)*a(3))/ a(1)**2
       bb4 =-((5*a(2)**3 - 5*a(1)*a(2)*a(3) + a(1)**2*a(4))/a(1)**3)
       bb5 =(14*a(2)**4 - 21*a(1)*a(2)**2*a(3) + 3*a(1)**2*a(3)**2 +
     #   6*a(1)**2*a(2)*a(4) - a(1)**3*a(5))/a(1)**4
       bb6 =(-42*a(2)**5 + 84*a(1)*a(2)**3*a(3)
     #-28*a(1)**2*a(2)**2*a(4)+7*a(1)**2*a(2)*(-4*a(3)**2+a(1)*a(5))+
     #a(1)**3*(7*a(3)*a(4) - a(1)*a(6)))/a(1)**5
      w=p-(p**2*a(2))/a(1) + (p**3*(2*a(2)**2 - a(1)*a(3)))/a(1)**2 -
     #(p**4*(5*a(2)**3 - 5*a(1)*a(2)*a(3) + a(1)**2*a(4)))/a(1)**3 +
     #(p**5*(14*a(2)**4 - 21*a(1)*a(2)**2*a(3) + 3*a(1)**2*a(3)**2 +
     #6*a(1)**2*a(2)*a(4) - a(1)**3*a(5)))/a(1)**4 +
     #(p**6*(-42*a(2)**5+84*a(1)*a(2)**3*a(3)-28*a(1)**2*a(2)**2*a(4)+
     #7*a(1)**2*a(2)*(-4*a(3)**2 + a(1)*a(5)) +
     #a(1)**3*(7*a(3)*a(4) - a(1)*a(6))))/a(1)**5


      p=z ! independent variable of the reversion
       h = p*bb1 + p**2*bb2 + p**3*bb3 + p**4*bb4+p**5*bb5+p**6*bb6
c      h = p*bb(1) + p**2*bb(2) + p**3*bb(3) + p**4*bb(4)
C      p**5*bb(5) + p**6*bb(6) + p**7*bb(7) + p**8*bb(8) + p**9*bb(9)
      write(*,202) z,h

      write(*,*) '  p  =',p, '  bb1 = ', bb1
       write(*,*) '  p**2*bb2  =',p**2*bb2, '  bb2 = ', bb2
       write(*,*) '  p**3*bb3  =',p**3*bb3, '  bb3 = ', bb3
       write(*,*) '  p**4*bb4  =',p**4*bb4, '  bb4 = ', bb4
       write(*,*) '  p**5*bb5  =',p**5*bb5, '  bb5 = ', bb5
       write(*,*) '  p**6*bb6  =',p**6*bb6, '  bb6 = ', bb6
      goto 100
      end
```

5.6 Listing of *RAY.FOR*

The following is a listing of the program **RAY.FOR**. It calculates the surface velocity as a function of the frequency for a multi-layered elastic system. *RAY.FOR*

5.6.1 Program listing

```
C      ** N-LAYERED RAYLEIGH WAVE PROGRAM.   MOD LEVEL 3. MAR 1970
C      THIS PROGRAM COMPUTES THE VALUES OF THE SURFACE PHASE VELOCITY OF
C      WAVES OF THE RAYLEIGH TYPE AT SELECTED WAVELENGTHS. ALL UNITS OF
C      LENGTH ARE DIMENSIONLESS, AND ARE MULTIPLES OF THE THICKNESS OF THE
C      FIRST(TOP) SURFACE LAYER. TEST DATA FOLLOWS:
C
C  comparison with single-layer e1=1,e2=100,nu1=nu2=0.4                    0.0
C  002        1.0.40     100.0.4
C     2.0   2.0   2.0
C  1.0
C  1.4          0.2          0.1    0.01   0.19   0.26
C
C
       PROGRAM RAYL
C
C     X003G4       *10       025 COGILL     TTI   TEXAS TRANSPORTATION INSTI
C     X003G4       *30       025 COGILL     TTI   TEXAS TRANSPORTATION INSTI
       REAL*8 PFREQ(100),PVEL(100),PWNGTH(100),PXN(100)
       REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),HH(6),
     1  FREINC
       REAL*8 SQRT,XLMDA(6)
       COMPLEX*16      XK,A(6),R(6),S(6),T(6),E(6,4,4),
     1 PM(6,4,4),CINV(6,4,4),XJ(4,4),DELXJ(2,2),
     2 T6,T7,G(4,4)
       COMPLEX*16  CB1RES,CINRES,WLNTHM,DELXN
       COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,CMPLX,XN,
     1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
C      COMPLEX*16   CDSQRT,CDCOS,CDSIN,SRH,SSH
       COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
       COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
     1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
     2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
       CHARACTER*4 TITLE(18)
       CHARACTER*4 ASTER
 101   FORMAT(I3,3X,12G6.0)
 102   FORMAT(6X,12F6.0)
 103   FORMAT(10F6.0)
 104   FORMAT(8F6.0)
 105   FORMAT(18A4,F8.0)
C      CALL ERRSET (208,0,-1,1,1,207)
        open(1,file='ray.dat',status='old',form='formatted')
        open(3,file='ray.prn',status='unknown',form='formatted')
        rewind 1
        rewind 3
 1001 READ(1,105,end = 999)(TITLE(I),I=1,18), WSWTCH
C
C         **WSWTCH** IS AN OUTPUT SWITCH WHICH CONTROLS THE
C      PRINTING OF THE CONVERGENCE PARAMETER 'VALUE'.
```

```
C          IF WSWTCH = 0 (OR LEFT BLANK) 'VALUE' IS NOT PRINTED
C             WSWTCH = 1.0 'VALUE' IS PRINTED WITH THE FINAL OUTPUT
C             WSWTCH = 2.0 'VALUE' AND ITS CORRESPONDING TRIAL VELOCITY
C             'CB1' ARE PRINTED AFTER EACH CYCLE OF CALCULATION.
C
      NPAGE = 1
      ASTER = '****'
      WRITE (*,204) ( ASTER,I = 1,5), (TITLE(I),I=1,18),(ASTER,I=1,5),
     2 NPAGE
      WRITE (3,204) ( ASTER,I = 1,5), (TITLE(I),I=1,18),(ASTER,I=1,5),
     2 NPAGE
      READ(1,101) NS,(EM(I),V(I), I=1,6)
      IF(NS.EQ.0) GO TO 9992
      READ(1,102) (RHO(I), I = 1,6)
C201  FORMAT(' TRIAL VELOCITY OUTSIDE ALLOWABLE RANGE')
C202  FORMAT (1H1)
 204  FORMAT(1H1//1H0,5A4,1X18A4,1X5A4,1X7H    PAGE,(I3))
 205  FORMAT(1H0,40X,'THE PROBLEM PARAMETERS ARE'//
     2      (1H ,3X,'LAYER',I3,'  HAS MODULUS ',F10.0,
     3      '   POISSON''S RATIO ',F5.3,'  DENSITY  ',F5.1,
     4      '   AND THICKNESS ',F6.3,' UNITS'))
 206  FORMAT(1H , 3X,'LAYER',I3,'  HAS MODULUS ',F10.0,
     2      '   POISSON''S RATIO ',F5.3,'  DENSITY  ',F5.1,
     3      '   AND IS SEMI-INFINITE.  '/)
 207  FORMAT(2X,'WAVELENGTH     VELOCITY ',11X,'FREQUENCY',9X,
     2 'WAVENUMBER   COUNTER REG.'/)
 209  FORMAT(1X,F8.4,8X,2F8.4,4XF8.4,8X, 2F8.4,3XI4,E18.4)
C     NLINE = 17 + NS
      IF(NS-6) 10,10,1
 1    READ(1,102) (EM(I),V(I),I=7,NS)
C      READ(1,102) (RHO(I), I = 7,NS)
 10   N=NS-1
      DO 1011 J = 1,6
 1011 HH(J) = 0.0
      READ(1,103) (HH(I),I=1,N)
      WRITE(*,205) (I,EM(I),V(I),RHO(I),HH(I), I = 1,N)
      WRITE(*,206) NS,EM(NS),V(NS),RHO(NS)
      WRITE(*,207)
      WRITE(3,205) (I,EM(I),V(I),RHO(I),HH(I), I = 1,N)
      WRITE(3,206) NS,EM(NS),V(NS),RHO(NS)
      WRITE(3,207)
      DO 120 I =1,100
      PFREQ(I) =0.0
      PVEL(I) =0.0
      PWNGTH(I) =0.0
      PXN(I) =0.0
 120  CONTINUE
C
C         THE FIVE CARDS STARTING WITH STATEMENT 126 ARE INSERTED
C     IN ORDER TO PERMIT THE OPERATION OF THE FREE PLATE OPTION.
```

```
C              ISWTCH = 1        FOR SEMI-INFINITE SYSTEM
C              ISWTCH = 2        FOR COMPOUND PLATE
C
 126  IF(EM(NS)) 127,127,128
 127  NS = NS - 1
      ISWTCH = 2
      GO TO 1111
 128  ISWTCH = 1
 1111 DO 11 I=1,NS
      XLMDA(I)=V(I)*EM(I)/((1.+V(I))*(1.-2.*V(I)))
      XMU(I)=EM(I)/(2.*(1.+V(I)))
 11   CONTINUE
C
C     WE SHALL HOLD THE WAVELENGTH CONSTANT DURING EACH ITERATION, AND
C     INTERPOLATE TO FIND THE PHASE VELOCITY 'CB1'.
C
      DO 111 I=1,NS
      BETA(I)=SQRT(XMU(I)/XMU(1))*SQRT(RHO(1)/RHO(I))
 111  ALPHA(I)=SQRT((XLMDA(I)+2.*XMU(I))/(XLMDA(1)+
     12.*XMU(1)))*SQRT((XLMDA(1)+2.*XMU(1))/XMU(1))*SQRT(RHO(1)/RHO(I)
     1)
C     THIS (DO 111) ROUTINE DETERMINES BETA(I), ALPHA(I) AS MULTIPLES
C     OF BETA(1)
      DELTAP = CMPLX(0.0,0.0)
      DELTAN = CMPLX(0.0,0.0)
      CVP    = CMPLX(0.0,0.0)
      CVN    = CMPLX(0.0,0.0)
C1212 READ(1,104) WLNGTH,FREQ1,FREINC
C     CB1TRL = WLNGTH * FREQ1
C     CINCR = WLNGTH * FREINC
      READ (1,104) WLNGTH,CB1TRL,CINCR,WLFAC,DELFAC
C     'WLNGTH' IS THE MAXIMUM WAVELENGTH TO BE CALCULATED
C     'WLFAC' IS THE RATIO OF THE MINIMUM TO THE MAXIMUM WAVELENGTH.
C     'DELFAC' IS THE WAVENUMBER INTERVAL AT WHICH POINTS ARE TO BE
C     CALCULATED.  THE QUOTIENT 1/DELFAC  GIVES THE NUMBER OF
C     POINTS TO BE CALCULATED, IN EACH UNIT OF WAVENUMBER. FOR EXAMPLE,
C     IF THE RANGE FROM THE MINIMUM TO THE MAXIMUM WAVENUMBER IS 6.0,
C     THE PROGRAM WILL RETURN 6.0/DELFAC POINTS ON THE FREQUENCY-DISPERSION
C     CURVE.
C
      IPLOT = 0
      CB1RES = CB1TRL
      CINRES = CINCR
      WLNTHM = WLFAC * WLNGTH
      DELXN = DELFAC
      GO TO 1213
 1212 WLNGTH = WLNGTH/(1 + DELXN*WLNGTH)
      IF(REAL(WLNGTH).LE. REAL(WLNTHM)) GO TO 9991
      CB1TRL = CB1RES
      CINCR = CINRES
```

```
 1213 CONTINUE
      IF(REAL(WLNGTH) .EQ.0.0 .AND. AIMAG(WLNGTH) .EQ. 0.0) GO TO 1001
      CALL TRAVEL(N,NS,HH,A,R,S,T,E,G,DELXJ,ITRACK)
C900  CB1 = 0.5*(CV1+CV2)
      XN = 1./WLNGTH
      FREQ = CB1 * XN
C     THE FOLLOWING 'IF' HAS BEEN DE-ACTIVATED TO SHORT CIRCUIT A
C     LOOP THROUGH THE SPIRAL SEARCH, WHICH CAN OCCUR IF THE
C     IMAGINARY PART OF 'CB1' IS ZERO.     WHC 71.024
C     IF(ABS(AIMAG(CB1)).LT. 0.0001 .AND. NTRLS .LT. 10) GO TO 1213
      IPLOT = IPLOT+ 1
      PFREQ(IPLOT) = FREQ
      PVEL(IPLOT) = CB1
      PWNGTH(IPLOT) = WLNGTH
      PXN(IPLOT) = XN
C     OUTPUT FORMAT BASED ON WHC33B
      IF (WSWTCH .EQ. 1.0) GO TO 901
      GO TO 902
 901  DIFF=AMIN1(CABS(DELTA1),CABS(DELTA2),CABS(DELTAN),CABS(DELTAP))
      WRITE (*,209) REAL(WLNGTH),CB1,REAL(FREQ),XN,NTRLS,DIFF
      WRITE (3,209) REAL(WLNGTH),CB1,REAL(FREQ),XN,NTRLS,DIFF
      GO TO 1212
 902  WRITE (3,209) REAL(WLNGTH),CB1,REAL(FREQ),XN,NTRLS
      WRITE (*,209) REAL(WLNGTH),CB1,REAL(FREQ),XN,NTRLS
      GO TO 1212
C999  WRITE (3,201)
 9991 CONTINUE
 9992 CONTINUE
C9991 CALL FPLOT (PVEL,PWNGTH,PFREQ,PXN,TITLE,IPLOT)
C9992 CALL CLOSE
      GO TO 1001
  999 stop
      END
      SUBROUTINE TRAVEL(N,NS,HH,A,R,S,T,E,G,DELXJ,ITRACK)          TRAVEL
      REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),HH(6),
     1  FREINC
      REAL*8 ABS,XLMDA(6)
      COMPLEX*16          XK,A(6),R(6),S(6),T(6),E(6,4,4),
     1 PM(6,4,4),CINV(6,4,4),XJ(4,4),VALUE,DELXJ(2,2),
     2 T6,T7,G(4,4)
      COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,CMPLX,XN,
     1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
      COMPLEX*16 CELTA,CELTA1,CELTA2
      COMPLEX*16 CB1REC
C     COMPLEX*16  CDSQRT,CDCOS,CDSIN,CRH,CSH,SRH,SSH
      COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
      COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
     1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
     2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
 203  FORMAT('   CB1 = ', 2E14.4)
```

```
 208  FORMAT(20X,'DELTA = ',2D14.4,I4,'   CB1REC = ',2E14.4)
 210  FORMAT('    RESIGNED AFTER  ',I4, '  TRIALS'/)
 211  FORMAT('        RESIGNED   NCIRC = ',I4, '    ATTEMPTS')
      NINTP = 0
      NTRLS=0
      ITRACK = 1
        NREV = 5
C
C        IF THE PROGRAM OSCILLATES, MAKE THE PREVIOUS CARD 'NREV = 7'.
C      IF 'NREV = 1',  THE PROGRAM IS SENSITIVE AND OSCILLATES READILY.
C
 121  CONTINUE
      IF(MOD(ITRACK,4).EQ. 1) ICYCLE = 1
      IF(MOD(ITRACK,4).EQ. 2) ICYCLE = 2
      IF(MOD(ITRACK,4).EQ. 3) ICYCLE = 3
      IF(MOD(ITRACK,4).EQ. 0) ICYCLE = 4
C     IF(REAL(CB1) .EQ. 1.0) GO TO 1201
      GO TO (1203,1204,1203,1204),ICYCLE
 1203 CB1 = CMPLX(REAL(CB1TRL)+REAL(CINCR),AIMAG(CB1TRL))
      GO TO 1205
 1204 CB1 = CMPLX(REAL(CB1TRL),AIMAG(CB1TRL)+AIMAG(CINCR))
 1205 CONTINUE
C     GO TO 122
C          STATEMENT 1201 IS A BACKTRACKER IN CASE EITHER A(I), R(I) OR
C     S(I) IS ZERO.  IF R(I) OR S(I) IS ZERO, A ZERO DIVIDE OCCURS IN
C     SUBROUTINE 'GMATRX'.  IF A(I) IS ZERO, THE RESULT IS NOT OF
C     INTEREST FOR THE PRESENT.
C          THE SEARCH FOR A VELOCITY BACKTRACKS TO THE MOST RECENT 'CB1'
C     YIELDING NON-ZERO A(I), R(I), S(I).  IT RE-COMMENCES FORWARD
C     TRACKING USING AN INTERVAL ONE TENTH OF THE PREVIOUS ONE.
C1201 CB1 = CB1TRL + 0.1 * CINCR
C     CINCR = 0.1 * CINCR
 122  CONTINUE
      WLNGTH = CMPLX(REAL(WLNGTH), REAL(WLNGTH)*AIMAG(CB1)/REAL(CB1))
      XK = 6.2831852/WLNGTH
      IF(NTRLS .GT.240) GO TO 8901
      IF(WSWTCH .EQ. 0.0) GO TO 123
      WRITE(3,203) CB1
      WRITE(*,203) CB1
C     IF(CB1.GT.1.0,0.0 .OR.CB1.LT.BETA(NS),0.0) GO TO 999
 123  CONTINUE
C
      DO 129 I = 1, NS
      A(I) = 0.0
      R(I) = 0.0
      S(I) = 0.0
      T(I) = 0.0
 129  CONTINUE
 131  CONTINUE
      DO 130 I = 1,NS
```

```
      A(I)=XK*XK*(1.-CB1*CB1/(2.*((BETA(I)/BETA(1))**2)))
      R(I)=CDSQRT( XK*XK*(1.-CB1*CB1/((ALPHA(I)/BETA(1))**2)))
      S(I)=CDSQRT( XK*XK*(1.-CB1*CB1/((BETA(I)/BETA(1))**2)))
      T(I)=XMU(I)/XMU(1)
C THE FOLLOWING 'IF' HAS BEEN DE-ACTIVATED TO OBVIATE ERROR MD-3   69004
C     IF(A(I).EQ.CMPLX(0.0,0.0).OR.R(I).EQ.CMPLX(0.0,0.0).OR.
C    1   S(I).EQ.CMPLX(0.0,0.0)) GO TO 1201
 130  CONTINUE
      CALL GMATRX   (A,R,S,T,G,NS)
C
C     SET UP THE 'E' MATRICES, THE PRODUCTS OF THE D'S AND THE INVERSE
C     'C' 'S.
C
      CALL EMATRX   (HH,A,R,S,T,E,N)
 137  CALL PROMAT(N,E,G,DELXJ)
C
C     WE NOW HAVE THROWER'S 'J' MATRIX COMPLETE, FOR THE PARTICULAR
C     WAVELENGTH 'WLNGTH' AND THE TRIAL VELOCITY 'CB1'.
C
      CALL CHECK(DELXJ,VALUE)
 138  CONTINUE
      IF(WSWTCH .EQ. 2.0)  GO TO 1371
      GO TO 139
 1371 WRITE (3,208) VALUE,ICYCLE,CB1REC
      WRITE (*,208) VALUE,ICYCLE,CB1REC
 139  IF(NTRLS .NE. 0) GO TO 140
      DELTA1= VALUE
      CV1 = CB1
      GO TO 160
 140  DELTA2= VALUE
      CV2 = CB1
      DIFVEL = CABS(CV2-CV1)
      SUMVEL = CABS(CV1 + CV2)
      GO TO (1412,1412,1411,1411),ICYCLE
 1411 PHI = PHI + PINCR
C
C        ENTER SPIRAL SEARCH FOR IMPROVED ROOT
C
C       EJECT IF CB1 IS PURELY REAL
 204  FORMAT('   RAD = ',E14.4)
      IF(ABS(AIMAG(CB1)).LT. 0.001)  GO TO 9009
      IF(NCIRC) 1416,1415,1416
 1415 RAD = AIMAG(CB1REC)
 1416 CONTINUE
      CELTA = VALUE
      IF(NCIRC.EQ.0) GO TO 1420
      CELTA1 = CELTA2
 1420 CELTA2 = CELTA
      IF(NCIRC .EQ. 0) GO TO 1429
C     SIGN CHANGE TEST FOLLOWS
```

```
      IF(SIGN(1.,REAL(CELTA1))-SIGN(1.,REAL(CELTA2)))1421,1429,1421
 1421 IF(SIGN(1.,AIMAG(CELTA1)) - SIGN(1.,AIMAG(CELTA2)))1422,1429,1422
 1422 PHI1 = PHI - 1.5*PINCR
      CINCR = CMPLX(XKONST*RAD*COS(PHI1),XKONST*RAD*SIN(PHI1))
      PHI = 2 * PI
C     'CV1', 'CV2' ARE NEEDED AT STATEMENT 900, IN THE MAIN PROGRAM.
      CV1 = CB1REC
      CB1REC = CB1REC + 0.3 * CINCR
      CV2 = CB1REC
      PINCR = -PINCR
 1429 NCIRC = NCIRC + 1
      IF(NCIRC .GT. 40) GO TO 9008
      IF(PHI - 1.9 * PI) 1417,1413,1413
 1417 IF(PHI+1.9*PI) 1413,1413,1414
 1413 XKONST =  0.7 * XKONST
      IF(XKONST*RAD .LT. 0.002)GO TO 9009
      PHI = 0.0
 1414 CINCR=CMPLX(XKONST*RAD*COS(PHI),XKONST*RAD*SIN(PHI))
      CB1 = CB1REC + CINCR
      GO TO 122
 1412 CONTINUE
      IF(DIFVEL .LT.0.0005    * SUMVEL) GO TO 900
      IF(NINTP .NE. 0) GO TO 300
      GO TO (1401,1402,1402,1401), ICYCLE
 1401 IF(SIGN(1.,REAL(DELTA2))-SIGN(1.,REAL(DELTA1)))  300,150,300
 1402 IF(SIGN(1.,AIMAG(DELTA2))-SIGN(1.,AIMAG(DELTA1))) 300,150,300
C1403 CONTINUE
 150  GO TO (1501,1502,1502,1501), ICYCLE
 1501 IF(ABS(REAL(DELTA1))-ABS(REAL(DELTA2))) 151,152,152
 1502 IF(ABS(AIMAG(DELTA1))-ABS(AIMAG(DELTA2))) 151,152,152
C1503 CONTINUE
C     ROUTINE TO OVERCOME SIGNIFICANCE ERROR
 151  NREV = NREV - 1
      IF(NREV) 1511,152,152
 1511 CINCR = - 2.*CINCR
 152  DELTA1 = DELTA2
      CV1 = CV2
 160  NTRLS = NTRLS + 1
      CB1TRL = CB1
C     ITRACK = ITRACK + 1
      GO TO 121
C
C     INTERPOLATE ROUTINE,  RETAINING THE MOST RECENT DELTAS
C     OF OPPOSITE SIGN
C
C
C     PREPARE FOR INTERPOLATION ROUTINE.  THE PROGRAM PERFORMS THE
C     FOLLOWING OPERATIONS IN A CYCLIC FORM.
C(1)TRACK 'CB1' ALONG ITS REAL*8 AXIS, SEEK ZERO ON REAL*8 AXIS OF 'DELTA'.
C(2)TRACK 'CB1' ALONG ITS IMAG AXIS, SEEK ZERO ON IMAG AXIS OF 'DELTA'.
```

```
C(3)TRACK 'CB1' ALONG ITS REAL*8 AXIS, SEEK ZERO ON IMAG AXIS OF 'DELTA'.
C(4)TRACK 'CB1' ALONG ITS IMAG AXIS, SEEK ZERO ON REAL*8 AXIS OF 'DELTA'.
C
C
C      FIRST CONNECT THE REAL*8 AND IMAGINARY PARTS OF THE INTERPOLATED
C      'DELTA1'.  THE SEQUENCE IS CONTROLLED BY 'ICYCLE', AND IS SET
C      BY THE 'MOD' SWITCH FOLLOWING STATEMENT 121.
C
 300  CONTINUE
      GO TO (3001,3002,3002,3001),ICYCLE
 3001 IF(REAL(DELTA1)) 301,302,302
 3002 IF(AIMAG(DELTA1)) 301,302,302
C3003 CONTINUE
 301  CONTINUE
      GO TO (3011,3012,3012,3011),ICYCLE
 3011 DELTAN = CMPLX(REAL(DELTA1),AIMAG(DELTAN))
      GO TO 3013
 3012 DELTAN = CMPLX(REAL(DELTAN),AIMAG(DELTA1))
 3013 CONTINUE
C      NOW CONNECT THE PARTS OF 'CV1' AND 'CVN'.
      GO TO (3014,3015,3014,3015),ICYCLE
 3014 CVN = CMPLX(REAL(CV1),AIMAG(CVN))
      GO TO 3016
 3015 CVN = CMPLX(REAL(CVN),AIMAG(CV1))
 3016 CONTINUE
      GO TO 303
 302  CONTINUE
      GO TO (3021,3022,3022,3021),ICYCLE
 3021 DELTAP = CMPLX(REAL(DELTA1),AIMAG(DELTAP))
      GO TO 3023
 3022 DELTAP = CMPLX(REAL(DELTAP),AIMAG(DELTA1))
 3023 CONTINUE
C      NOW CONNECT THE PARTS OF 'CV1' AND 'CVP'.
      GO TO (3024,3025,3024,3025),ICYCLE
 3024 CVP = CMPLX(REAL(CV1),AIMAG(CVP))
      GO TO 3026
 3025 CVP = CMPLX(REAL(CVP),AIMAG(CV1))
 3026 CONTINUE
C
C      NOW CONNECT THE REAL*8 AND IMAGINARY PARTS OF THE INTERPOLATED
C      'DELTA2'
C
 303  CONTINUE
      GO TO (3031,3032,3032,3031),ICYCLE
 3031 IF(REAL(DELTA2)) 305,306,306
 3032 IF(AIMAG(DELTA2)) 305,306,306
C3033 CONTINUE
 305  CONTINUE
      GO TO (3051,3052,3052,3051),ICYCLE
 3051 DELTAN = CMPLX(REAL(DELTA2),AIMAG(DELTAN))
```

```
      GO TO 3053
 3052 DELTAN = CMPLX(REAL(DELTAN),AIMAG(DELTA2))
 3053 CONTINUE
C     NOW CONNECT THE PARTS OF 'CV2' AND 'CVN'.
      GO TO (3054,3055,3054,3055),ICYCLE
 3054 CVN = CMPLX(REAL(CV2),AIMAG(CVN))
      GO TO 3056
 3055 CVN = CMPLX(REAL(CVN),AIMAG(CV2))
 3056 CONTINUE
      GO TO 307
 306  CONTINUE
      GO TO (3061,3062,3062,3061),ICYCLE
 3061 DELTAP = CMPLX(REAL(DELTA2),AIMAG(DELTAP))
      GO TO 3063
 3062 DELTAP = CMPLX(REAL(DELTAP),AIMAG(DELTA2))
 3063 CONTINUE
C     NOW CONNECT THE PARTS OF 'CV2' AND 'CVP'.
      GO TO (3064,3065,3064,3065),ICYCLE
 3064 CVP = CMPLX(REAL(CV2),AIMAG(CVP))
      GO TO 3066
 3065 CVP = CMPLX(REAL(CVP),AIMAG(CV2))
 3066 CONTINUE
 307  NINTP = 1
      GO TO (3071,3072,3072,3071),ICYCLE
 3071 DP = REAL(DELTAP)
      DN = REAL(DELTAN)
      GO TO 3073
 3072 DP = AIMAG(DELTAP)
      DN = AIMAG(DELTAN)
 3073 CONTINUE
      GO TO (3074,3075,3074,3075),ICYCLE
 3074 CP = REAL(CVP)
      CN = REAL(CVN)
      GO TO 3076
 3075 CP = AIMAG(CVP)
      CN = AIMAG(CVN)
 3076 CONTINUE
      IF(CP.EQ.0.0.AND.CN.EQ.0.0.OR.DP.EQ.0.0.AND.DN.EQ.0.0)GO TO 900
      FN = XTERPL(DP,DN,CP,CN)
      GO TO (3077,3078,3077,3078),ICYCLE
 3077 CB1 = CMPLX(FN,AIMAG(CB1))
      GO TO 3079
 3078 CB1 = CMPLX(REAL(CB1),FN)
 3079 CONTINUE
      NREV = 3
      NTRLS = NTRLS + 1
      DELTA1 = DELTA2
      CV1 = CV2
      GO TO 122
C
```

```
 900  CONTINUE
      NINTP = 0
      GO TO (3082,3081,3082,3082) , ICYCLE
 3081 CB1REC = CB1
      CB1TRL = CB1
      NCIRC = 0
      XKONST = 1.0
      PHI = 0
      PI = 3.14159265
      PINCR = PI/4
 3082 CONTINUE
      GO TO (9081,9084,9081,9082),ICYCLE
 9081 CINCR = CMPLX(-.5*REAL(CINCR),AIMAG(CINCR))
      GO TO 9083
 9082 CINCR = CMPLX(REAL(CINCR),-.5*AIMAG(CINCR))
      GO TO 9083
 9084 CINCR = CMPLX(AIMAG(CB1REC),AIMAG(CINCR))
 9083 CONTINUE
      GO TO (9091,9092,9091,9092),ICYCLE
 9091 CB1TRL = CMPLX(REAL(CB1)              ,AIMAG(CB1TRL))
      GO TO 9093
 9092 CB1TRL = CMPLX(REAL(CB1TRL),AIMAG(CB1))
 9093 CONTINUE
      GO TO (9001,9001,9001,9009), ICYCLE
 9001 ITRACK = ITRACK + 1
      GO TO 121
 9008 WRITE(3,211) NCIRC
      WRITE(*,211) NCIRC
      RETURN
 9009 CONTINUE
        RETURN
 8901 WRITE(3,210) NTRLS
      WRITE(*,210) NTRLS
      RETURN
C     DEBUG TRACE,SUBCHK,SUBTRACE,INIT(CV1,CV2,CB1REC,CB1TRL,VALUE,NCIRC
C     *,CELTA,CELTA1,CELTA2,PHI,CINCR,PHI1,DIFVEL,SUMVEL)
C     AT 121
C     TRACE ON
C     AT 123
C     TRACE OFF
C     AT 140
C     TRACE ON
C     AT 1412
C     TRACE OFF
C     AT 900
C     TRACE ON
C     AT 9009
C     TRACE OFF
      END
      SUBROUTINE GMATRX    (A,R,S,T,G,NS)                        GMATRX
```

```
      REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),
     1  FREINC
      COMPLEX*16          XK,A(6),R(6),S(6),T(6),
     1 PM(6,4,4),CINV(6,4,4),XJ(4,4),
     2 T6,T7,G(4,4)
      REAL*8 XLMDA(6)
      COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,XN,
     1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
      COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
      COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
     1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
     2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
C
C     DEFINITION OF MATRIX 'G', AS GIVEN IN THROWER'S PAPER
C
      DO 1 J = 1,4
      DO 1 I = 1,4
  1   G(I,J) = 0.0
C11   DO 10 I = NS,NS
      I=NS
C
C
      G(1,1) =        1./(2.*T(I)*(A(I)-XK**2))
      G(1,2)    =    XK/(2.*T(I)*R(I)*(A(I)-XK**2))
      G(1,3)    =    A(I)/(R(I)*(A(I)-XK**2))
      G(1,4)    =   XK/(A(I) - XK**2)
      G(2,1)  =      XK/(2.*T(I)*S(I)*(A(I)-XK**2))
      G(2,2) = G(1,1)
      G(2,3) = G(1,4)
      G(2,4) =       A(I)/(S(I)*(A(I)-XK**2))
 10   CONTINUE
      RETURN
      END
      SUBROUTINE EMATRX   (HH,A,R,S,T,E,N)                    EMATRX
      REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),HH(6),
     1 FREINC
      REAL*8 XLMDA(6)
      COMPLEX*16          XK,A(6),R(6),S(6),T(6),E(6,4,4),
     1 PM(6,4,4),CINV(6,4,4),XJ(4,4),
     2 T6,T7
      COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,XN,
     1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
      COMPLEX*16  CRH,CSH,SRH,SSH,CEXP
      COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
      COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
     1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
     2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
C     SET UP THE ELEMENTS OF THROWER'S "E" MATRIX
C
      DO 401 I = 1, 6
```

```
      DO 401 J = 1, 4
      DO 401 K = 1, 4
 401  E(I,J,K) = 0.0
 402  DO 499 I = 1, N
      AIXK = A(I) - XK*XK
C     T(I) =T(I)
C     S(I) =S(I)
      R1 = CEXP(R(I)*HH(I))
      R2 = CEXP(-R(I)*HH(I))
      CRH = 0.5*(R1 + R2)
      SRH = 0.5*(R1 - R2)
      S1 = CEXP(S(I)*HH(I))
      S2 = CEXP(-S(I)*HH(I))
      CSH =0.5*(S1+S2)
      SSH =0.5*(S1-S2)
      E(I,1,1) = (A(I)*CRH - XK*XK*CSH)/AIXK
      E(I,1,2)=XK*(A(I)*SRH/R(I)-S(I)*SSH)/AIXK
      E(I,1,3)= 2.*T(I)*(A(I)*A(I)*SRH/R(I)-XK*XK*S(I)*SSH)/AIXK
      E(I,1,4) = 2.*T(I)*XK*A(I)*(CRH - CSH)/AIXK
      E(I,2,1) = -XK*(R(I)*SRH - A(I)*SSH/S(I))/AIXK
      E(I,2,2) = (A(I)*CSH - XK*XK*CRH)/AIXK
      E(I,2,3) = -2.*T(I)*XK*A(I)*(CRH - CSH) /AIXK
      E(I,2,4) = -2.*T(I)*(XK*XK*R(I)*SRH - A(I)*A(I)*SSH/S(I))/AIXK
      E(I,3,1) = (R(I)*SRH - XK*XK*SSH/S(I))/(2.*T(I)*AIXK)
      E(I,3,2) = XK*(CRH-CSH)/(2.*T(I)*AIXK)
      E(I,3,3) = (A(I)*CRH - XK*XK*CSH)/AIXK
      E(I,3,4) = XK*(R(I)*SRH - A(I)*SSH/S(I))/AIXK
      E(I,4,1) = -XK*(CRH - CSH)/(2.*T(I)*AIXK)
      E(I,4,2) = -(XK*XK*SRH/R(I)-S(I)*SSH)/(2.*T(I)*AIXK)
      E(I,4,3) = - XK*(A(I)*SRH/R(I) -S(I)*SSH)/AIXK
      E(I,4,4) = -(XK*XK*CRH-A(I)*CSH)/AIXK
 499  CONTINUE
      RETURN
      END
      SUBROUTINE PROMAT(N,E,G,DELXJ)                                PROMAT
      REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),
     1  FREINC
      REAL*8 XLMDA(6)
      COMPLEX*16          XK,E(6,4,4),
     1 PM(6,4,4),CINV(6,4,4),XJ(4,4),DELXJ(2,2),
     2 T6,T7,G(4,4)
      COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,XN,
     1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
      COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
      COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
     1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
     2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
C201  FORMAT('XJ(',I2,',',I2,') = ', D14.8)
C
C         CONTINUED PRODUCT OF THE 'E' MATRICES.
```

```
C
C     START SETTING UP THE PRODUCT MATRICES "PM".   THEY ARE OBTAINED
C     USING THE CONTINUED PRODUCT OF THE 'E' MATRICES.
C     IS PRE-MULTIPLIED BY THE 'G' MATRIX FOR THE SEMI-INFINITE MEDIUM.
C                      NS  =  NO. OF MEDIA
C                      N   =  NO. OF LAYERS
C                          =  NS - 1 .
      DO 26 K = 1,N
      DO 20 J = 1,4
      DO 20 M = 1,4
 20   PM(K,M,J) = 0.0
      IF(K-1) 21,21,23
C     PM(1,I,J) = E(1,I,J) IS THE 'E' MATRIX FOR THE TOP LAYER
 21   DO 22 J = 1,2
      DO 22 I = 1,4
 22   PM(K,I,J) = E(K,I,J+2)
      GO TO 26
 23   CONTINUE
C
C     PRE-MULTIPLY THE PRODUCT MATRIX THUS FAR ESTABLISHED BY THE 'E'
C     MATRICES FOR THE SUCCESSIVELY LOWER LAYERS
C
C     FOR AN EXPLANATION OF THE STRANGE APPEARANCE OF 'PM' AT THIS STAGE
C     SEE THROWER'S EQUATION (19).  AS THE INITIAL 'PM' MATRIX
C     (THE CONTRIBUTION FROM THE TOP LAYER) IS ONLY A 4(ROW) X 2(COLUMN)
C     MATRIX, SUBSEQUENT 'PM' MATRICES ARE ONLY 4(ROW) X 2(COLUMN)
C     MATRICES.  THIS CONSTITUTES THE RATHER ASTOUNDING ECONOMY
C     OF THE METHOD.
C
      DO 25 J = 1,2
      DO 25 I = 1,4
      T6 = E(K,I,1)*PM(K-1,1,J) + E(K,I,2)*PM(K-1,2,J)
     1    +E(K,I,3)*PM(K-1,3,J) + E(K,I,4)*PM(K-1,4,J)
      PM(K,I,J) = T6
 25   CONTINUE
C
C     NOW PRE-MULTIPLY THE PRODUCT MATRIX 'PM' BY THE 'G' MATRIX,
C     A 2 X 4 RECTANGULAR MATRIX DEFINED IN THROWER'S PAPER.  THIS
C     YIELDS THE 'TEST' DETERMINANT DENOTED BY 'DELXJ'.
C
 26   CONTINUE
      GO TO (27,36), ISWTCH
 27   DO 33 I = 1,2
      DO 33 J = 1,2
 33   DELXJ(I,J) = 0.0
      DO 35 J = 1,2
      DO 35 I = 1,2
      T7 = G(I,1)*PM(N,1,J) + G(I,2)*PM(N,2,J)
     1    +G(I,3)*PM(N,3,J) + G(I,4)*PM(N,4,J)
      DELXJ(I,J) = T7
```

```
 35    CONTINUE
       RETURN
 36    DELXJ(1,1) = PM(N,1,1)
       DELXJ(2,2) = PM(N,2,2)
       DELXJ(1,2) = PM(N,1,2)
       DELXJ(2,1) = PM(N,2,1)
       RETURN
       END
       SUBROUTINE CHECK (DELXJ,VALUE)                                CHECK
       REAL*8 EM(6),V(6),XMU(6),BETA(6),ALPHA(6),RHO(6),
      1  FREINC
       REAL*8 XLMDA(6)
       COMPLEX*16           XK,
      1 PM(6,4,4),CINV(6,4,4),XJ(4,4),VALUE,DELXJ(2,2),
      2 T6,T7
       COMPLEX*16  CB1,CB1TRL,CINCR,WLNGTH,CV1,CV2,XN,
      1 DELTA1,DELTA2,DELTAN,DELTAP,FREQ,CVP,CVN
c      COMPLEX*16  CDSQRT,CDCOS,CDSIN,CRH,CSH,SRH,SSH
       COMMON ALPHA,BETA,XK,XMU,XLMDA,PM,T1,T7
       COMMON CB1,DELTA1,DELTA2,CV1,CV2,CB1TRL,CINCR,DELTAN,DELTAP,
      1 CVN,CVP,WLNGTH,WSWTCH,NTRLS,ICYCLE,ISWTCH,T6,K,M,EM,V,
      2 XN,XJ,FREQ,FREINC,SLOPE,CINV,RHO
       VALUE = DELXJ(1,1)*DELXJ(2,2) - DELXJ(1,2)*DELXJ(2,1)
       RETURN
C      DEBUG TRACE, SUBTRACE, INIT
C     1                (EM,V,XLMDA,XMU,BETA,ALPHA,XK,A,R,S,T,E,PM,CINV,
C     1XJ,CRH,CSH,SRH,SSH,TEST,VALUE,HH,I,J,K,K1)
       END
       FUNCTION XTERPL(DELTA1,DELTA2,CV1,CV2)                        XTERPL
       SLOPE=(DELTA2-DELTA1)/(CV2-CV1)
       XTERPL=CV1-DELTA1/SLOPE
       RETURN
       END
```

5.7 Listing of *PREP.FOR*

The following is a listing of the program **PREP.FOR**. It prepares the datafile **RAY.DAT** as input to the program **RAY.FOR**. *PREP.FOR*

5.7.1 Program listing

```
        PROGRAM PREP
C         THIS PROGRAM PREPARES DATA FOR RAY. THE INPUT IS IN FILE PREP.DAT
C    THE DATA LINES INPREP.DAT ARE TITLE,
C         NS,(EM(I),V(I), I=1,NS),SWTCH
C         (RHO(I),I=1,NS)
C         (HH(I),I=1,N)
```

```
C       WLNGTH,CD1TRA,CINCR ALLCOMPLEX; WLFAC,DELFAC BOTH REAL
C
        DIMENSION EM(5),V(5),RHO(5),HH(5),TITLE(20)
        COMPLEX WLNGTH,CB1TRL,CINCR
C       OPEN FILE PREP.DAT WHICH CONTAINS DATA IN FREE FORMAT.
        OPEN(1,FILE='PREP.DAT',STATUS='OLD')
C       OPEN FILE RAY.DAT TO ACCEPT DATA IN FIXED FORMAT.
        OPEN(3,FILE='RAY.DAT',STATUS='UNKNOWN')
        REWIND 3
  101   FORMAT(20A4)
  300         WRITE (*,*) 'ENTER TITLE'
        READ(1,101,END=999) (TITLE(I), I = 1,19)
              WRITE(*,*)'NS, (EM(I),V(I),I=1,NS),SWTCH'
        READ(1,*)NS,(EM(I),V(I),I=1,NS),SWTCH
              WRITE(*,*)'(RHO(I),I=1,NS)'
        READ(1,*)(RHO(I),I=1,NS)
              WRITE(*,*) 'HH(I),I=1,N)'
        N = NS-1
        READ(1,*)(HH(I),I=1,N)
             WRITE(*,*)'WLNGTH,CB1TRL,CINCR,COMPLEX,WLFAC,DELFAC,REAL'
c            WRITE(*,*)'ENTER WLNGTH,CB1TRL,CINCR ALL COMPLEX'/
c    1 'WLFAC,DELFAC BOTH REAL'
        READ (1,*) WLNGTH,CB1TRL,CINCR,WLFAC,DELFAC
C      'WLNGTH' IS THE MAXIMUM WAVELENGTH TO BE CALCULATED
C      'WLFAC' IS THE RATIO OF THE MINIMUM TO THE MAXIMUM WAVELENGTH.
C      'DELFAC' IS THE WAVENUMBER INTERVAL AT WHICH POINTS ARE TO BE
C      CALCULATED. THE QUOTIENT 1/DELFAC GIVES THE NUMBER OF
C      POINTS TO BE CALCULATED, IN EACH UNIT OF WAVENUMBER. FOR EXAMPLE,
C      IF THE RANGE FROM THE MINIMUM TO THE MAXIMUM WAVENUMBER IS 6.0,
C      THE PROGRAM WILL RETURN 6.0/DELFAC POINTS ON THE FREQUENCY-DISPERSION
C      CURVE.
C
 2001   FORMAT(19A4,F4.0)
 201    FORMAT(I3,3X,6(F6.0,F6.2))
 202    FORMAT (6X,12F6.0)
 203    FORMAT (10F6.2)
 204    FORMAT (8F6.4)
        WRITE (3,2001)(TITLE(I),I=1,19),SWTCH
        WRITE (3,201) NS,(EM(I),V(I),I=1,NS)
        WRITE (3,202) (RHO(I),I=1,NS)
        N = NS - 1
        WRITE (3,203) (HH(I),I=1,N)
        WRITE (3,204) WLNGTH,CB1TRL,CINCR,WLFAC,DELFAC
C        GO TO 300
 999    CONTINUE
        CLOSE(UNIT=3)
              STOP 'END OF FILE'
        END
```

5.7.1.1 Program listing

Sample data follows for program *PREP.FOR* .

```
MARTINZEC55
E1=11100,E2=8400,E3=280;H1=0.13,H2=0.25,NU1=NU2=NU3=0.25 5 11100.
.25 84000.   .25   20000.   .25 40000.   .25   280. .25   0.0
      2.    2.    2.   2.    2.
0.05 0.08 0.25 0.10 (5, 0.0) (0.4, 0.00)  (0.05, 0.01)  0.0001 0.1
```

5.8 Listing of *HARDK.MWS*

The following is a listing of the Maple worksheet **HARDK.MWS**. It can be used to plot the response of a system consisting of a layer of material having a high stiffness overlying a semi-infinite medium containing a material having a lower stiffness.

5.8.1 Listing of worksheet

This file uses implicitplot to plot the relationship between <k> and <w> given that **xi1*eta2-xi28eta1=0.**

```
>  eq1:={a1 = 25,a2 = 2.5,b1 = 10.0,b2 = 1.0,mu1 = 100.0,mu2
= 1.0}:
>  eq2:={c=w/k,kb1 = w / b1,kb2 = w / b2}:
> Use <k> as a dependent variable.

>  eq3:={r1 = w*(1/c**2 - 1/a1**2)**(1/2),s1 =  w*(1/c**2
-
>  1/b1**2)**(1/2),
>  r2 =  w*(- 1/c**2 + 1/a2**2)** (1/2), s2 =  w*(- 1/c**2
+
>  1/b2**2)  **(1/2)}:
>  eq4:={ww = 2 * ( mu2/mu1-1 ),xx = mu2/mu1 * kb2**2/k**2
- 2 *
>  ( mu2/mu1 -1 ),
>  yy = kb1**2/k**2 + 2 * ( mu2/mu1 -1 ), zz = mu2/mu1 *
>  kb2**2/k**2 - kb1**2 /k**2 - 2 * ( mu2/mu1 -1 )}:
>  eq3:=subs(eq1,eq2,eq3):
>  eq4 := subs(eq2,eq1,eq4):
> Substitute the independent variables in equation (4)
```

```
>  h:= 0.1:
> Set the thickness of the layer equal to unity
>  x1 := ((2-kb1^2/k^2) *(xx* cosh( r1* h) + r2/r1* yy* sinh(
r1* h) )
>  -2* s1/k* (r2/k* ww* sinh( s1* h) + k/s1* zz* cosh( s1*
h)
>  )):
>  x2 := ((2-kb1^2/k^2)* (s2/k* ww* cosh( r1* h) + k/r1*
zz* sinh( r1*
>  h) )
>  -2* s1/k* (xx* sinh( s1* h) + s2/s1* yy* cosh( s1* h)
)):
>  e1 := ((2-kb1^2/k^2)* (r2/k* ww* cosh( s1* h) + k/s1*
zz* sinh( s1*
>  h) )
>  -2* r1/k* ( xx* sinh(w* r1* h) + r2/r1* yy* cosh(  r1*
h)
>  )):
>  e2 := ((2-kb1^2/k^2)* (xx* cosh(  s1* h) +s2/s1* yy* sinh(
s1* h))
>  -2* r1/k* (s2/k* ww* sinh(  r1* h) + k/r1* zz* cosh( r1*
h)
>  )):
>  x1 := subs(eq4,eq3,eq2,eq1,x1):
> Substitute the independent variables in xi and the eta
>  x2 := subs(eq4,eq3,eq2,eq1,x2):
>  e1 := subs(eq4,eq3,eq2,eq1,e1):
>  e2 := subs(eq4,eq3,eq2,eq1,e2):
>  simplify(x1*e2-x2*e1):
> Use this statement as a debugger
>  with(plots):

Warning, the name changecoords has been redefined
>  implicitplot(x1*e2=x2*e1,w=0.1..2.0,k=0.002..0.4,font=[TIMES,BOLD,1
>  ,title="Plot of <k> vs <w> for hard over
>  soft",labeldirections=[HORIZONTAL,VERTICAL],labels=["Frequency
>  omega","Reciprocal wavelength k"]);
> Plot of x1*e2-x2*e1=0, showing <k> and <w>
```

5.9 Listing of *SOFTK.MWS*

The following is a listing of the Maple worksheet **SOFTK.MWS**. It can be used to plot the response of a system consisting of a layer of material having a low stiffness overlying a semi-infinite medium containing a material having a higher stiffness.

5.9.1 Listing of worksheet

```
>  eq1:={a1 = 2.5,a2 = 25,b1 = 1.0,b2 = 10.0,mu1 = 1.0,mu2
= 100.0}:

>  eq2:={c=k/w,kb1 = w / b1,kb2 = w / b2}:

>  eq3:={r1 = w* ( - 1/c**2 + 1/a1**2) ** (1/2),s1 =  w*(
-

>  1/c**2 + 1/b1**2  ) **(1/2),

>  r2 =  w*(  1/c**2 - 1/a2**2) ** (1/2), s2 =  w*(  1/c**2
-

>  1/b2**2  ) **(1/2)}:

>  eq4:={ww = 2 * ( mu2/mu1 -1 ),xx = mu2/mu1 * kb2**2/k**2
- 2 *

>  ( mu2/mu1 -1 ),

>  yy = kb1**2/k**2 + 2 * ( mu2/mu1 -1 ), zz = mu2/mu1 *

>  kb2**2/k**2 - kb1**2 /k**2 - 2 * ( mu2/mu1 -1 )}:

>  eq4:= subs(eq2,eq1,eq4):  # This validates eq4

>  eq3:=subs(eq2,eq1,eq3):



>  x1 := ((2-kb1^2/k^2) *(xx* cos( r1* h) + r2/r1* yy* sin(
r1* h) )

>  +2* s1/k* (r2/k* ww* sin( s1* h) - k/s1* zz* cos( s1*
h) )):

>  x2 := ((2-kb1^2/k^2)* (s2/k* ww* cos( r1* h) + k/r1* zz*
sin( r1* h)

>  )

>  +2* s1/k* (xx* sin( s1* h) - s2/s1* yy* cos( s1* h) )):
```

```
>  e1 := ((2-kb1^2/k^2)* (r2/k* ww* cos( s1* h) + k/s1* zz*
sin( s1* h)
>  )
>  +2* r1/k* ( xx* sin(w* r1* h) - r2/r1* yy* cos(  r1* h)
)):
>  e2 := ((2-kb1^2/k^2)* (xx* cos(  s1* h) +s2/s1* yy* sin(
s1* h))
>  +2* r1/k* (s2/k* ww* sin(  r1* h) - k/r1* zz* cos( r1*
h)
>  )):
>  x1 := subs(eq4,eq3,eq2,eq1,x1):
> Update the xi and the eta
>  x2 := subs(eq4,eq3,eq2,eq1,x2):
>  e1 := subs(eq4,eq3,eq2,eq1,e1):
>  e2 := subs(eq4,eq3,eq2,eq1,e2):
>  h:=1.0:
>  eq:=x1*e2-x2*e1:
>  ##solve(eq,c);
>  with(plots):
>  implicitplot(x1*e2=x2*e1,w=0.001..0.03,k=0.005..0.1,font=[TIMES,BOI
>  14],title="Plot of <k> vs <w> for soft over
>  hard",labeldirections=[HORIZONTAL,VERTICAL],labels=["Frequency
>  omega","Reciprocal wavelength k"]);
> Plot of x1*e2-x2*e1=0, showing <k> and <w>
```

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