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Estimation of Acoustic Source Strength by Inverse Methods: Part II: Methods for Choosing Regularisation Parameters

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ISVR Technical Report No 279

October 1998

University of Southampton

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UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

FLUID DYNAMICS AND ACOUSTICS GROUP

Estimation of Acoustic Source Strength by Inverse Methods: Part II: Methods for Choosing Regularisation Parameters

by

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October 1998

Authorized for issue by
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Group Chairman

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ABSTRACT

Two regularisation methods, Tikhonov regularisation and singular value discarding, are used to improve the accuracy of reconstruction of acoustic source strength by inverse techniques. In this report, some methods are investigated for choosing the Tikhonov regularisation parameter and the singular values to be discarded. Of these, we concentrate on the use of ordinary cross-validation and generalised cross-validation. These methods can provide an appropriate regularisation parameter without prior knowledge of either the acoustic source strength or the contaminating measurement noise. Some computer simulation results obtained using a model of a randomly vibrating simply supported plate mounted in an infinite baffle are presented to illustrate the performance of the methods for choosing the regularisation parameters.

1. INTRODUCTION

In the reconstruction of acoustic source strength by inverse techniques, the accuracy of reconstruction is largely determined by the conditioning of the matrix to be inverted. This matrix can often be ill-conditioned. This can occur despite the choice of geometrical arrangement of microphones being made to account for the behaviour of the condition number of the acoustic transfer function matrix. This is described in full in reference [1]. For example, when we wish to identify the contribution to the total acoustic field made by a vibrating surface (e.g., a plate-like structure excited by dynamic forces) from the viewpoint of a discrete inverse problem, this surface has to be discretised into a large number of small elements each of which is regarded as an acoustic source. In this case, we need also a large number of microphones, which is at least equal to the number of acoustic sources. However, such a system can result in ill-conditioning because the condition number of the matrix to be inverted increases as the dimension of this matrix increases. This ill-conditioning makes the problem ill-posed. In the ill-conditioned problem, finding a good estimate of acoustic source strength by the least squares method becomes problematic. This is because the effect of measurement noise and/or modelling error appears in the reconstructed acoustic source strength, giving a large deviation from the desired values [1]. For this reason, we demonstrated in reference [1] that Tikhonov regularisation and singular value discarding could be used to enhance the accuracy of reconstruction of acoustic source strength. In this case, the success of these regularisation methods depends on the appropriate choice of the Tikhonov regularisation parameter and the singular values to be discarded. (Note that in the field of inverse problems both Tikhonov regularisation and singular value discarding are referred to as "regularisation methods", but here we take the term "regularisation parameter" to mean the Tikhonov regularisation parameter).

This report deals with the methods for choosing these values. In association with the choice of the regularisation parameter, one method for choosing this is based on the minimisation of the mean squared error between the desired ("true") source strength and that deduced from the Tikhonov regularised solution. Similarly we can use the method based on the minimisation of the mean squared error between the desired acoustic pressures and the Tikhonov regularised acoustic pressures. However, these methods require prior knowledge of either the acoustic source strength or the noise. For this reason, we introduce the techniques of ordinary cross-validation and generalised cross-validation for choosing the regularisation parameter. These methods have mainly been applied in the field of statistical data analysis and image restoration (see references [2, 3] for example). The main advantage of these techniques is that they do not require prior knowledge of the source strengths to be reconstructed or the contaminating noise and thus they can be used in practical situations. Regarding the methods for determining the singular values to be discarded, we can make use of the singular value distribution of the matrix to be inverted, the ordinary cross-validation technique and the generalised cross-validation technique. Of these methods, our emphasis is put on the use of ordinary cross-validation and generalised cross-validation. The theoretical bases for these methods are developed in order to obtain the formulae which enable the acoustic source strengths to be recovered with improved accuracy.

In order to explore the main features of the theories presented, we undertake a series of computer simulations in which we reconstruct the volume velocities of a randomly vibrating simply supported plate mounted in an infinite baffle. The initial restoration of acoustic source strengths is conducted for perfect data in which no measurement noise is assumed. Then we investigate the influence of measurement noise on the reconstruction accuracy. Finally we explore the performance of Tikhonov regularisation and singular value discarding which incorporate the generalised cross-validation technique for choosing the regularisation parameter and the singular values to be discarded.

2. METHODS FOR CHOOSING THE TIKHONOV REGULARISATION PARAMETER

2.1 INTRODUCTION

In reference [1] we showed that the Tikhonov regularised solution for the acoustic source strength vector could be written as

$$\mathbf{q}_R = (\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\hat{\mathbf{p}}, \tag{1}$$

where we have used the subscript R to denote "Tikhonov regularised". In this solution $\mathbf{H}$ is the matrix of acoustic transfer functions relating the vector $\mathbf{p}$ of desired (or modelled) acoustic pressures to the vector $\mathbf{q}$ of desired (or modelled) acoustic source strengths and $\hat{\mathbf{p}}$ is the vector of measured acoustic pressures. The regularisation parameter is denoted by β. Similarly, we showed that for acoustic sources having a stationary random time history, the Tikhonov regularised solution for the source strength cross-spectral matrix could be written as

$$\mathbf{S}_{qqR} = \{(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\}\mathbf{S}_{\hat{p}\hat{p}}\{(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\}^{\mathrm{H}}, \tag{2}$$

where $\mathbf{S}_{\hat{p}\hat{p}}$ is the matrix of cross-spectra of measured acoustic pressures. In this report, we first describe established techniques for choosing β in the solution for source strength given by equation (1). We then proceed to apply the same techniques to the solution given by equation (2) on the grounds that a good solution for $\mathbf{q}_R$ will lead to a good solution for $\mathbf{S}_{qqR}$.

In applying Tikhonov regularisation, we have to consider two factors: the manner by which the result should be regularised and the amount of regularisation. The method of regularisation will be determined by the choice of "regularisation operator" which here is chosen as the identity matrix $\mathbf{I}$, although this can be replaced by a number of linear matrix operators. Also, the degree of regularisation is determined by the regularisation parameter β. We shall here concentrate on only the method for choosing the regularisation parameter. Further discussion regarding the choice of the regularisation operator is given in reference [3]. The choice of a good value of regularisation parameter in Tikhonov regularisation has received a great deal of attention [2-8]. Here we will introduce some established methods with the modifications necessary to enable them to be oriented towards our problem.

2.2 METHODS REQUIRING PRIOR KNOWLEDGE

One of the methods requiring prior knowledge is to minimise the mean squared error (MSE) between the desired solution $\mathbf{q}$ and the Tikhonov regularised solution $\mathbf{q}_R(\beta)$. This is defined by

$$\mathrm{MSE}(\beta)=E[\|\mathbf{q}-\mathbf{q}_R(\beta)\|^2], \tag{3}$$

where E denotes the expectation operator. We here denote the regularisation parameter β minimising MSE(β) by β_{MSE}.

Another choice of the regularisation parameter can be made by minimising the predicted mean squared error (PMSE) defined by

$$\mathrm{PMSE}(\beta) = E[\|\mathbf{Hq} - \mathbf{Hq}_R(\beta)\|^2], \tag{4}$$

and we define β_{PMSE} as the regularisation parameter minimising PMSE(β). It is clear from equations (3) and (4), that both methods require prior knowledge of the desired value $\mathbf{q}$ to be reconstructed. However, for most inverse problems, this knowledge is not available. It is thus necessary to make an attempt to have an alternative method for choosing a satisfactory regularisation parameter, without such *a priori* knowledge. A further discussion of the relationship between the regularisation parameters β_{MSE} and β_{PMSE} is given by Thompson *et al* [6].

2.3 ORDINARY CROSS-VALIDATION

Several methods not requiring prior knowledge of either of the value to be restored or the noise have been proposed in determining the regularisation parameter, such as the cross-validation technique [4], the generalised cross-validation technique [2], the L-curve method [7], and the maximum likelihood method [8]. However, here we focus on the use of the cross-validation technique and the generalised cross-validation technique.

Allen [4] proposed a method, called *Allen's PRESS* (which stands for predicted sum of squares), in order to select a good ridge parameter in ridge regression analysis. This parameter is tantamount to the regularisation parameter in our problem. This method usually referred to as the ordinary cross-validation (OCV) technique. This was also suggested by Wahba and Wold [9] in the context of smoothing splines. This is a widely recognised method in the field of statistical data analysis. The advantage of this approach is that it does not require prior knowledge of either the value to be restored or the noise.

The essence of the method, when viewed in the current context, is to first find the vector of complex acoustic source strengths $\mathbf{q}_R(\beta,k)$ which minimises the cost function

$$J(\beta,k) = \sum_{\substack{m=1 \\ m\neq k}}^{M} [\hat{p}_m - p_m]^2 + \beta \mathbf{q}^{\mathrm{H}} \mathbf{q}, \qquad (5)$$

which is the usual cost function for minimisation as described in reference [1], but with the k'th values of measured and modelled pressures ($\hat{p}_k$ and p_k) omitted from the calculation. (In this sense the ordinary cross-validation technique is also sometimes referred to as "the leaving-one-out method"). Having arrived at the source strength vector $\mathbf{q}_R(\beta,k)$ that minimises this function, we then evaluate the effectiveness of this vector in predicting the value of the measured pressure $\hat{p}_k$ that was "left out" of the calculation of the cost function. We denote the predicted value of $\hat{p}_k$ by $p_k(\beta,k)$ which is the k'th component of $\mathbf{Hq}_R(\beta,k)$. The ordinary cross-validation function $V_o(\beta)$ is then defined in order to measure the success of this prediction when the "leaving-one-out" process is repeated for all the available data points $\hat{p}_k$ (i.e., for k=1 to M). The ordinary cross-validation function is thus defined by

$$V_o(\beta) = \frac{1}{M} \sum_{k=1}^{M} [\hat{p}_k - p_k(\beta,k)]^2 . \qquad (6)$$

The value of β which leads to the minimisation of this function is then deemed "optimal"; it is the value of the regularisation parameter β which ensures the best prediction of each of the measured pressures from knowledge of all the other measured pressures.

A notationally convenient means for expressing this cost function can also be developed by following the procedure presented by Craven and Wahba [10]. One first has to recognise that the solution to the minimisation of $J(\beta,k)$ can be found by solving the full M-point minimisation problem but with the data point $\hat{p}_k$ given by the k'th component of the solution to the problem, i.e., we set $\hat{p}_k = p_k(\beta,k)$. A rigorous proof of this observation is given by Craven and Wahba [10]. Here we illustrate this method of approach and write the cost function given by equation (5) as

$$J(\beta,k) = [\hat{\mathbf{p}}(k) - \mathbf{p}]^{\mathrm{H}}[\hat{\mathbf{p}}(k) - \mathbf{p}] + \beta\mathbf{q}^{\mathrm{H}}\mathbf{q}\,, \tag{7}$$

where the vector of measured pressures $\hat{\mathbf{p}}(k)$ is given by

$$\hat{\mathbf{p}}^{\mathrm{T}}(k) = [\hat{p}_1\ \hat{p}_2\ \cdots\ \hat{p}_{k-1}\ p_k(\beta,k)\ \hat{p}_{k+1}\ \cdots\ \hat{p}_M]\,. \tag{8}$$

The solution for the source strength vector $\mathbf{q}_R(\beta,k)$ that minimises this function is given by

$$\mathbf{q}_R(\beta,k) = (\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\hat{\mathbf{p}}(k)\,. \tag{9}$$

Thus we can write the vector of estimated pressures that minimise $J(\beta,k)$ as

$$\mathbf{p}(\beta,k) = \mathbf{H}\mathbf{q}_R(\beta,k) = \mathbf{H}(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\hat{\mathbf{p}}(k) = \mathbf{B}(\beta)\hat{\mathbf{p}}(k)\,, \tag{10}$$

where the matrix $\mathbf{B}(\beta)$ (the "influence matrix") is given by $\mathbf{H}(\mathbf{H}^{\mathrm{H}}\mathbf{H}+\beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}$. This relationship, when written in full, thus takes the form

$$\begin{bmatrix} p_1(\beta,k) \\ p_2(\beta,k) \\ \vdots \\ p_k(\beta,k) \\ \vdots \\ p_M(\beta,k) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1M} \\ b_{21} & b_{22} & \cdots & b_{2M} \\ \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kM} \\ \vdots & \vdots & & \vdots \\ b_{M1} & b_{M2} & \cdots & b_{MM} \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \vdots \\ p_k(\beta,k) \\ \vdots \\ \hat{p}_M \end{bmatrix}, \tag{11}$$

where b_{ij} are the components of matrix $\mathbf{B}(\beta)$. Now note that the solution to the M-point minimisation problem with no changes made to the vector of measured pressures $\hat{\mathbf{p}}$ can be written as

$$\mathbf{q}_R(\beta) = (\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}\hat{\mathbf{p}}, \tag{12}$$

and thus the corresponding set of estimated pressures can be written as $\mathbf{p}(\beta)=\mathbf{H}\mathbf{q}_R(\beta)=\mathbf{B}(\beta)\hat{\mathbf{p}}$. When written in full, this relationship can be expressed as

$$\begin{bmatrix} p_1(\beta) \\ p_2(\beta) \\ \vdots \\ p_k(\beta) \\ \vdots \\ p_M(\beta) \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1M} \\ b_{21} & b_{22} & \cdots & b_{2M} \\ \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kM} \\ \vdots & \vdots & & \vdots \\ b_{M1} & b_{M2} & \cdots & b_{MM} \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \vdots \\ \hat{p}_k \\ \vdots \\ \hat{p}_M \end{bmatrix}. \tag{13}$$

From the above two matrix expressions we can derive a relationship between $p_k(\beta,k)$ and $p_k(\beta)$. It follows that

$$p_k(\beta,k) = b_{k1}\hat{p}_1 + b_{k2}\hat{p}_2 + \ \ldots \ + b_{kk}p_k(\beta,k) + \ \ldots \ + b_{kM}\hat{p}_M, \tag{14}$$

and also that

$$p_k(\beta) = b_{k1}\hat{p}_1 + b_{k2}\hat{p}_2 + \ \ldots \ + b_{kk}\hat{p}_k + \ \ldots \ + b_{kM}\hat{p}_M. \tag{15}$$

Taking the difference between these two equations then shows that

$$(1 - b_{kk})p_k(\beta,k) = p_k(\beta) - b_{kk}\hat{p}_k. \tag{16}$$

It therefore follows, after some algebra, that

$$\hat{p}_k - p_k(\beta,k) = \frac{\hat{p}_k - p_k(\beta)}{1 - b_{kk}}. \tag{17}$$

This enables the expressions for the ordinary cross-validation function $V_o(\beta)$ given by equation (6) to be written as

$$V_o(\beta) = \frac{1}{M}\sum_{k=1}^{M}\left[\frac{\hat{p}_k - p_k(\beta)}{1 - b_{kk}}\right]^2. \tag{18}$$

Note also that since $\mathbf{p}(\beta)=\mathbf{H}\mathbf{q}_R(\beta)=\mathbf{B}(\beta)\hat{\mathbf{p}}$ then we may write this expression as

$$V_o(\beta) = \frac{1}{M}\left\|\mathbf{C}(\mathbf{I} - \mathbf{B}(\beta))\hat{\mathbf{p}}\right\|_e^2, \tag{19}$$

where $\|\ \|_e$ denotes the Euclidean norm of a vector or a matrix and **C** is the diagonal matrix whose entries are also given by $1/(1-b_{kk})$. Note that $V_o(\beta)$ is not a function of either the source strength to be restored or the noise but a function of only **H** the assumed transfer function matrix, $\hat{\mathbf{p}}$ the measured pressure vector, and, of course, β the regularisation parameter. We denote the value of β that minimises the function $V_o(\beta)$ by β_{OCV}.

2.4 GENERALISED CROSS-VALIDATION

It was pointed out by Golub *et al* [2] that the ordinary cross-validation technique described above may be expected to fail in cases where the matrix $\mathbf{B}(\beta)$ is close to diagonal. (This would be the case, for example, in our acoustical problem where sensors were placed symmetrically with respect to the source array and close to source array such that **H** becomes close to diagonal). It is clear that if $\mathbf{B}(\beta)$ is diagonal, then the cross-validation function $V_o(\beta)$ given above reduces simply to $(1/M)\|\hat{\mathbf{p}}\|_e^2$ (i.e., $1/M$ times the sum of squared measured pressures) which is entirely independent of the choice of β. This shortcoming led Golub *et al* [2] to suggest the use of "generalised cross-validation" (GCV) which follows from their argument that any good choice of β should be "invariant under rotation of the measurement coordinate system". Here we follows the analysis put forward by Golub *et al* in deriving the GCV function appropriate to the case considered here.

Firstly we employ the SVD of the matrix **H** [1] to write the relationship $\hat{\mathbf{p}}=\mathbf{Hq}+\mathbf{e}$ in the form

$$\hat{\mathbf{p}}=\mathbf{U\Sigma V}^{\mathrm{H}}\mathbf{q}+\mathbf{e}\,, \tag{20}$$

and since the unitary matrix **U** has the property $\mathbf{U}^{\mathrm{H}}\mathbf{U}=\mathbf{I}$, then pre-multiplication of this equation by $\mathbf{U}^{\mathrm{H}}$ results in

$$\mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}}=\mathbf{\Sigma V}^{\mathrm{H}}\mathbf{q}+\mathbf{U}^{\mathrm{H}}\mathbf{e}\,. \tag{21}$$

Note that this equation describes the relationship between the "transformed" pressure $\tilde{\mathbf{p}}=\mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}}$ and the "transformed" source strength $\tilde{\mathbf{q}}=\mathbf{V}^{\mathrm{H}}\mathbf{q}$ as described in reference [1]. This equation is further transformed by pre-multiplication by the matrix **W** which has the *ik*'th entry given by

$$W_{ik} = \frac{1}{\sqrt{M}} \mathrm{e}^{2\pi jik/M} \quad , \quad i,k = 1,2, \ldots, M , \tag{22}$$

where $j = \sqrt{-1}$. The matrix $\mathbf{W}$ is a unitary matrix ($\mathbf{W}^{\mathrm{H}}\mathbf{W}$=$\mathbf{I}$) and has the effect of applying a discrete Fourier transform to the vectors $\mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}}$ etc. appearing in equation (21). This then becomes

$$\mathbf{W}\mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}} = \mathbf{W}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{H}}\mathbf{q} + \mathbf{W}\mathbf{U}^{\mathrm{H}}\mathbf{e} . \tag{23}$$

The transformed model is thus written as

$$\hat{\mathbf{p}}_{\mathrm{trans}} = \mathbf{H}_{\mathrm{trans}}\mathbf{q}_{\mathrm{trans}} + \mathbf{e}_{\mathrm{trans}} , \tag{24}$$

where $\hat{\mathbf{p}}_{\mathrm{trans}}$=$\mathbf{W}\mathbf{U}^{\mathrm{H}}\hat{\mathbf{p}}$ is the vector of transformed measured pressures, $\mathbf{q}_{\mathrm{trans}}$=$\mathbf{W}\mathbf{V}^{\mathrm{H}}\mathbf{q}$ is the vector of transformed source strengths, $\mathbf{e}_{\mathrm{trans}}$=$\mathbf{W}\mathbf{U}^{\mathrm{H}}\mathbf{e}$ is the vector of transformed noise components and $\mathbf{H}_{\mathrm{trans}}$=$\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}^{\mathrm{H}}$.

The procedure adopted by Golub *et al* is then to apply ordinary cross-validation to this transformed model. Firstly the transformed influence matrix $\mathbf{B}_{\mathrm{trans}}$ is defined by $\mathbf{H}_{\mathrm{trans}}(\mathbf{H}^{\mathrm{H}}_{\mathrm{trans}}\mathbf{H}_{\mathrm{trans}}+\beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}_{\mathrm{trans}}$ and $\hat{\mathbf{p}}_{\mathrm{trans}}$ is substituted into the expression for the ordinary cross-validation function given by equation (19). Noting that $\mathbf{B}_{\mathrm{trans}}$ is a circulant matrix [11] and thus constant down the diagonals then shows that the generalised cross-validation function can be written as

$$V(\beta) = \frac{(1/M)\left\|(\mathbf{I} - \mathbf{B}_{\mathrm{trans}})\hat{\mathbf{p}}_{\mathrm{trans}}\right\|_e^2}{[(1/M)\,\mathrm{Tr}(\mathbf{I} - \mathbf{B}_{\mathrm{trans}})]^2} , \tag{25}$$

where Tr denotes the trace (sum of diagonal entries) of a matrix. By expressing $V(\beta)$ in terms of the eigenvalues of $\mathbf{B}_{\mathrm{trans}}$ and noting that these are the same as the eigenvalues of $\mathbf{B}(\beta)$ (see Golub *et al* [2]) also shows that the generalised cross-validation function can be written as

$$V(\beta)=\frac{(1/M)\|(\mathbf{I}-\mathbf{B}(\beta))\hat{\mathbf{p}}\|_e^2}{[(1/M)\ \mathrm{Tr}(\mathbf{I}-\mathbf{B}(\beta))]^2}\ . \tag{26}$$

Furthermore, it can be shown [12] that $V(\beta)$ is a weighted version of the ordinary cross-validation function $V_o(\beta)$ (equation (6)) such that

$$V(\beta)=\frac{1}{M}\sum_{k=1}^{M}\left[\frac{\hat{p}_k-p_k(\beta)}{1-b_{kk}}\right]^2 w_{kk}\ , \tag{27}$$

where

$$w_{kk}=\left(\frac{1-b_{kk}}{1-(1/M)\ \mathrm{Tr}\mathbf{B}(\beta)}\right)^2 . \tag{28}$$

Note that the term $(1/M)\mathrm{Tr}\mathbf{B}(\beta)$ evaluates the average of the sum of all diagonal elements of the influence matrix $\mathbf{B}(\beta)$. Thus, the weighting factor $w_{kk}(\beta)$ represents the contribution of each diagonal element relative to the sum of all diagonal elements of $\mathbf{B}(\beta)$. The generalised cross-validatory choice of the regularisation parameter β_{GCV} is made by minimising the function $V(\beta)$. Also note that when **B** is a circulant matrix (i.e., when **H** is circulant), $V(\beta)$ is identical to $V_o(\beta)$ since in this case w_{kk}=1. Finally consider the denominator and numerator of the function $V(\beta)$. The denominator is given by $[(1/M)\mathrm{Tr}[\mathbf{I}-\mathbf{H}(\mathbf{H}^{\mathrm{H}}\mathbf{H}+\beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}]]^2$ and thus evaluates the perturbation in the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ caused by the presence of regularisation parameter in improving the conditioning of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$. As β is increased, the denominator will become progressively smaller than unity, thus tending to increase $V(\beta)$. Also, the numerator is given by $(1/M)\|\hat{\mathbf{p}}-\mathbf{H}\mathbf{q}_R(\beta)\|_e^2$ and thus represents the residual sum of squares. Therefore, the function $V(\beta)$ evaluates both the error in the solution and the inaccuracy introduced into the matrix to be inverted by the inclusion of the regularisation parameter.

Since the GCV technique does not demand any prior knowledge, this seems to be a promising method for choosing the regularisation parameter in practical situations. However, in closing this section it should be mentioned that the GCV technique does not always produce satisfactory results. Some workers have made cautionary remarks as to the use of this technique. Wahba [12] stated that the GCV was quite likely to produce an unsatisfactory result if the noise components contaminating the measured data are highly correlated. This is a reflection of the basic assumption regarding the noise made at the starting point of the derivation of $V(\beta)$ [2], in which the noise components are assumed to be independent and identically distributed, with zero mean and the same variance. In addition, from a simulation study, Thompson *et al* [13] showed some empirical evidence of the possibilities for the unsatisfactory choice of the regularisation parameter when the GCV technique is employed. They stated that the potential problems with the $V(\beta)$ function fall into one or a combination of the following groups: $V(\beta)$ with multiple minima, $V(\beta)$ with no minimum for $\beta>0$, $V(\beta)$ for which it is hard to locate the global minimum numerically, and $V(\beta)$ whose global minimum produces an unsatisfactory β_{GCV}. It is therefore desirable to bear these points in mind when using the GCV technique.

2.5 AN EXAMPLE OF THE APPLICATION OF CROSS-VALIDATION TECHNIQUES

By way of introduction to the use of these techniques, we illustrate their use with a simple model (Figure 1) which was considered in reference [1] and consists of 9 point monopole sources and 9 microphones. It is assumed that only the source located central has unit volume velocity (say $1m^3/s$) and 10% measurement noise is added to acoustic pressures sensed at the microphones. The graphs of Figure 2 show the values of functions MSE(β), $V_o(\beta)$ and $V(\beta)$ defined by equations (3), (19) and (26) plotted against the regularisation parameters for the cases in which the source-to-microphone distance (r_{ms}) is different. The first row in figure 2 shows the results of the simulation when the microphone array is placed very close to the source array ($r_{ms} = 0.5\ r_{ss}$). As a consequence, the problem is very well conditioned and there is no real need for regularisation and $MSE(\beta)$ is in any case very small. The second, third and fourth rows show the results as the microphone array is moved progressively further away from the source array ($r_{ms} = 2r_{ss}$, $3r_{ss}$, $30r_{ss}$) and in these cases, clear minima are observed in the functions $V_0(\beta)$ and $V(\beta)$ (corresponding roughly to values of β of 10^{-3}, 10^{-4} and 10^{-5}). In all three cases, the values of β that minimise $V(\beta)$ are similar to those that minimise $V_0(\beta)$. In none of the cases is there a clear correspondence between the behaviour of $MSE(\beta)$ with $V(\beta)$ and $V_0(\beta)$, although in the second and third rows of results $MSE(\beta)$ obviously increases once the optimal values of are β exceeded. Reference to Figure 14 of reference [1] shows that the optimal values of β identified from the minima of $V(\beta)$ and $V_0(\beta)$ do indeed however give a reasonable estimate of the values of β necessary to best resolve the source distribution.

It is of course impossible to draw general conclusions from the results of this illustrative simulation, but it does at least demonstrate that for the type of case considered here, clear minima of $V_0(\beta)$ and $V(\beta)$ do exist and that the values deduced have the correct order of magnitude. Again, although in these cases there are no clear differences between the behaviour of $V_0(\beta)$ and $V(\beta)$, we shall generally continue to use the generalised cross validation function $V(\beta)$ since its superiority has been thoroughly argued by other workers [2, 10]. Furthermore, we will demonstrate with the computer simulations presented below, that the use of the GCV function $V(\beta)$ is highly effective in restoring the strength of acoustic sources when the inverse problem is badly conditioned.

3. THE CHOICE OF TIKHONOV REGULARISATION PARAMETER USED FOR RECONSTRUCTION OF SOURCE STRENGTH CROSS-SPECTRA

3.1 METHODS REQUIRING PRIOR KNOWLEDGE

Firstly, by analogy with the method based on the minimisation of the mean squared error given by equation (3), we can consider the residual $R_{qq}(\omega,\beta)$ between $\mathbf{S}_{qq}$ (the desired solution) and $\mathbf{S}_{qqR}$ (the Tikhonov regularised solution) defined by

$$R_{qq}(\omega,\beta) = \|\mathbf{S}_{qq}(\omega) - \mathbf{S}_{qqR}(\omega,\beta)\|_e . \qquad (29)$$

Hence a good regularisation parameter will be the value that minimises this residual. We denote this regularisation parameter by β_{qq}. Note that it is relatively easy to compute the value of the optimal regularisation parameter. Figure 3 shows an algorithm for finding β_{qq}. To start with, we read $\mathbf{H}(\omega)$ and $\mathbf{S}_{\hat{p}\hat{p}}(\omega)$ at the lower frequency ω_l of interest. Then the first trial regularisation parameter is given by the initial weight W_i multiplied by the minimum singular value of $\mathbf{H}^{\mathrm{H}}\mathbf{H}$. The reason for the use of the minimum singular value is that this is used in the denominator of the equation defining the condition number (see equations (35) and (39) in reference [1]). The condition number thus is more sensitive to the variation of the minimum singular value than that of other singular values. Using these data, the values of $\mathbf{S}_{qqR}(\omega,\beta)$ and $R_{qq}(\omega,\beta)$ are computed. After that, $\mathbf{S}_{qqR}(\omega,\beta)$ and $R_{qq}(\omega,\beta)$ are recursively computed by successively changing the value of weight W by a suitable stepsize ΔW from the initial value W_i to the final value W_f. Among the values of $R_{qq}(\omega,\beta)$, the minimum is sought and at this point the associated value of β becomes the regularisation parameter β_{qq} which we wish to find. Using the selected β_{qq}, we eventually obtain $\mathbf{S}_{qqR}$. This process is repeated until the upper frequency ω_u of interest.

With regard to this method, however, it should be noted that *a priori* knowledge of the contaminating noise is required. Using equation (2) and $\mathbf{S}_{\hat{p}\hat{p}} = \mathbf{S}_{pp} + \mathbf{S}_{nn}$, equation (29) can be developed into the form

$$R_{qq}(\omega,\beta) = \| \mathbf{S}_{qq} - \mathbf{D}(\mathbf{S}_{pp}+\mathbf{S}_{nn})\mathbf{D}^{\mathrm{H}}\|_e = \| \mathbf{S}_{qq} - \mathbf{D}\mathbf{S}_{pp}\mathbf{D}^{\mathrm{H}} - \mathbf{D}\mathbf{S}_{nn}\mathbf{D}^{\mathrm{H}}\|_e\,, \qquad (30)$$

where $\mathbf{D}=(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}$ and measurement noise $\mathbf{n}$ is assumed to be uncorrelated with the acoustic pressure $\mathbf{p}$. In this equation $\mathbf{D}\mathbf{S}_{pp}\mathbf{D}^{\mathrm{H}}$ does not contain measurement noise $\mathbf{S}_{nn}$ and in such a case regularisation is not required. Thus

$$\mathbf{D}\mathbf{S}_{pp}\mathbf{D}^{\mathrm{H}} = \{(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\,\mathbf{H}^{\mathrm{H}}\}\mathbf{S}_{pp}\{(\mathbf{H}^{\mathrm{H}}\mathbf{H})^{-1}\,\mathbf{H}^{\mathrm{H}}\}^{\mathrm{H}} = \mathbf{S}_{qq}\,. \qquad (31)$$

Accordingly, equation (30) becomes

$$R_{qq}(\omega,\beta) = \|-\mathbf{D}\mathbf{S}_{nn}\mathbf{D}^{\mathrm{H}}\|_e = \|\mathbf{D}\mathbf{S}_{nn}\mathbf{D}^{\mathrm{H}}\|_e\,. \tag{32}$$

Substituting $\mathbf{D}=(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\mathbf{H}^{\mathrm{H}}$ into equation (32) produces

$$R_{qq}(\omega,\beta) = \|\{(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\,\mathbf{H}^{\mathrm{H}}\}\mathbf{S}_{nn}\{(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\,\mathbf{H}^{\mathrm{H}}\}^{\mathrm{H}}\,\|_e\,. \tag{33}$$

It is obvious from this expression that we need *a priori* knowledge of the noise $\mathbf{S}_{nn}$ in order to find β_{qq} by minimising $R_{qq}(\omega,\beta)$. However, it is not guaranteed that this knowledge will be available in practical situations and therefore this approach is limited in its practical limitation.

As an alternative method for finding the regularisation parameter, we can consider another residual defined by

$$R_{pp}(\omega,\beta) \;=\; \|\mathbf{S}_{pp}(\omega) - \mathbf{H}\mathbf{S}_{qqR}\mathbf{H}^{\mathrm{H}}(\omega,\beta)\|_e\,. \tag{34}$$

Note that this approach is analogous to the method using the minimisation of the predicted mean squared error given by equation (4). In this approach the choice of a satisfactory regularisation parameter can also be made by minimising this residual $R_{pp}(\omega,\beta)$. We shall denote this regularisation parameter as β_{pp}. The detailed procedure for finding β_{pp} is the same as that described previously (i.e., as in Figure 3) except using $R_{pp}(\omega,\beta)$ instead of $R_{qq}(\omega,\beta)$. Also note that

$$R_{pp}(\omega,\beta) = \|\mathbf{H}(\mathbf{S}_{qq} - \mathbf{S}_{qqR})\mathbf{H}^{\mathrm{H}}\|_e\,, \tag{35}$$

and since it follows from equation (33) that

$$R_{pp}(\omega,\beta) = \|\{\mathbf{H}(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\,\mathbf{H}^{\mathrm{H}}\}\mathbf{S}_{nn}\{\mathbf{H}(\mathbf{H}^{\mathrm{H}}\mathbf{H} + \beta\mathbf{I})^{-1}\,\mathbf{H}^{\mathrm{H}}\}^{\mathrm{H}}\,\|_e\,, \tag{36}$$

it is also evident that this method also requires prior knowledge of the noise $\mathbf{S}_{nn}$.

3.2 APPLICATION OF GENERALISED CROSS-VALIDATION

Since the two previous methods require *a priori* knowledge of noise, we now consider the application of the GCV technique discussed previously to the restoration of $\mathbf{S}_{qq}$. The idea behind the adoption of the function $V(\beta)$ is that a regularisation parameter producing a satisfactory recovery of $\mathbf{q}$ should also yield a good reconstruction of $\mathbf{S}_{qq}$. Accordingly, after estimating the regularisation parameter based on the minimisation of the function $V(\beta)$ given by equation (26), we substitute this into equation (2). We employ the procedure shown in Figure 4 as an algorithm for identifying β_{GCV}. This procedure is similar to that providing β_{qq} of Figure 3. We first read $\mathbf{H}(\omega)$ and $\hat{\mathbf{p}}(\omega)$ at the lower frequency ω_l of interest. Then the first trial regularisation parameter is given by the initial weight W_i multiplied by the minimum singular value of $\mathbf{H}^{\mathrm{H}}\mathbf{H}$. Using these data, the value the generalised cross-validation function $V(\beta)$ is computed. After that, $V(\beta)$ are recursively computed by successively changing the value of weight W by a suitable stepsize ΔW from the initial value W_i to the final value W_f. From the values of $V(\beta)$, we find a particular β which produces the minimum $V(\beta)$. At this point, the value β becomes the regularisation parameter β_{GCV} which we wish to identify. Using the selected β_{GCV}, we eventually obtain $\mathbf{S}_{qqR}$. This process is repeated until the upper frequency ω_u of interest.

As already pointed out, even though β_{qq} and β_{pp} obtained by minimising R_{qq} and R_{pp} (equations (29) and (34)) require prior knowledge of the noise, we can use these and the resulting value of $\mathbf{S}_{qqR}(\beta_{qq})$ and $\mathbf{S}_{qqR}(\beta_{pp})$ as comparators to check the performance of β_{GCV} and the resulting $\mathbf{S}_{qqR}(\beta_{\mathrm{GCV}})$ reconstructed by the GCV technique.

4. DETERMINATION OF THE SINGULAR VALUES TO BE DISCARDED

As an alternative way to cope with the ill-conditioned acoustic inverse problem, the singular value discarding technique can be applied [1]. In this case, the singular value discarded solution estimating the vector of acoustic source is given by

$$\mathbf{q}_D = \mathbf{H}_D^+ \hat{\mathbf{p}} = (\mathbf{V}\Sigma_D^+\mathbf{U}^{\mathrm{H}})\,\hat{\mathbf{p}}, \tag{37}$$

where **U** and **V** are the matrices consisting of the left and right singular vectors of **H**, Σ_D is the matrix of singular values left after discarding some singular values, the superscript $^+$ denotes the pseudo-inverse, and $\mathbf{H}_D = \mathbf{U}\Sigma_D\mathbf{V}^{\mathrm{H}}$. Note that it is also possible to construct the cross-spectral matrix of acoustic source strengths from the expression

$$\mathbf{S}_{qqD} = \mathbf{H}_D^+ \mathbf{S}_{\hat{p}\hat{p}} (\mathbf{H}_D^+)^{\mathrm{H}} = (\mathbf{V}\Sigma_D^+\mathbf{U}^{\mathrm{H}})\,\mathbf{S}_{\hat{p}\hat{p}}\,(\mathbf{V}\Sigma_D^+\mathbf{U}^{\mathrm{H}})^{\mathrm{H}}, \tag{38}$$

In connection with the determination of the singular values to be truncated, some researchers suggest that it is suitable to discard the singular values below "machine epsilon" (as pointed out by Rothwell and Drachman [14], for example). However, it is important to recognise the fact that the conditioning of the matrix to be inverted is determined by not the absolute magnitudes of the singular values but the ratio of the largest to the smallest singular value. Poor conditioning can arise even if the smallest singular value is much larger than machine epsilon. It is therefore clear that this guide is not desirable. Powell and Seering [15] used the singular value discarding method when researching the problem of identifying multiple input forces (which served as sources of structural vibration) from multiple vibration measurements and the transfer functions between forces and vibration signals. In this study they rejected the singular values smaller than the error computed from coherence functions in association with the frequency response measurements. More recently, Krzanowski and Kline [16] proposed a way of determining the number of significant components in principal component analysis [17] by the use of the cross-validation technique. The principal components of principal component analysis are the same as the singular values of singular value decomposition. In this sense, the cross-validation technique used to determine the significant principal components can be applied to deciding the singular values to be eliminated in our problem.

Another way to determine the singular values to be removed is through the use of GCV which was also used in choosing a good regularisation parameter (section 2).

Golub *et al* [2] proposed the adoption of the GCV technique as a tool for choosing the significant principal components in association with principal component analysis. Consider the SVD of the M x N matrix $\mathbf{H}$, i.e., $\mathbf{H}=\mathbf{U}\Sigma\mathbf{V}^{\mathrm{H}}$ in which $\mathbf{U}$ is of order M x M, Σ is M x N, and $\mathbf{V}$ is N x N. From this matrix $\mathbf{H}$, a matrix $\mathbf{H}_v=\mathbf{U}\Sigma_v\mathbf{V}^{\mathrm{H}}$ is constructed in which Σ_v is the M x N diagonal matrix consisting of the 1st to the vth nonzero singular values of $\mathbf{H}$ (the (v+1)'th to the N'th singular values are set to zero). Thus

$$\Sigma_v = \left[\begin{array}{cccccc} s_1 & & & & & \\ & s_2 & & 0 & & \\ & & \ddots & & & \\ & & & s_v & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & 0 \\ \hdashline & & & 0 & & \end{array}\right] \left.\begin{array}{l} \left.\begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} v\text{x}N \\ \vphantom{\begin{array}{c}1\\1\\1\end{array}} \end{array}\right\} N\text{x}\mathrm{N} \\ \left.\vphantom{1}\right\}(M-N)\text{x}N \end{array}\right\} M\text{x}N \ . \qquad (39)$$

Based on this matrix $\mathbf{H}_v$, we construct a matrix $\mathbf{B}_v$ defined by

$$\mathbf{B}_v = \mathbf{H}_v(\mathbf{H}_v^{\mathrm{H}}\mathbf{H}_v)^{+}\mathbf{H}_v^{\mathrm{H}} \ . \qquad (40)$$

Note that the matrix $\mathbf{B}_v$ is similar to the influence matrix $\mathbf{B}(\beta)$ which was used in association with the determination of the regularisation parameter β_{GCV}. Equation (40) can be expanded out as

$$\mathbf{B}_v = \mathbf{U}\Sigma_v\mathbf{V}^{\mathrm{H}}(\mathbf{V}\Sigma_v^{\mathrm{H}}\mathbf{U}^{\mathrm{H}}\mathbf{U}\Sigma_v\mathbf{V}^{\mathrm{H}})^{+}\mathbf{V}\Sigma_v^{\mathrm{H}}\mathbf{U}^{\mathrm{H}} , \qquad (41)$$

which results in

$$\mathbf{B}_v = \mathbf{U}\Sigma_v\mathbf{V}^{\mathrm{H}}\mathbf{V}(\Sigma_v^{\mathrm{H}}\Sigma_v)^{+}\ \mathbf{V}^{\mathrm{H}}\mathbf{V}\Sigma_v^{\mathrm{H}}\mathbf{U}^{\mathrm{H}} , \qquad (42)$$

and can thus be written as

$$\mathbf{B}_v = \mathbf{U}\Sigma_v(\Sigma_v^{\mathrm{H}}\Sigma_v)^{+}\Sigma_v^{\mathrm{H}}\mathbf{U}^{\mathrm{H}} , \qquad (43)$$

by using the relations $(\mathbf{UAV})^{+}=\mathbf{V}^{\mathrm{H}}\mathbf{A}^{+}\mathbf{U}^{\mathrm{H}}$, $\mathbf{U}^{\mathrm{H}}\mathbf{U}=\mathbf{I}$ and $\mathbf{V}^{\mathrm{H}}\mathbf{V}=\mathbf{I}$ [11] and noting that $\mathbf{A}$ is an M x N matrix. The matrix $\mathbf{B}_v$ can therefore be expressed as

$$\mathbf{B}_v = \mathbf{U}\mathbf{I}_v\mathbf{U}^{\mathrm{H}}, \tag{44}$$

where the matrix $\mathbf{I}_v$ is given by

$$\mathbf{I}_v = \Sigma_v(\Sigma_v^{\mathrm{H}}\Sigma_v)^{+}\Sigma_v^{\mathrm{H}}$$

$$
= \left[\begin{array}{ccccccc}
1 & & & & & & \\
& 1 & & & 0 & & \\
& & \ddots & & & & \\
& & & 1 & & & \\
& & & & 0 & & \\
& & & & & \ddots & \\
& & & & & & 0 \\
\hdashline
& & 0 & & & 0 & \\
& & & & & & \ddots \\
& & & & & & 0
\end{array}\right]
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\ \ddots\\1\end{array}}\right\} v\mathrm{x}N \\
\left.\vphantom{\begin{array}{c}0\\ \ddots\\0\end{array}}\right\} N\mathrm{x}N \\
\left.\vphantom{\begin{array}{c}0\\ \ddots\\0\end{array}}\right\} M\mathrm{x}M
\end{array}
\tag{45}
$$

Note in this equation that the (N+1)'th to the M'th diagonal components are originally zeros, because $\mathbf{H}$ is an M x N rectangular matrix and thus its singular value matrix Σ has (M-N) x N zero components as shown in equation (39). (For the case of an M x M square $\mathbf{H}$ matrix, there are no (M-N) x N zero components). Finally define a function V_v of $\mathbf{B}_v$ as

$$V_v = \frac{(1/M)\|[\mathbf{I}-\mathbf{B}_v]\hat{\mathbf{p}}\|_e^2}{[(1/M)\,\mathrm{Tr}[\mathbf{I}-\mathbf{B}_v]]^2}, \tag{46}$$

which is the same with $V(\beta)$ given by equation (26) except that $\mathbf{B}_v$ is used instead of $\mathbf{B}(\beta)$. Golub *et al* [2] suggested that the significant principal components could be decided by choosing $\mathbf{B}_v$ for which V_v was smallest. According to this idea, we can apply the GCV technique to the determination of the singular values to be eliminated in our problem. That is to say, we can discard some singular values as follows. At first we calculate the value of V_v by using $\mathbf{B}_v$ in which all diagonal components of $\mathbf{I}_v$ are unity except the N'th diagonal component set equal to zero. Then the same calculation is undertaken again by using another $\mathbf{B}_v$ where the (N-1)'th and N'th diagonal components of $\mathbf{I}_v$ are replaced by zeros. This process is repeated up to the case of $\mathbf{B}_v$ in which the 3rd to the N'th diagonal components of $\mathbf{I}_v$ are set to zero. After that, among the calculated values (i.e., N-2) of V_v, we find the minimum. Finally, the number of the smallest singular values to be discarded from the matrix $\mathbf{H}$ is made equal to the number of zeros contained in the 1st to the N'th diagonal components of the matrix $\mathbf{I}_v$ associated with the minimum value of V_v. Also note that using the properties of trace of the matrix [11], we can express $\mathrm{Tr}[\mathbf{I}\text{-}\mathbf{B}_v]=\mathrm{Tr}[\mathbf{I}]\text{-}\mathrm{Tr}[\mathbf{B}_v]$, $\mathrm{Tr}[\mathbf{I}]=M$ (where $\mathbf{I}$ is M-dimensional identity matrix), and $\mathrm{Tr}[\mathbf{B}_v]=\mathrm{Tr}[\mathbf{U}\mathbf{I}_v\mathbf{U}^{\mathrm{H}}]=\mathrm{Tr}[\mathbf{I}_v\mathbf{U}^{\mathrm{H}}\mathbf{U}]=\ \mathrm{Tr}[\mathbf{I}_v]=v$. Thus, equation (46) can be written as

$$V_v = \frac{(1/M)\|[\mathbf{I}-\mathbf{B}_v]\hat{\mathbf{p}}\|_e^2}{[1-v/M]^2}\ . \qquad (47)$$

The denominator of this expression evaluates the inaccuracy caused by discarding singular values to improve the conditioning of matrix $\mathbf{H}$. Also, the numerator evaluates the residual sum of squares produced by the perturbation in the matrix $\mathbf{H}$, since this is given by $(1/M)\|[\mathbf{I}-\mathbf{H}_v(\mathbf{H}_v^{\mathrm{H}}\mathbf{H}_v)^{+}\mathbf{H}_v^{\mathrm{H}}]\hat{\mathbf{p}}\|_e^2$. It is therefore clear that the function V_v determines the singular values to be discarded by considering both these factors in a similar way to the GCV function above.

Another possible discarding technique is based on the singular value distribution. That is to say, the determination of the singular values to be removed is made by looking into the relative magnitude of the singular values. Thus, this method demands empirical experience to some extent. This utilises only the singular value distribution obtainable from the transfer function matrix **H**. We will refer to this as the *singular value distribution based discarding* technique and we will describe this in more detail in the next section.

5. COMPUTER SIMULATION OF THE RECONSTRUCTION OF VOLUME VELOCITIES

5.1 VOLUME VELOCITY OF A RANDOMLY VIBRATING PLATE

We now wish to investigate how the theories developed above can be used to reconstruct the volume velocities of a randomly vibrating simply supported rectangular plate mounted in an infinite baffle (Figure 5). In order to check the accuracy of the volume velocities reconstructed by inverse techniques, the values to be used as comparators are computed directly as follows. The simply supported rectangular plate illustrated in Figure 5 is divided into 144 (12x12 along x- and y-directions) contiguous rectangular piston elements of the same area. The real plate model can be considered to be made up from a number of point monopole sources. Firstly, in order to find the local surface velocity at the individual centre points of 144 elements resulting from the structural vibration of this plate, we excite the plate by a normally distributed random force at the excitation point (Figure 5). For harmonic excitation $Fe^{j\omega t}$ at a point (x_o, y_o), the surface velocity v_s at an arbitrary position (x, y) of the simply supported plate is expressed as [18]

$$v_s(x,y,\omega) = -\frac{j4\omega F}{M_p}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin(m\pi x_o / L_x)\sin(m\pi x / L_x)\sin(n\pi y_o / L_y)\sin(n\pi y / L_y)}{\omega^2 - \omega_{mn}^2} \tag{48}$$

where M_p is the total mass of the plate, m and n are the number of half-waves of the modal shapes along x- and y-directions, and ω_{mn} is the natural frequency. Then the volume velocity q_i of each element is computed by

$$q_i = v_s \Delta L_x \Delta L_y\ , \tag{49}$$

where ΔL_x (here $L_x/12$) and ΔL_y (here $L_y/12$) denote the dimensions of a rectangular piston element in the x- and y-directions. Based on this discretised model, we here construct two simulation models as shown in Figure 6(a) and (b): 4 and 144 volume velocity source and microphone models. Whilst the individual volume velocities of the 144 source model are obtained by equation (49), those of the 4 source model are found from

$$q_i = \sum_{j=1}^{36} (v_s)_j \Delta L_x \Delta L_y \,. \tag{50}$$

Needless to say, these volume velocities can be found more accurately by discretising the plate into smaller rectangular piston elements. In the meantime, these two models are built on the notion of equivalent point monopole sources and so the highest frequency to be analysed is confined by the condition of ka<<1, where $k=\omega/c_0$ is the wavenumber and a is the typical dimension of the source element. That is, the highest frequencies to be analysed by using the point monopole models should be below 574Hz ($a=L_x/4$) and 3448Hz ($a=L_x/24$) for 4 and 144 volume velocity source models, respectively.

5.2 METHODS FOR COMPUTING THE ACOUSTIC PRESSURE FIELD

As can be seen from equation (2), we need the acoustic pressure auto- and cross-spectra to reconstruct the volume velocity auto- and cross-spectra. In order to obtain the acoustic pressure field radiated by a randomly vibrating simply supported plate mounted in an infinite baffle, we can use the Rayleigh integral [19] employing the modal behaviour of the plate, the method using the wavenumber Fourier transform [18], and Green function approach. Of these, we employ the Green function approach. The acoustic pressure field is found by first multiplying the Green function with the volume velocities of individual rectangular piston elements which are regarded as point monopole sources and then adding contributions from all the piston elements. This approach can be expressed by **p=Hq** where **H** contains the Green functions relating each source element to each measurement position.

5.3 RECONSTRUCTION BY THE LEAST SQUARES METHOD

We first assume that there is no measurement noise corrupting the acoustic pressures and there is no error included in the acoustic transfer functions. Under these conditions, the performance of the least squares technique is tested. Figure 6 shows the geometrical arrangement of the simply supported plate and microphones to detect the acoustic field for the two simulation models. This placement is chosen to make the condition number of the matrix $\mathbf{H}^H\mathbf{H}$ as small as possible, utilising the results of the study of the behaviour of the condition number in the acoustic inverse problem [1]. That is to say, the microphone array is placed symmetrically with respect to the discretised source array, with the same pattern as the discretised source array, and with the microphone-to-microphone distance set equal to the source-to-source distance. Figure 7 illustrates the condition numbers of the matrix $\mathbf{H}^H\mathbf{H}$ for the two models.

The volume velocities reconstructed by using the least squares method (equation (31) in reference [1]) for the 4 source and 4 microphone model are compared with the desired values in Figure 8. Their magnitudes and phases are in extremely good agreement with the desired values. (Note that these results are normalised by unit force input to the plate). The results of Figure 9(a) and (b) compare the spatial distribution of the reconstructed volume velocity auto-spectra with the desired values at the two resonant frequencies, for the case of the 144 source and 144 microphone model. The volume velocity auto-spectra coincide with the principal diagonal elements of the 144-by-144 matrices S_{qq} and S_{qqo}. Note that in Figure 9(a) and (b) the individual nodal points correspond to the centre points of each of the 144 rectangular elements and the magnitudes are the squares of volume velocity (per unit force). Thus these shapes do not match exactly to the structural modes. The recovered phases of S_{qqo} obtained by using the least squares method are compared with the desired values S_{qq} in Figure 9(c) and (d). It is clear from Figure 9 that both magnitude and phase are recovered with extremely good accuracy. The rest of results not shown here also revealed excellent reconstruction. The least squares method enables the volume velocities to be reconstructed extremely well as long as it is assumed that measurement noise is not present in the acoustic pressures and error is not included in the acoustic transfer functions.

The effect of measurement noise on the degradation of the restoration of volume velocities is now investigated. Each signal-to-noise ratio (SNR) for the two models is plotted in Figure 10. Of these, Figure 10(b) shows the SNR of the 144 source and microphone model only at the resonant frequencies. With the presence of noise, volume velocities are recovered using the same geometrical configurations of the plate and microphones as previously. The respective outcomes for the two models are illustrated in Figures 11 and 12. Measurement noise forces the reconstructed results to deviate from the desired values and the amount of discrepancy is larger at lower frequencies since the conditioning (Figure 7) of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ becomes worse as the frequency decreases. Observing the results of Figure 11, the reconstruction is made better in the neighbourhood of the structural resonant frequencies than in the neighbourhood of the anti-resonant frequencies, because of the relatively small amount of measurement noise (see Figure 10(a)). This therefore illustrates that the measure of degradation of the accuracy of reconstruction by the simple least squares method is proportional to the combination of the conditioning of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ to be inverted and the amount of noise.

5.4 RECONSTRUCTION BY TIKHONOV REGULARISATION

In order to enhance the results reconstructed poorly by the least squares method, we first apply Tikhonov regularisation. Among the methods for choosing the regularisation parameters proposed in section 2, we use the methods based on the minimisation of the residual R_{qq} (equation (29)) and the generalised cross-validation function $V(\beta)$ (equation (26)). Utilising these methods, we find β_{qq} and β_{GCV} by following the procedure presented in Figures 3 and 4, respectively. Although the regularisation parameter β_{qq} can be chosen only when having prior knowledge of either the acoustic source strength or the noise, this is used here as a comparator to check the performance of β_{GCV} which is determined without such prior knowledge. The results of Figure 13 plot the regularisation parameters for the two simulation models. As we have noted in section 2, the determination of β_{qq} by minimising R_{qq} is affected by the degree of noise (refer to equation (33)). For this reason, it is clearly observed from Figures 10 and 13 that the regularisation parameters in the frequency range in which SNR is low are large relative to the other frequency range. Also, since the regularisation parameter is chosen based on the minimum singular values of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ at the individual frequencies, the overall trend of the optimal regularisation parameter increases as frequency increases. Note that the singular values of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ increase as frequency increases (see Figure 17, for example). The reason for this can be clearly understood from equation (47) in reference [1] which expresses the eigenvalues of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ (i.e., the singular values of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$) for the two point source and two microphone model. This shows that the eigenvalues depend on the product of frequency and distance factors. The plot of β_{qq} shows discontinuity at some frequencies, indicating that no regularisation is recommended because the volume velocities have already been restored close to the desired values at these frequencies (see Figure 11). The use of regularisation parameters at these frequencies makes the condition numbers of the matrix $\mathbf{H}^{\mathrm{H}}\mathbf{H}$ decrease as shown in Figure 14.

By the virtue of the improved conditioning, the results of Figure 15 demonstrate that the reconstructed magnitudes and phases of volume velocities for the 4 source and 4 microphone model approach much closer to the desired values, compared with those achieved by the unregularised least squares solution. However, the use of β_{GCV} degrades the accuracy beyond about 420Hz. This suggests that the GCV technique will not always produce a satisfactory reconstruction. Observing the results of Figure 15 also reveals that whilst there is a good agreement in the neighbourhood of some anti-resonant frequencies, there also is still unwanted deviation at the others. This is caused by the relatively large condition number (the SNRs are similar at the anti-resonant frequencies, see Figure 10(a)).

The results of Figure 16 compare the volume velocity auto-spectra and the phases of cross-spectra reconstructed by the unregularised least squares solution and the Tikhonov regularised solution using β_{qq} and β_{GCV} for the 144 source 144 microphone model. The results shown are for the (2,2) resonant frequency of the plate.

It can be seen that Tikhonov regularisation improves the accuracy of reconstruction achieved poorly by the unregularised solution. From the overall point of view, in the two models, the volume velocities recovered by both β_{qq} and β_{GCV} are similar to each other. Judging from the above results, we can recognise that the GCV technique is available in practical situations to enhance the reconstruction accuracy, since β_{GCV} is chosen without prior knowledge of either the volume velocities or the noise.

5.5 RECONSTRUCTION BY SINGULAR VALUE DISCARDING

Now we investigate the performance of the singular value discarding method in restoring the volume velocities. In order to determine which singular values should be removed, we make use of both the singular value distribution of the matrix to be inverted and the GCV function which was proposed in section 4. We denote the matrices consisting of the singular values which are left after discarding based on these methods as $\mathbf{H}_D$ and $\mathbf{H}_v$, respectively.

Plotted in Figure 17 are the singular values of the matrix $\mathbf{H}$ as a function of frequency for the two simulation models (recall that the number of singular values is equal to that of the discretised sources). For the simulation model consisting of 4 sources and 4 microphones, the singular values below $5\text{x}10^2$ are eliminated. The reason for doing this is in connection with the results (Figure 11) reconstructed by the least squares method. That is, these restored results are not good below about 260Hz and it was anticipated that if we truncate the singular values below $5\text{x}10^2$, then the reconstruction accuracy will be improved up to this frequency. As a result, the conditioning of the matrix to be inverted is improved to $\kappa(\mathbf{H}_D)$ from $\kappa(\mathbf{H})$, as graphed in Figure 18(a). Subsequently the singular value discarded solution $\mathbf{S}_{qqD}$ (equation (38) in reference [1]) enhances the magnitudes and phases of volume velocities restored unsatisfactorily by the simple least squares solution $\mathbf{S}_{qqo}$, as shown in Figure 19. The conditioning of the 144 source and 144 microphone model is ameliorated (Figure 18(b)) by eliminating the 26th to 144th singular values (Figure 17(b)). For this reason, the acoustic source strength distributions at 444Hz corresponding to the (2,2) structural mode are well reconstructed (Figure 20). Other results which are not presented here showed similar reconstruction. The above results which are based on the singular value distribution of the matrix **H** are used as the comparator to examine the capability of the singular value discarding technique using the function V_v given by equation (46).

The application of the GCV technique is undertaken as follows. Firstly, the last diagonal component of the identity matrix $\mathbf{I}_v$ (equation (45)) is removed. Note that the last components of $\mathbf{I}_v$ of the two models are 4 and 144, respectively. Then the value of the function V_v is calculated. This calculation is carried out again for the case in which two of the last diagonal components of $\mathbf{I}_v$ are set by zero. This process is repeated until the third to last diagonal components of $\mathbf{I}_v$ are replaced by zero. Of the calculated values of the function V_v (i.e., the 2 and 142 values of the function V_v for the respective models), we select the minimum value. At this point, if the associated $\mathbf{I}_v$ producing the minimum V_v has v unities from the 1st to the v'th element on its diagonal, then the (v+1)'th to the last singular values of the matrix **H** (M x N, $M \geq N$) are truncated. Thus, the matrix **H** transforms into $\mathbf{H}_v$. This is repeated at each component of frequency to be analysed. The results of Figure 18 show the condition number $\kappa(\mathbf{H}_v)$ obtained after this process. The results of Figures 19 and 20 illustrate a part of the reconstructed results of acoustic source strengths for the two models. The results of Figure 19 for the 4 source and microphone model approach the desired values very closely, compared to those achieved by the least squares method. Interestingly, the magnitudes and phases of acoustic source strengths recovered by using $\mathbf{H}_v$ are similar to those obtained by using $\mathbf{H}_D$. Of course, this is due to the similar conditioning of the matrices $\mathbf{H}_D$ and $\mathbf{H}_v$ (Figure 18(a)). Finally, the results obtained by using β_{GCV} for the 144 source and 144 microphone model reveal that whereas the magnitude reconstruction is acceptable, the phase reconstruction is still undesirable (Figure 20).

6. CONCLUSIONS

In association with the choice of regularisation parameter, we have suggested the use of three regularisation parameters, namely, β_{qq}, β_{pp}, and β_{GCV}. Of these, the determination of β_{qq} and β_{pp} is based on the minimisation of the difference between the "desired" acoustic source strength auto- and cross-spectra and the "estimated" values and the "desired" acoustic pressure auto- and cross-spectra and the "estimated" values, respectively. They therefore need prior knowledge of either the acoustic source or the noise. Nevertheless, they are useful as comparators to check the performance of the regularisation parameters β_{OCV} and β_{GCV} which are chosen by using the ordinary cross-validation and the generalised cross-validation technique which do not require such prior knowledge. Regarding the truncation of singular values, we have presented the method using the singular value distribution of the matrix to be inverted, ordinary cross-validation and generalised cross-validation. Through simulations using a simply supported plate mounted in an infinite baffle, we have confirmed the fact that the least squares method can reconstruct the volume velocity distributions and their interactions extremely well if and only if no measurement noise and no error included in the transfer functions are assumed. For the more realistic models including the effect of noise, it has been observed that the application of Tikhonov regularisation or singular value discarding to an ill-conditioned system can provide considerable improvement in the accuracy of reconstruction. Also, the GCV technique has been seen to be a practical tool for choosing properly the regularisation parameter and the singular values to be truncated. However, stress should be laid on the fact that the GCV technique does not always produce a successful choice of those values. More work is required to clearly delineate the appropriateness of the GCV technique in the current context.

ACKNOWLEDGEMENT

This research was financially supported by the Daewoo Motor Company, Korea.

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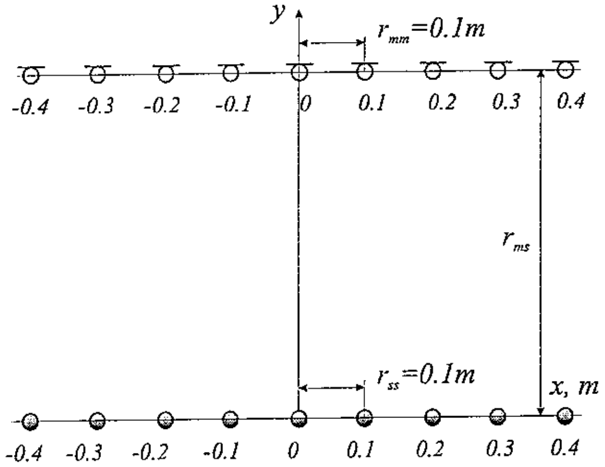

Figure 1. A geometrical arrangement of 9 point monopole sources and 9 microphones.

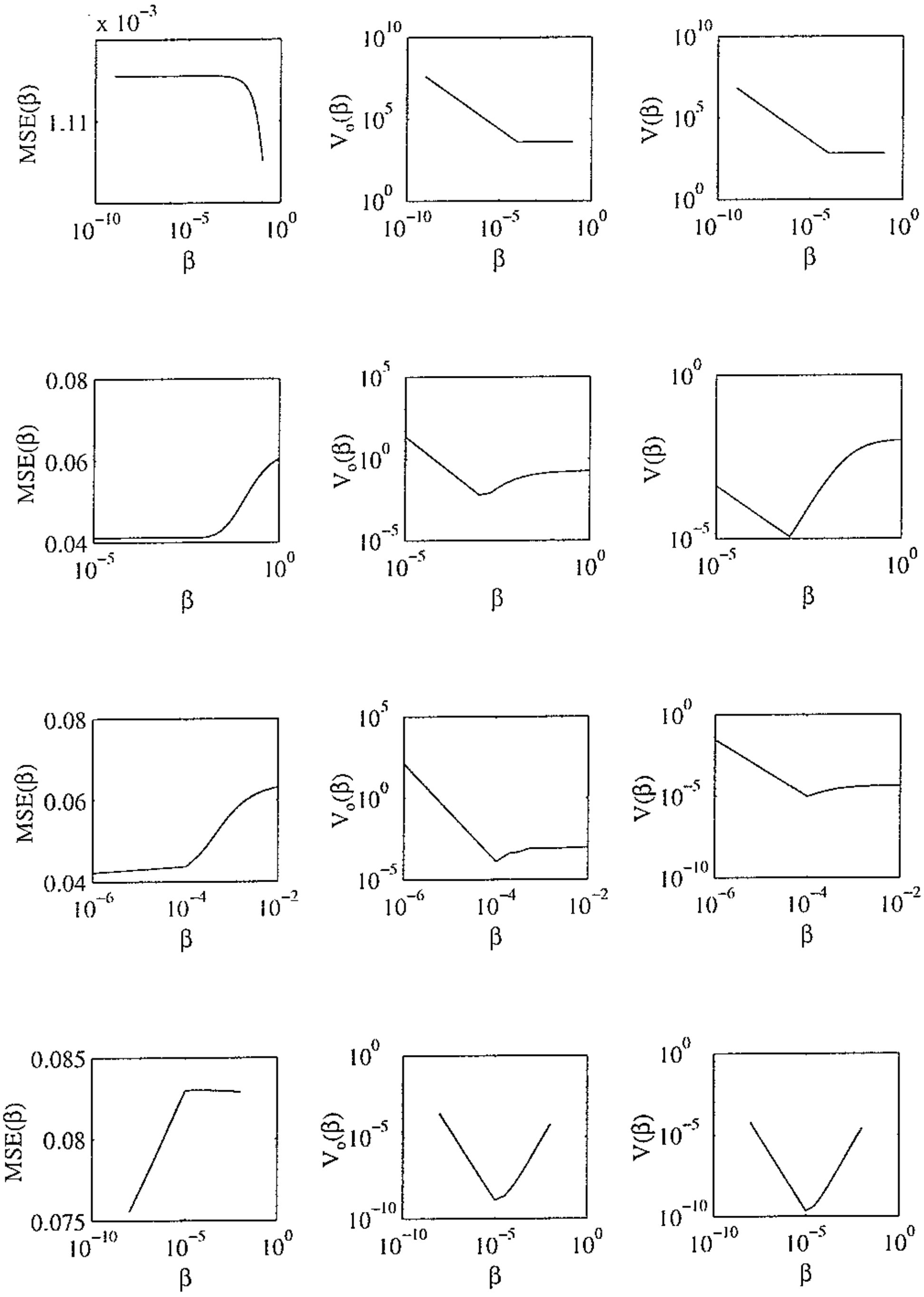

Figure 2. Comparison of MSE(β), $V_o(\beta)$ and $V(\beta)$ for the model of Figure 1: r_{ms}=0.5r_{ss} (the 1st row), 2r_{ss} (the 2nd row), 3r_{ss} (the 3rd row). 30r_{ss} (the 4th row), r_{ss}=0.1m.

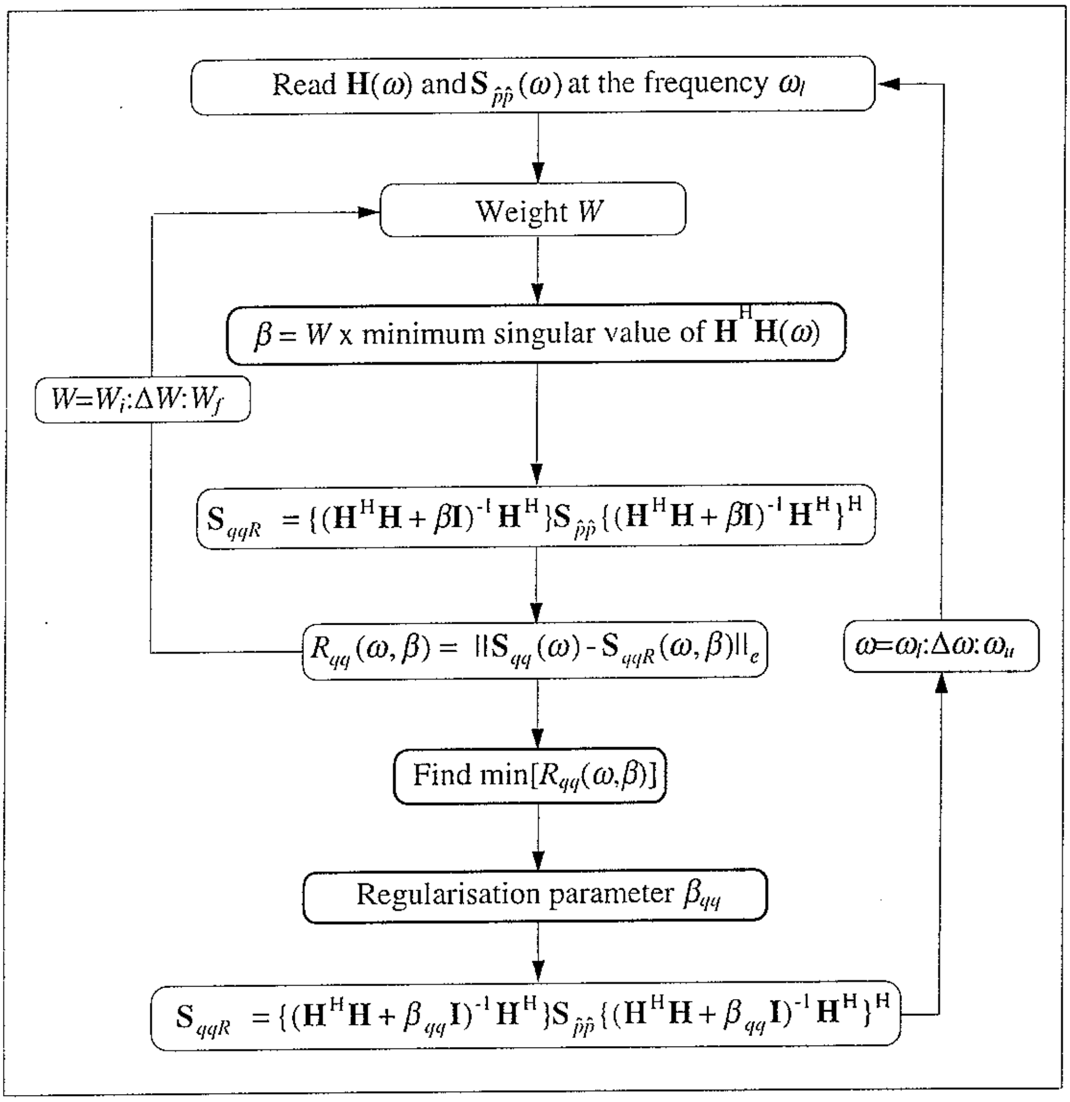

Figure 3. A method of finding the regularisation parameter β_{qq} by minimising the residual $R_{qq}(\omega,\beta)$ between the desired solution S_{qq} and the regularised solution S_{qqR}.

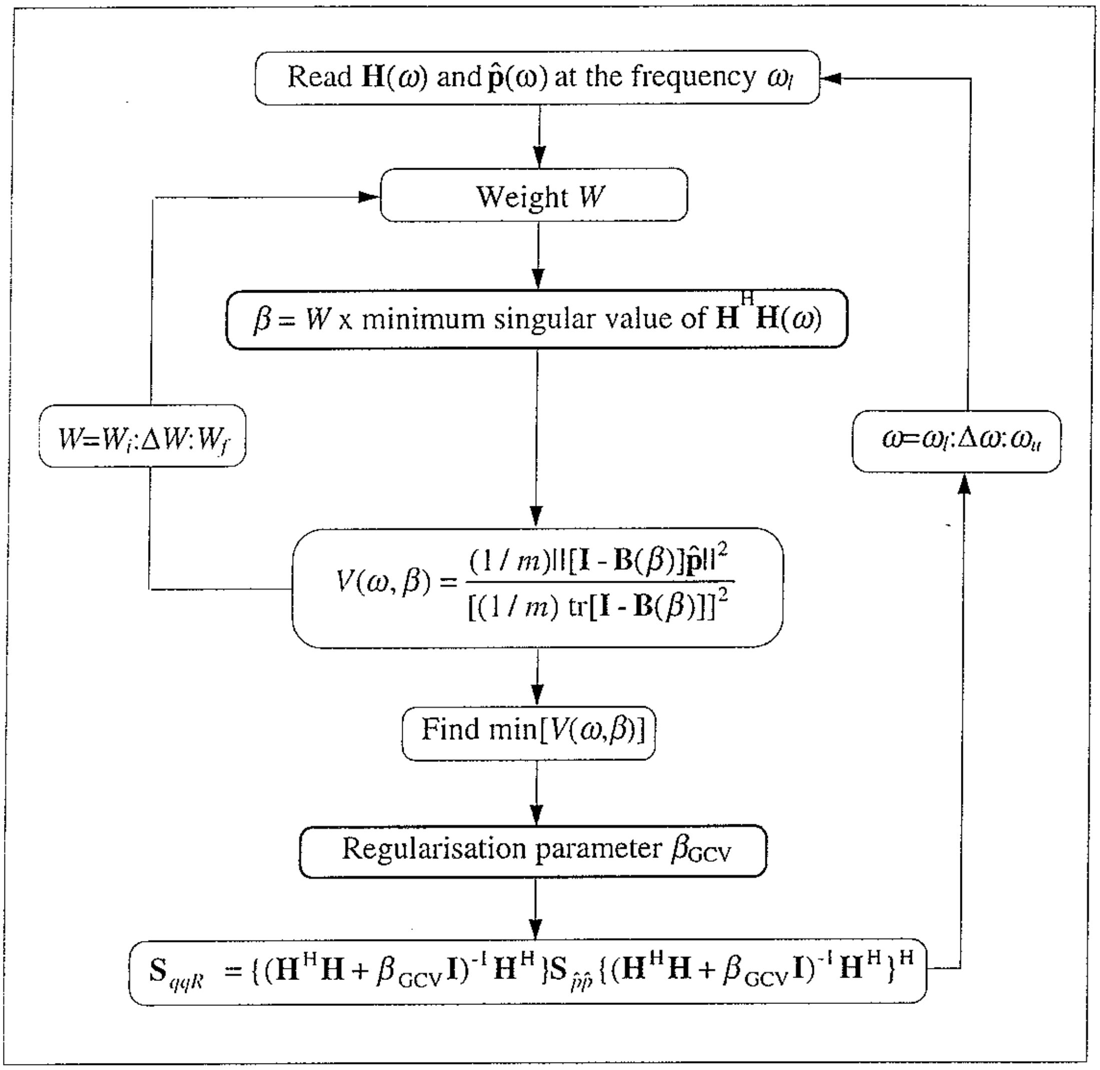

Figure 4. A method of finding the regularisation parameter β_{GCV} by minimising the generalised cross-validation function $V(\beta)$.

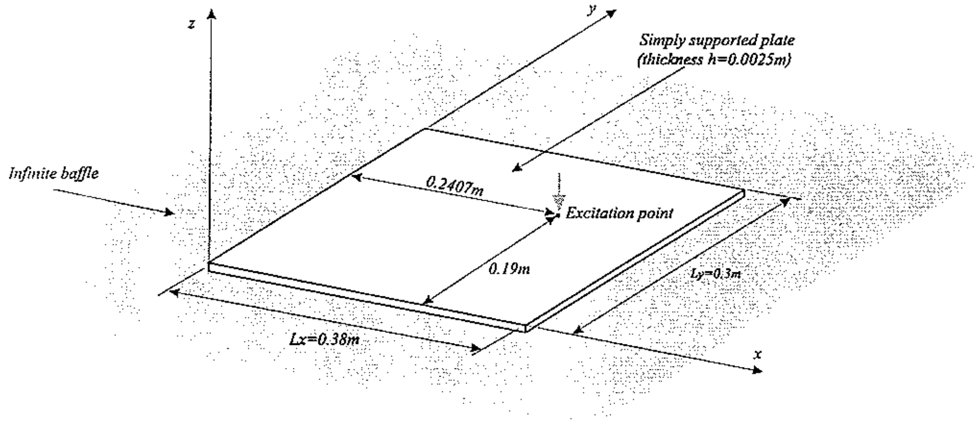

Figure 5. Geometry of a simply supported rectangular plate mounted in an infinite baffle used for the computer simulation.

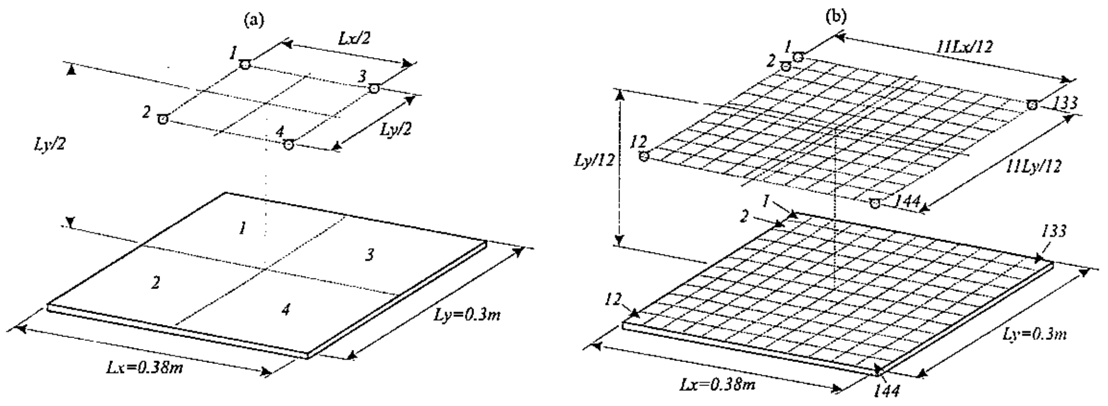

Figure 6. Geometrical arrangement of the discretised plate and microphones: (a) 4 source and 4 microphone model and (b) 144 source 144 microphone model.

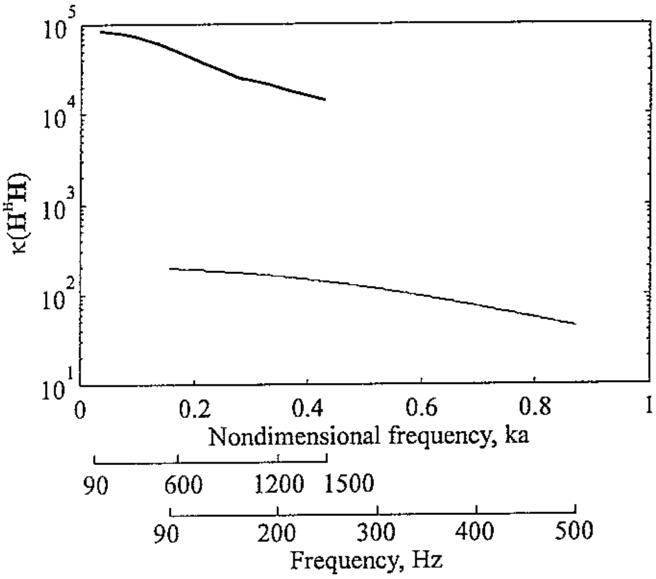

Figure 7. Condition numbers of the matrix $\mathbf{H}^H\mathbf{H}$ of 4 source and 4 microphone model (thin solid, $a=L_x/4$) and 144 source and 144 microphone model (thick solid, $a=L_x/24$).

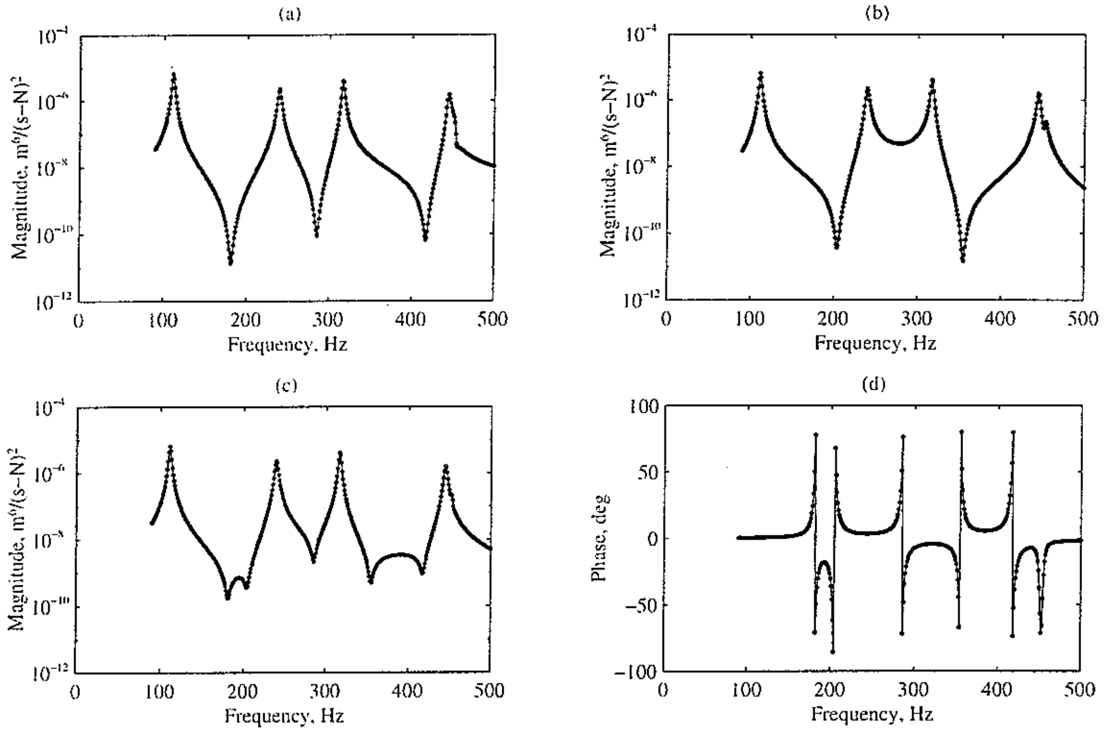

Figure 8. A comparison of the desired (solid) and reconstructed (by the least squares solution, dotted) volume velocity (per unit force) auto-spectra of sources (a) 3, (b) 4 and cross-spectra between sources 3 and 4 ((c) magnitude, (d) phase) of 4 sources and 4 microphones. No measurement noise is assumed.

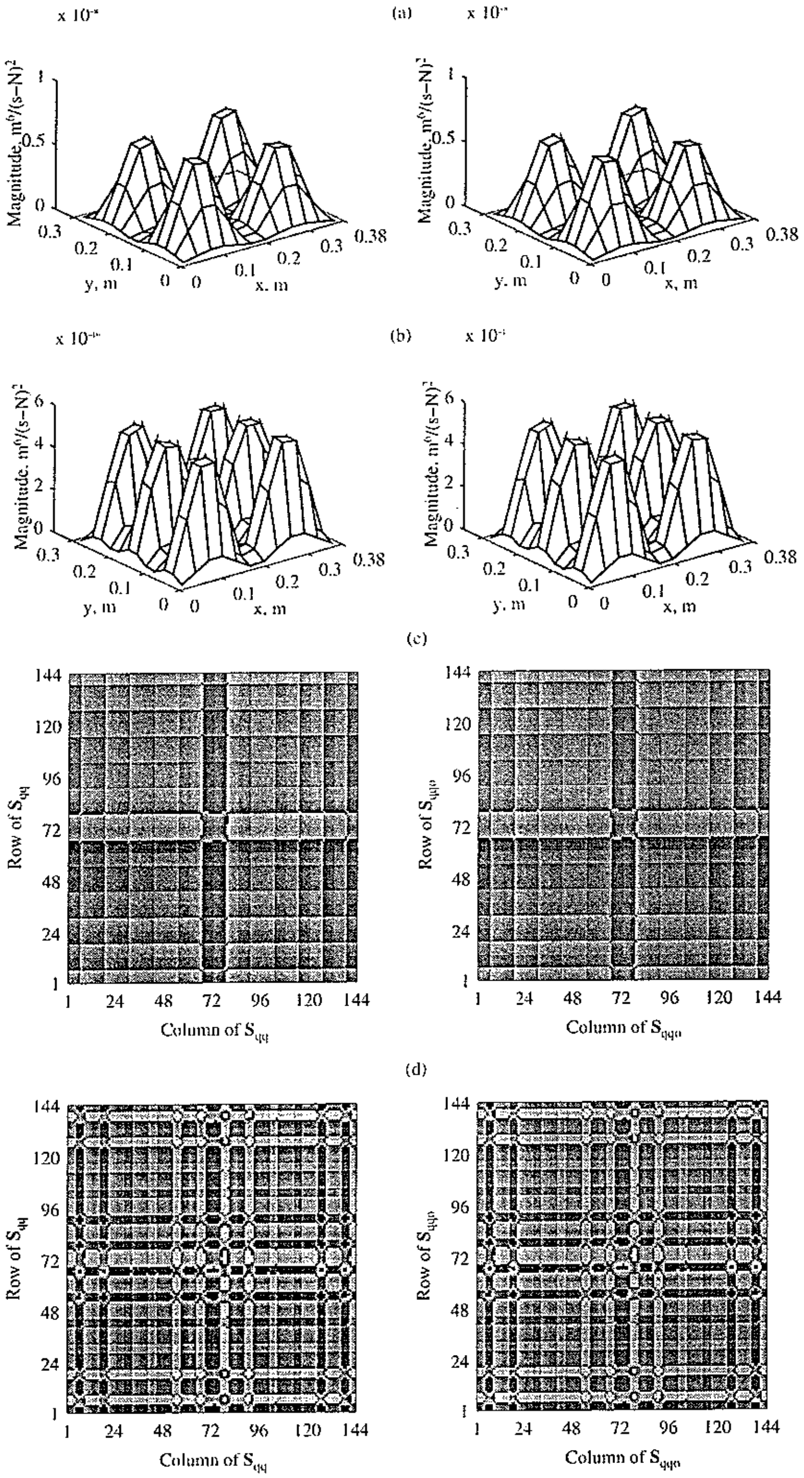

Figure 9. A comparison of the desired (left) and reconstructed (by the least squares solution, right) volume velocities (per unit force) of the 144 source and 144 microphone model. No measurement noise is assumed: (a), (c) auto-spectra and phase of cross-spectra at the frequency of (2,2) (444Hz, *ka*=0.129) structural mode, (b), (d) auto-spectra and phase of cross-spectra at the frequency of (2,3) (786Hz, *ka*=0.228) structural mode.

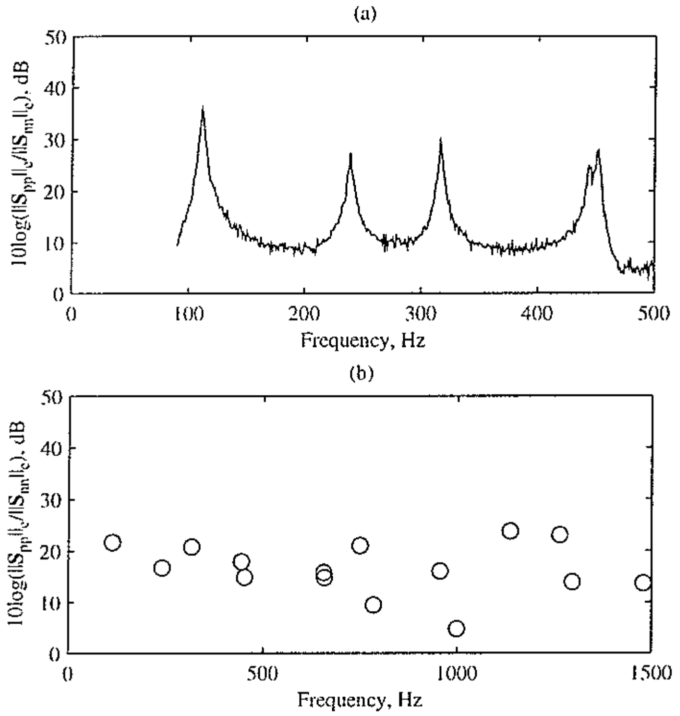

Figure 10. The ratio of acoustic pressure to noise for (a) 4 source and 4 microphone model and (b) 144 source and 144 microphone model.

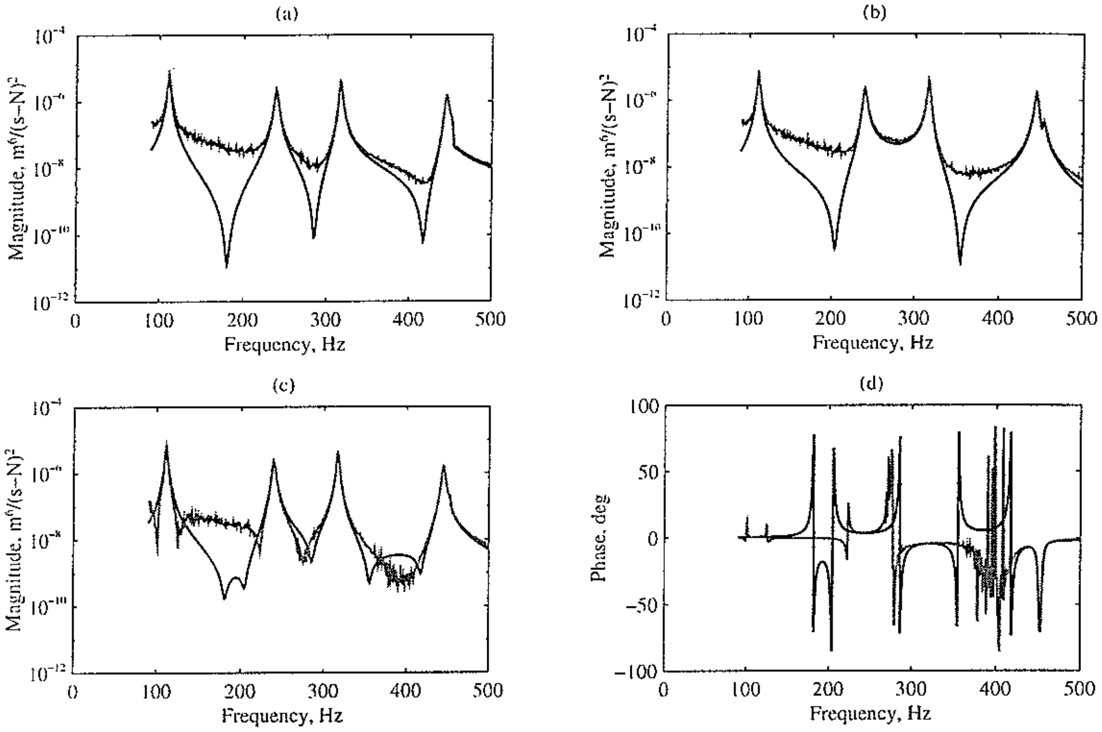

Figure 11. A comparison of the desired (black) and reconstructed (by the least squares solution, grey) volume velocity (per unit force) auto-spectra of sources (a) 3, (b) 4 and cross-spectra between sources 3 and 4 ((c) magnitude, (d) phase) of 4 source and 4 microphone model. Measurement noise (Figure 10(a)) is present.

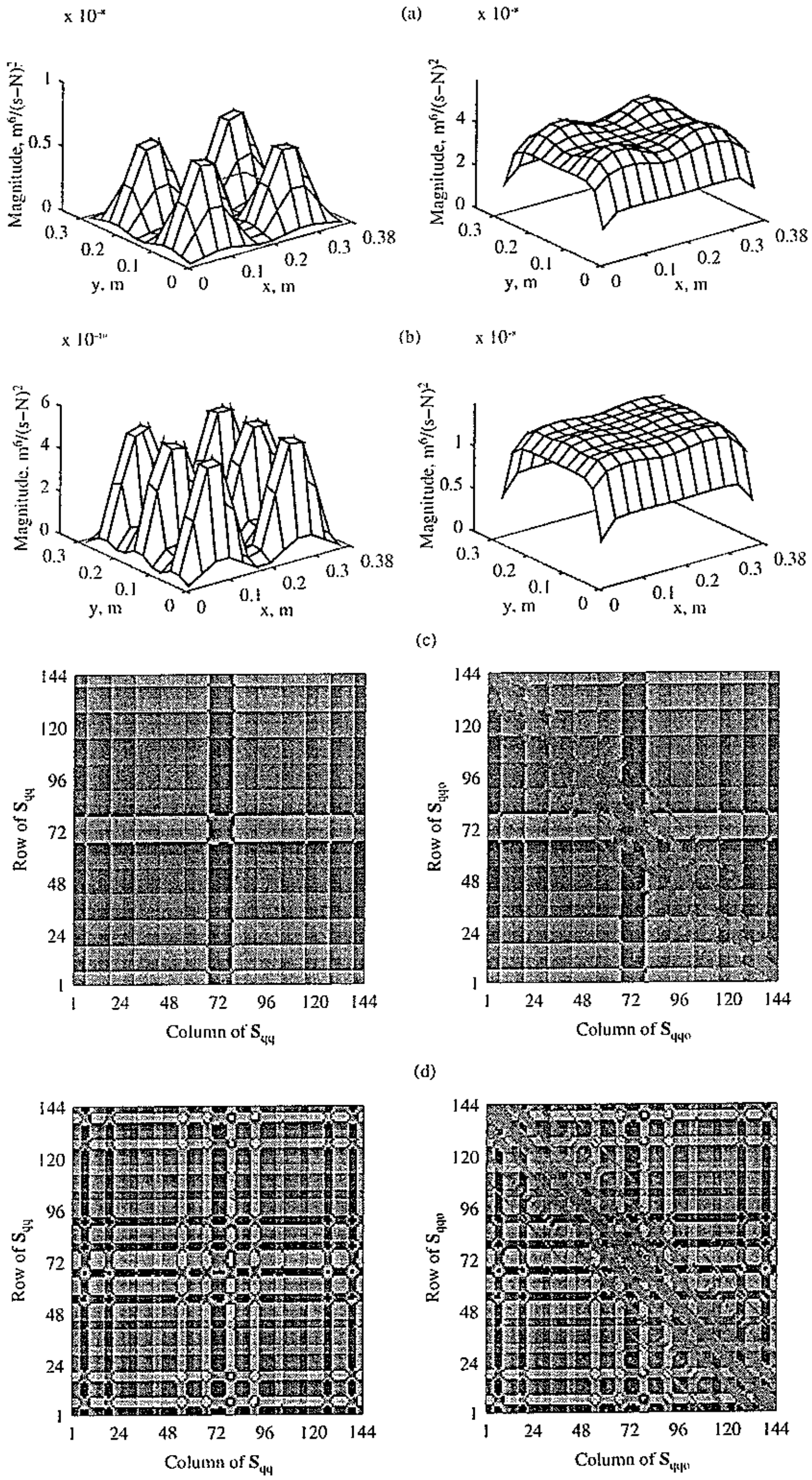

Figure 12. A comparison of the desired (left) and reconstructed (by the least squares solution, right) volume velocity (per unit force) auto-spectra of the 144 source and 144 microphone model. Measurement noise (Figure 10(b)) is assumed: (a), (c) magnitude and phase at the frequency of (2,2) (444Hz, *ka*=0.129) structural mode, (b), (d) magnitude and phase at the frequency of (2,3) (786Hz, *ka*=0.228) structural mode.

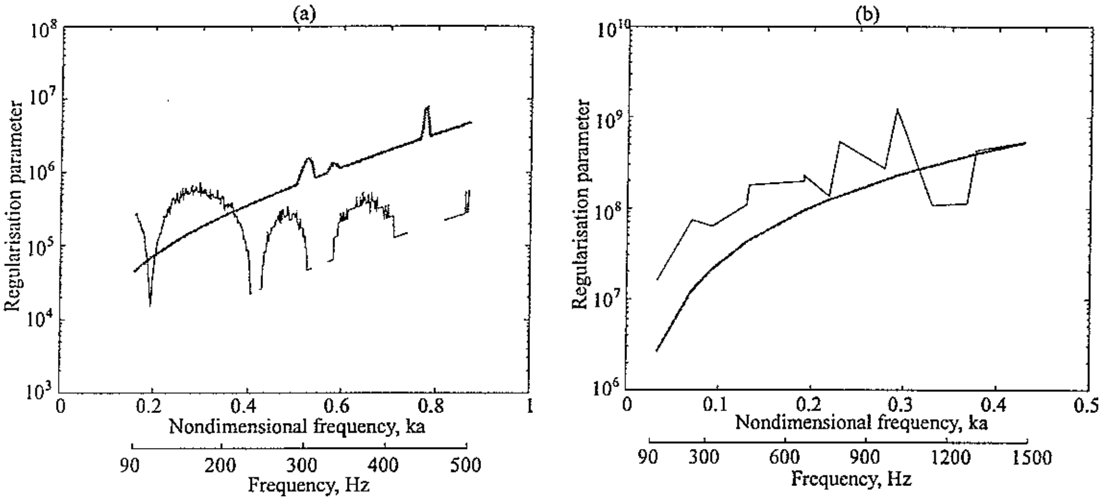

Figure 13. The regularisation parameters varying with frequencies of (a) 4 source and 4 microphone model, (b) 144 source and 144 microphone model: β_{qq} (black thin) and β_{GCV} (grey thick).

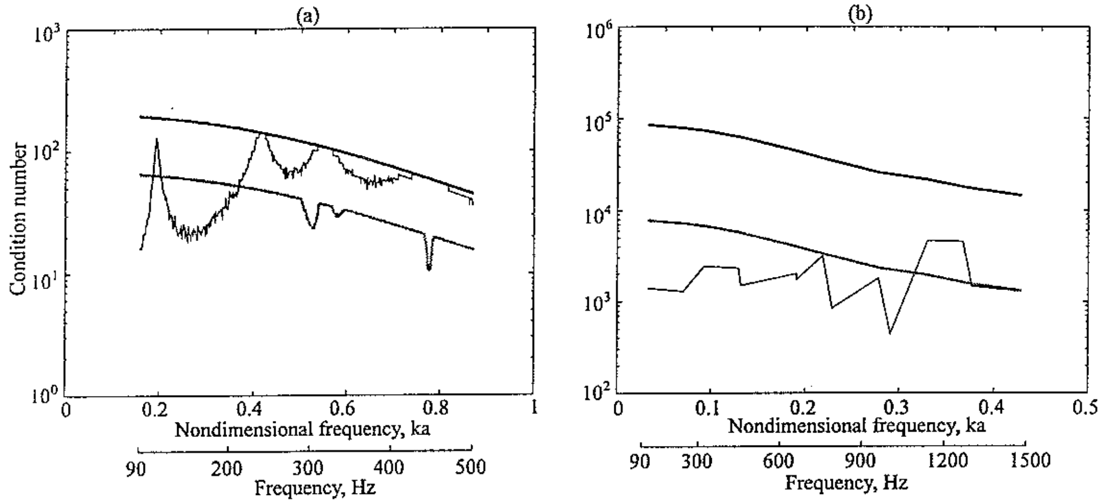

Figure 14. Condition numbers $\kappa(\mathbf{H}^H\mathbf{H})$ (black thick), $\kappa(\mathbf{H}^H\mathbf{H}+\beta_{qq}\mathbf{I})$ (black thin), and $\kappa(\mathbf{H}^H\mathbf{H}+\beta_{GCV}\mathbf{I})$ (grey thick) of the matrices to be inverted of (a) 4 source and 4 microphone model and (b) 144 source and 144 microphone model.

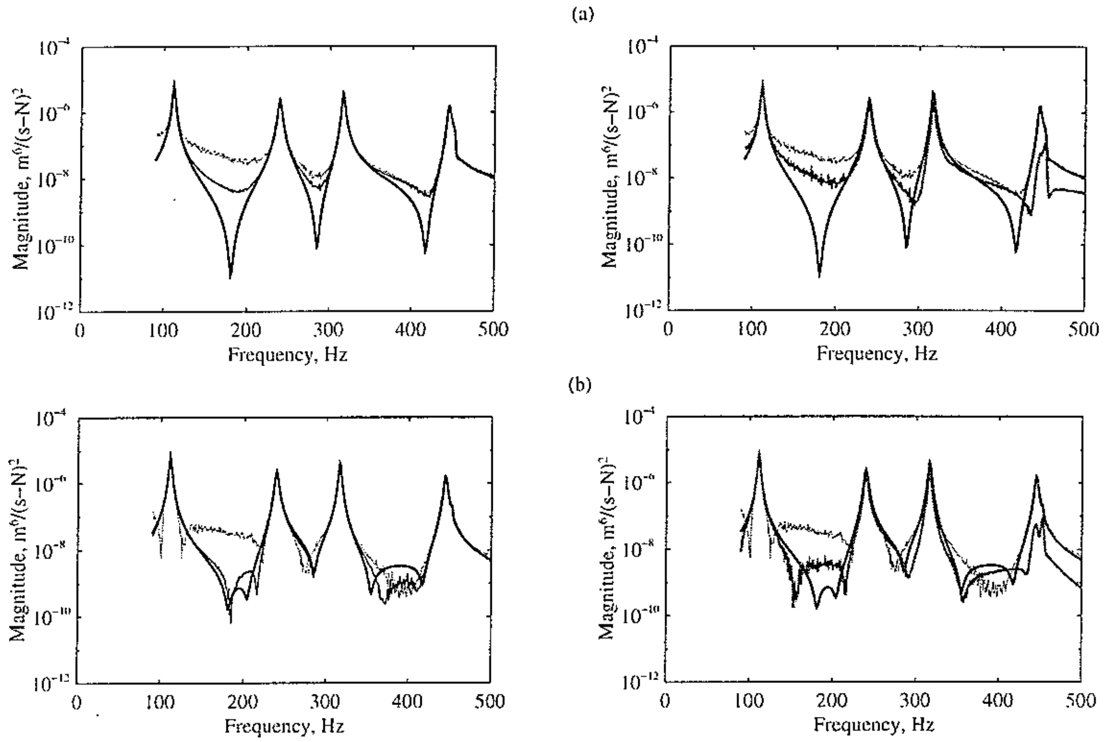

Figure 15A. (a) Volume velocity (per unit force) auto-spectra of source 3 and (b) cross-spectra between sources 3 and 4 of the 4 source and 4 microphone model when measurement noise (Figure 10(a)) is present: desired (thick black), unregularised (thin grey), regularised by β_{qq} (thin black), regularised by β_{GCV} (thick grey).

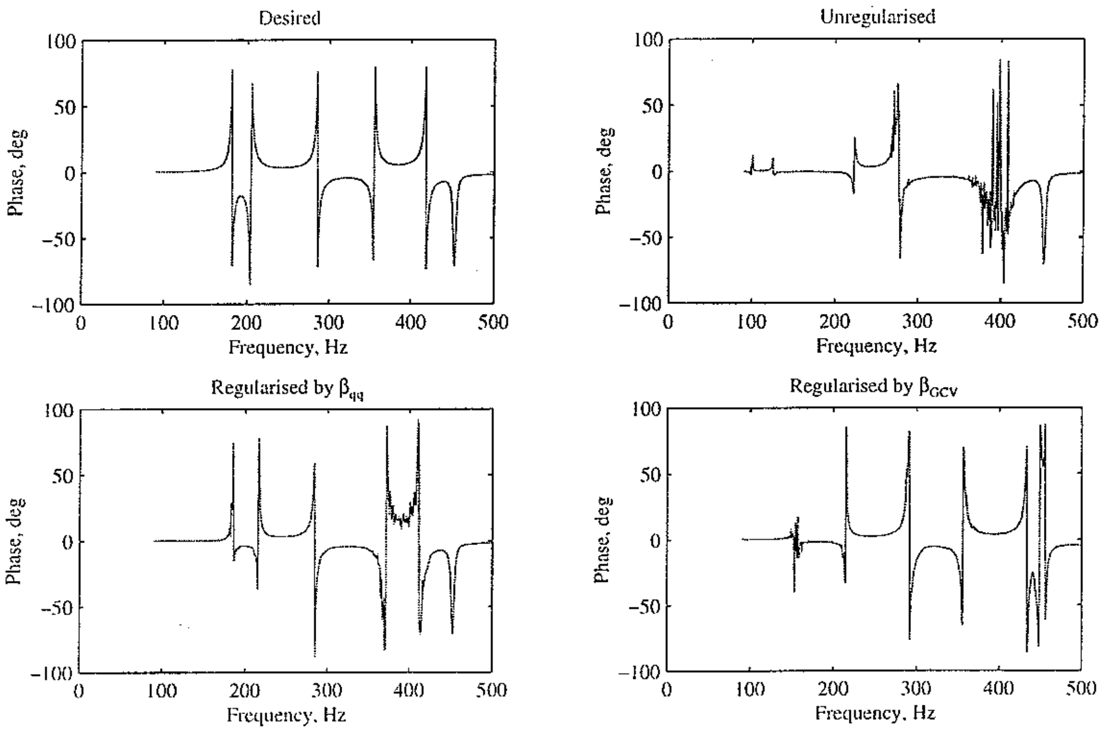

Figure 15B. Phases of the volume velocity (per unit force) cross-spectra between sources 3 and 4 of the 4 source and 4 microphone model when measurement noise (Figure 10(a)) is present.

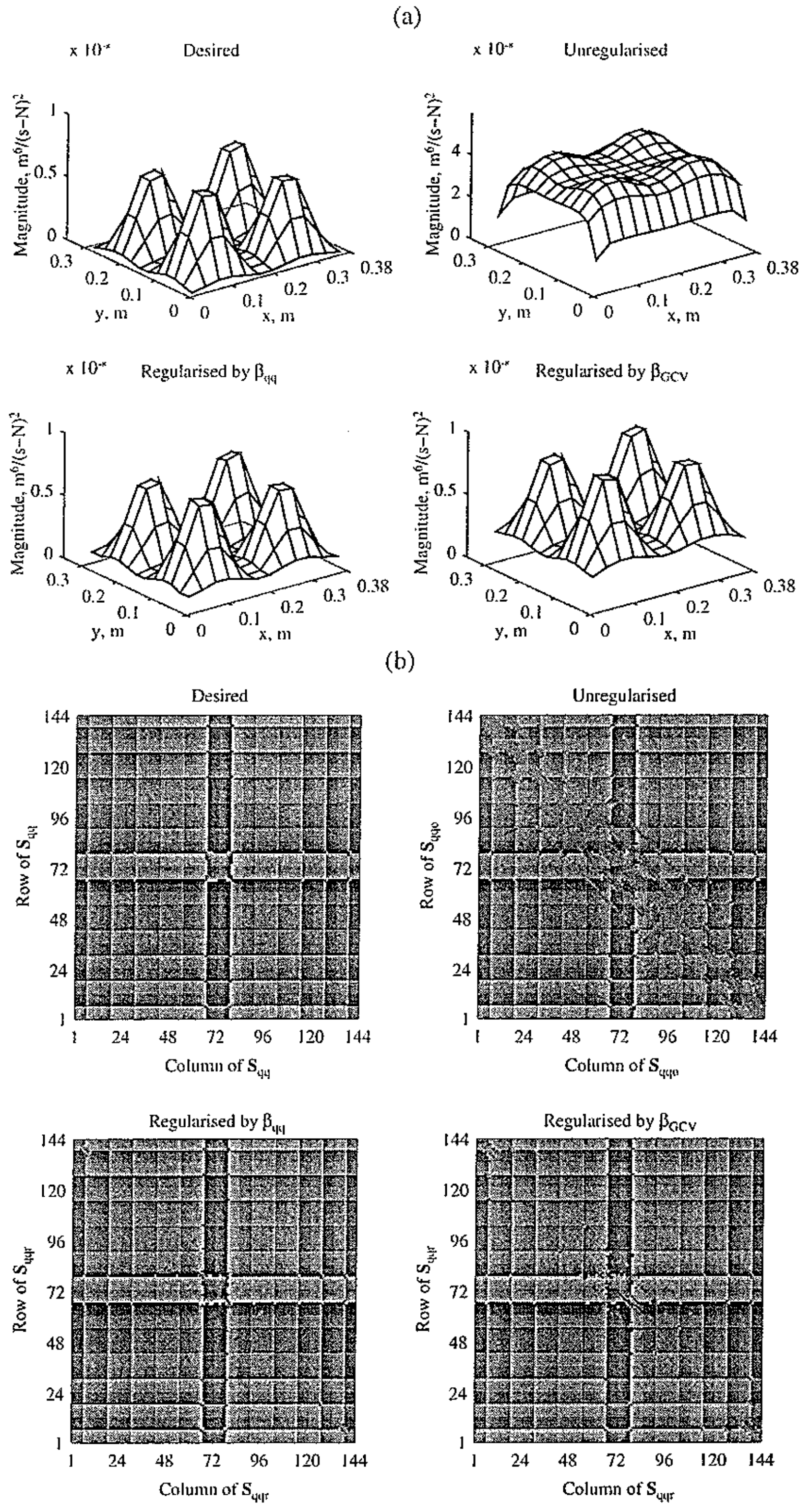

Figure 16. A comparison of the desired and reconstructed (by Tikhonov regularisation) volume velocities (per unit force) of 144 source and 144 microphone model at the frequency of structural mode (2,2) (444Hz, ka=0.129) when measurement noise (Figure 10(b)) is present: (a) auto-spectra and (b) phases of cross-spectra.

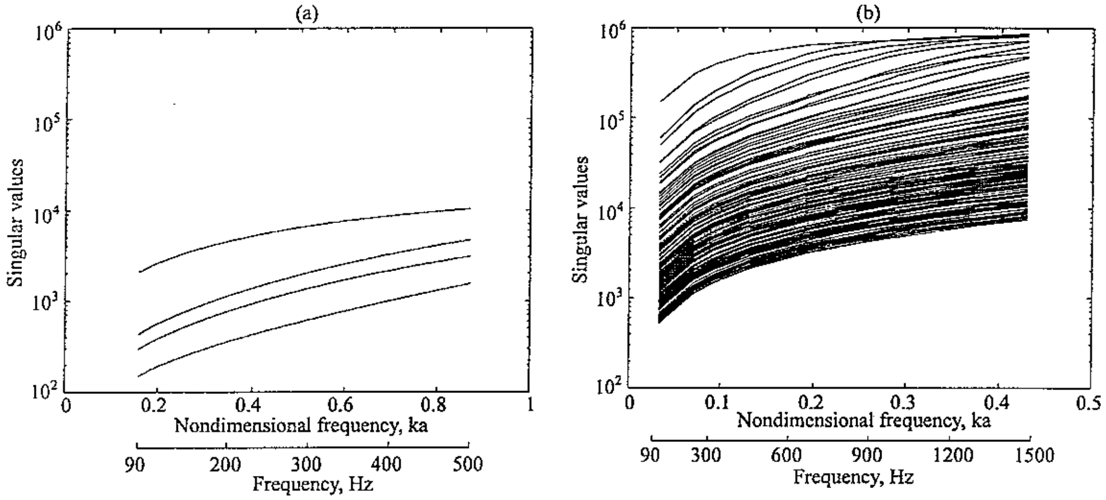

Figure 17. The singular values of the transfer function matrix **H** of (a) 4 source and 4 microphone model and (b) 144 source and 144 microphone model.

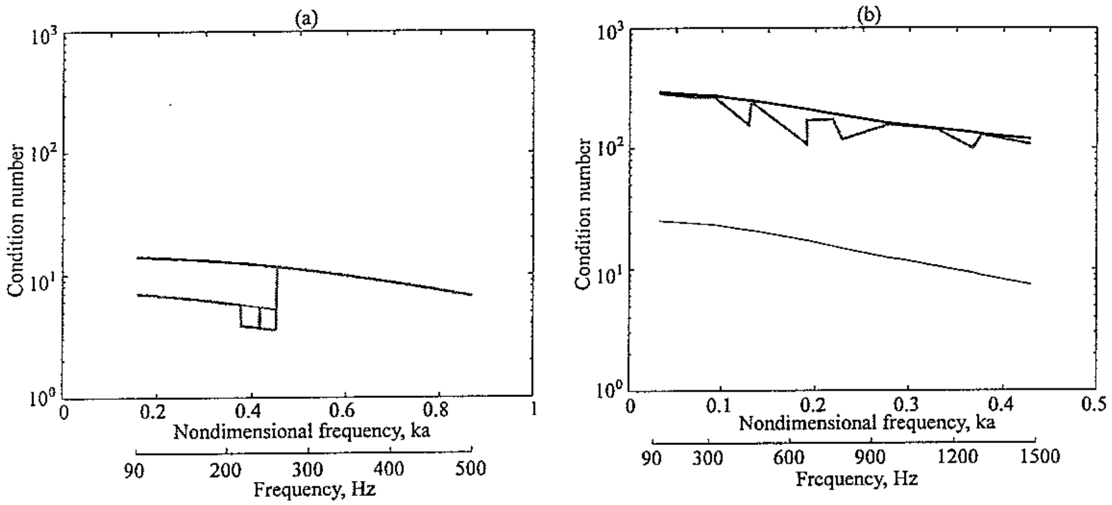

Figure 18. Condition numbers $\kappa(\mathbf{H})$ (black thick), $\kappa(\mathbf{H}_D)$ (black thin), and $\kappa(\mathbf{H}_v)$ (grey thick) of the matrices to be inverted of (a) 4 source and 4 microphone model and (b) 144 source and 144 microphone model.

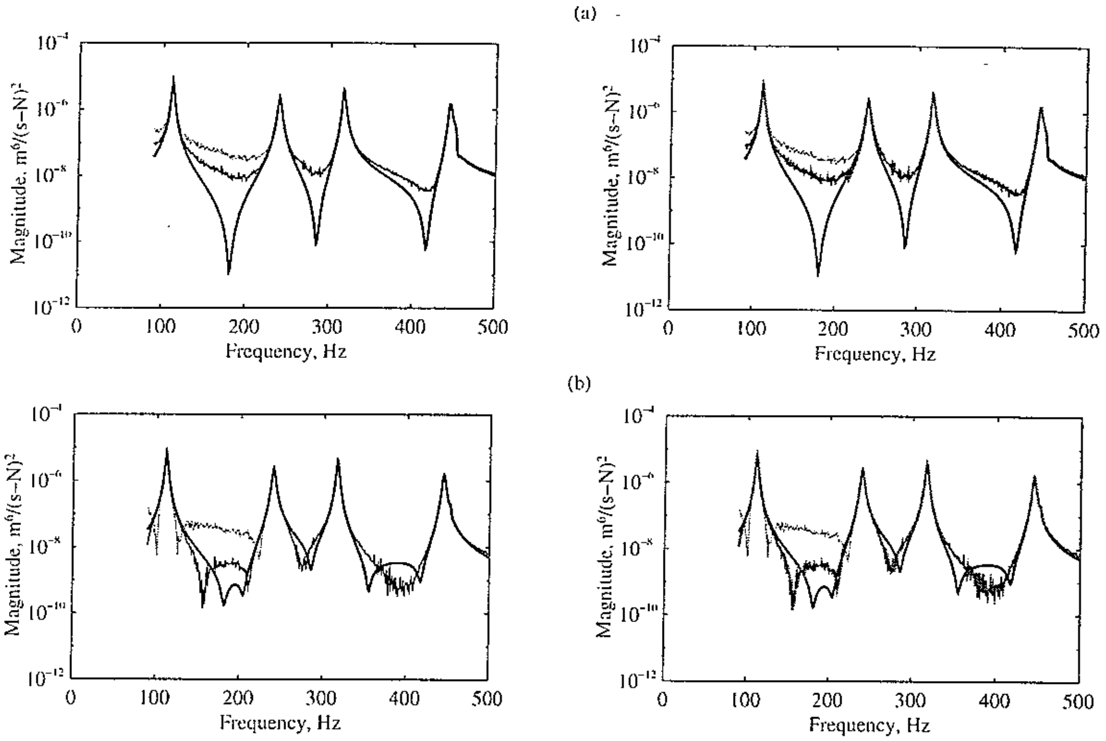

Figure 19A. (a) Volume velocity (per unit force) auto-spectra of source 3 and (b) cross-spectra between sources 3 and 4 of the 4 source and 4 microphone model when measurement noise (Figure 10(a)) is present: desired (thick black), reconstructed by **H** (thin grey), reconstructed by $\mathbf{H}_D$ (thin black), regularised by $\mathbf{H}_v$ (thick grey).

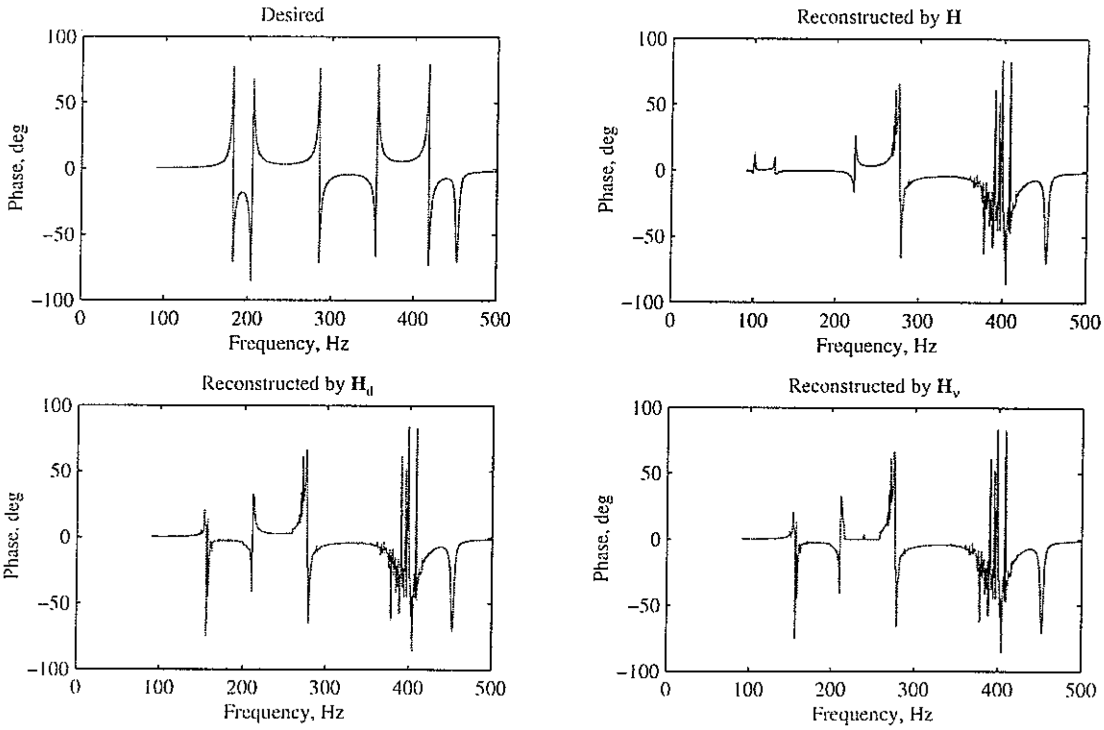

Figure 19B. Phases of the volume velocity (per unit force) cross-spectra between sources 3 and 4 of the 4 source and 4 microphone model when measurement noise (Figure 10(a)) is present.

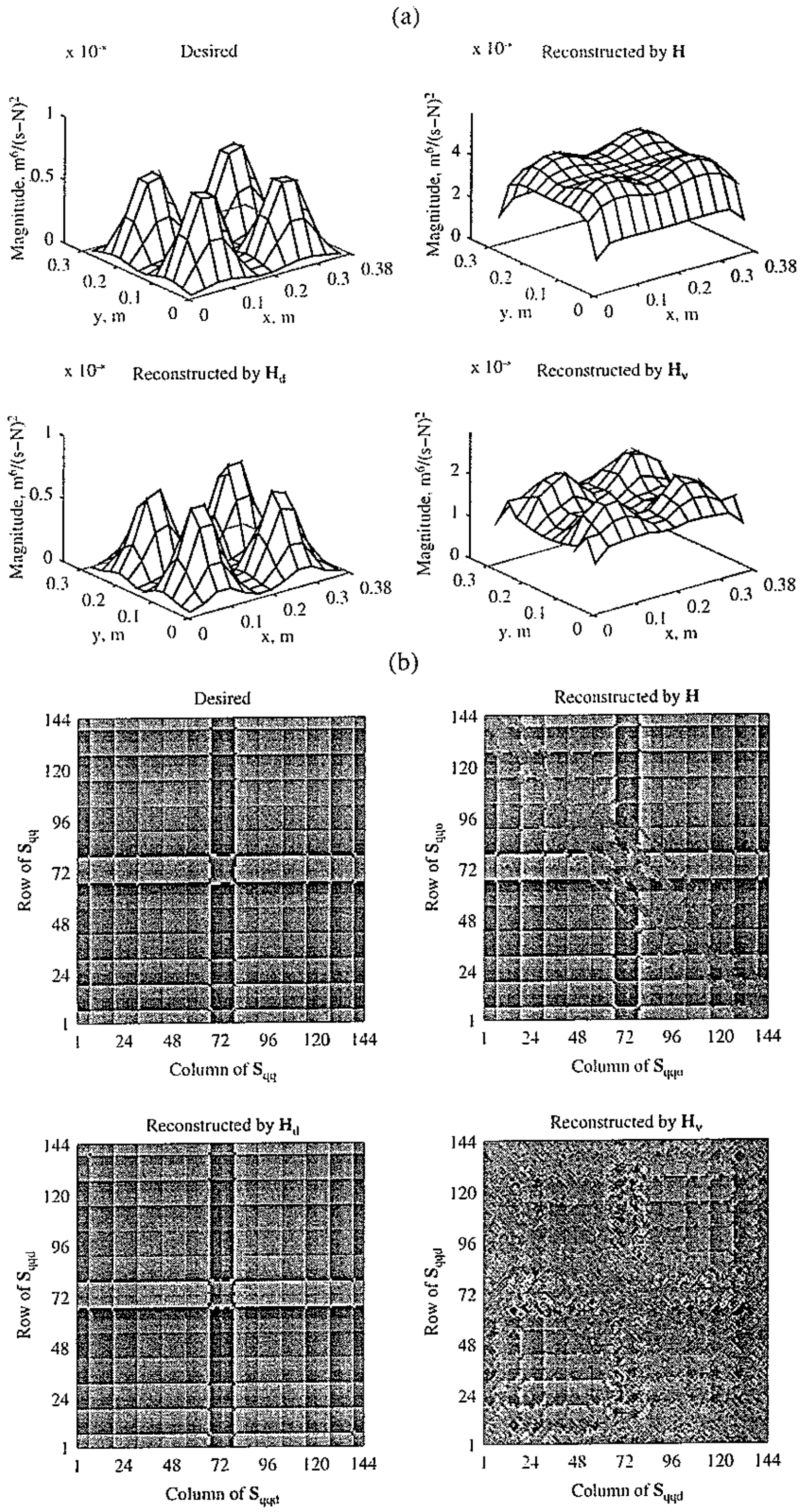

Figure 20. A comparison of the desired and reconstructed (by singular value discarding) volume velocities (per unit force) of 144 source and 144 microphone model at the frequency of structural mode (2,2) (444Hz, *ka*=0.129) when measurement noise (Figure 10(b)) is present: (a) auto-spectra and (b) phases of cross-spectra.