Institute of Sound and Vibration Research

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UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

Wave Reflection and Transmission Through a String-Beam Junction

by

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Abstract

Tensegrity structures involve a continuous network of thin strings or cables under tension that hold thin structural beams or rods in compression. The tension guarantees the shape of the static structure. They were first developed in the 1940's and have mainly been used as artistic sculptures. However, some applications now include large arenas and roof coverings, and increasing interest is being shown for use in deployable configurations for space structures.

Typically the global low frequency characteristics and vibration control are amenable to analysis techniques such as the Finite Element method. The study reported here investigates wave propagation phenomena at a junction between the tension and compression members and demonstrates the wave filtering properties of the junction, which are especially significant at the higher frequencies. The junction examined consists of a semi-infinite string attached to a semi-infinite Euler-Bernoulli beam that can support both flexural and longitudinal wave motion. Wave propagation through the junction is considered and theoretical expressions for the reflection and transmission coefficients are derived.

It is shown that maximum power transmission occurs when the impedance of the string is equal to one over the square root of two times the real part of the impedance of the beam in flexure, i.e when the beam is attached in-line with the string; above this frequency transmission is reduced. When longitudinal motion is excited, i.e. the beam is at 90° to the string, maximum transmission in the beam occurs when the string impedance and the in-plane beam impedance are equal. Reduced transmission is as a result of impedance mismatching.

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List of Symbols

Symbol	Use	Units
A	Wave amplitude	m
C	Damping	kgs^{-1}
E	Young's modulus	N/m^2
I	Second moment of area	m^4
K	Stiffness	kgs^{-2}
M	Mass	kg
Q	Shear force	N
S	Cross-sectional area	m^2
T	Tension	N
Z	Impedance	N/ ms^{-1}
c	Phase velocity	ms^{-1}
f	Force	N
j	Complex number notation	
k	Wavenumber	m^{-1}
m	Moment	Nm
p	Power	Js^{-1}
t	Time	s
u	Displacement	m
x	Displacement	m
y	Displacement	m
Ω	Non-dimensional frequency	
ε	Strain	
θ	Angle	degrees
ρ	Density	kg/m^3
σ	Stress	N/m^2

Subscripts

b	Refers to the beam	
C	Damping	
f	Flexural wave	

i	Incident wave
K	Stiffness
L	Longitudinal wave
M	Mass
r	Reflected wave
s	Refers to the string
x	x-direction
2,4	Evanescent, propagating wave in beam

1. Introduction

Tensegrity is a portmanteau word from tension and integrity. A loose definition of a tensegrity structure was given by Fuller [1], "a structural relationship in which the structural shape is guaranteed by the interaction between a continuous network of members in tension and a set of members in compression". The type of tensegrity structure considered in this report is one that is composed of a continuous network of tensioned strings that hold beams in compression and hence in position, but the beams do not connect to each other. The purpose of this report is to examine the wave filtering properties of a single junction between a string and a beam typically found in a tensegrity structure.

Static properties of tensegrity structures have been investigated by several researchers. Examples include, Motro [2], who described tensegrity structures and their main properties more precisely, and Kanchanasaratool and Williamson [3] who used particle dynamics as a means of modelling tensegrity orientation. However, previous work on the dynamic properties of tensegrity structures is limited. One example of such work is by Kahla, Moussa and Pons [4] who explored the non-linear dynamic analysis of tensegrity structures. Wave modelling through different element junctions in non-tensegrity structures has been extensively researched. Mace [5] considers wave reflection and transmission in beams with discontinuities, and Von Flotow [6] has examined disturbance propagation in networks of various structural elements. Power flow in structures has also been considered as a means of controlling structural vibration, [7-8].

A junction in a tensegrity structure consists of a minimum of three strings and a single beam. The work reported here involves the analysis of the dynamic properties of a simplified junction, consisting of a single string to a single beam, each of semi-infinite length. Dynamic analysis of these members is based upon the work of Kinsler et al [9] and Mead [10]. The analysis of the junction consists of initiating a transverse propagating wave in the string and obtaining the transmission and reflection coefficients when the string and beam are joined in three configurations, namely perpendicular, in-line and at an arbitrary angle.

Following this introduction, the point impedance of a semi-infinite string is derived and the properties of a reflected wave when the string is incident upon a general impedance boundary. Examples of a mass, stiffness and damping boundary are used and the amplitude and phase of the reflected wave in each case are derived. In the subsequent section, the flexural point impedance of a beam is derived and compared to that of the mass, stiffness and damping impedances. This leads to the analysis of the reflected and transmitted waves in the string and beam when they are joined in-line. Following this, the longitudinal point impedance of a beam is derived and compared to that of the mass, stiffness and damping impedances. This leads to the analysis of the reflected and transmitted waves when the string and beam are joined at right angles. The analysis of the string and beam attached at an arbitrary angle is then conducted by imposing a sliding boundary condition and combining the analysis for the two particular junctions. The final section is a conclusion to the report.

2. A string connected to an arbitrary impedance

2.1 Introduction

Tensegrity structures involve a network of strings in tension that hold beams in compression. The type of tensegrity model considered in this report does not allow the beams to interact, hence, it is described as a continuous tension – discontinuous compression structure. The simplest tensegrity structure has a junction that consists of three strings and a beam.

This section is concerned with a simplified junction between a semi-infinite string in-line with a semi-infinite beam. A semi-infinite length is chosen to allow the waves at the junctions to be free from reflected waves from another junction that would arise in a finite system, and hence the junction properties can be considered in isolation.

The first section investigates wave propagation in the string, and determines the impedance of the string and the effect of attaching a general impedance to the end of the string. In the next section, the flexural impedance of the beam is derived together with expressions for relative transmitted and reflected wave amplitudes when the string is attached to the beam in-line. To confirm the expressions, the power that propagates through, and reflects from, the junction is considered in the final section. A power balance check is then made.

2.2 The impedance of a string

A semi-infinite string, under a static tension force, T, is subject to a transverse harmonic force, $Fe^{j\omega t}$ with frequency ω, at the end of the string, defined as $x = 0$ as shown in Figure 1. This end is assumed not to move in the x-direction but is free to move in the y-direction. Because of the force, a wave, of amplitude A, propagates along the string away from the end. As the disturbance is at $x = 0$ and the string is infinitely long to the left of that point, waves will only propagate to the left. Thus the transverse motion of the string as a function of time, t, is given by

$$y(x,t) = Ae^{j(\omega t + k_S x)} \tag{2.1}$$

where the wavenumber, $k_S = \dfrac{\omega}{c_S}$, and c_S is the phase velocity, which is the speed of the disturbance along the string, given by $c_S = \sqrt{\dfrac{T}{\rho_s S_s}}$, where S_s is the cross sectional area of the string, ρ_s is the density and the subscript "s" refers to the string. Hence $\rho_s S_s$ is the mass per unit length of the string. The derivation of equation (2.1) is given in Appendix A.

Equating the forces acting at $x = 0$ results in a relationship between the wave amplitude A and the force F in the frequency domain, i.e. the amplitude of the applied force has to equal the resolved component of the tension force. The end of the string is considered massless, therefore all the components of force in the y-direction must vanish, i.e. force equilibrium applies to give

$$f = T\sin\theta \tag{2.2}$$

For small θ, sinθ ≈ tanθ, where tanθ is $\frac{\partial y}{\partial x}$ which can be substituted into equation (2.2) to give,

$$f = T\left(\frac{\partial y}{\partial x}\right)_{x=0} \tag{2.3}$$

Substituting for $f = Fe^{j\omega t}$ and differentiating equation (2.1) with respect to x, and combining with equation (2.3) gives:

$$Fe^{j\omega t} = jk_s TAe^{j\omega t} \tag{2.4}$$

which can be re-arranged to give,

$$A = -\frac{jF}{k_s T} \tag{2.5}$$

Hence, the lateral motion of the string is given by,

$$y(x,t) = -\frac{jF}{k_s T}e^{j(\omega t + k_s x)} \tag{2.6}$$

Differentiating this expression with respect to time and evaluating at $x = 0$ gives the transverse velocity at this position. The amplitude of the velocity is

$$\dot{Y}(0) = \frac{\omega F}{k_s T} \tag{2.7}$$

The impedance is defined as the harmonic force $f = Fe^{j\omega t}$ divided by the resulting harmonic velocity $\dot{y}(0) = \dot{Y}(0)e^{j\omega t}$ at the point where the force is acting and hence the point impedance for a semi-infinite string is given by,

$$Z_s = \frac{k_s T}{\omega} = \frac{T}{c_s} = \sqrt{T\rho_s S_s} \tag{2.8}$$

Note that this is a real quantity and is independent of frequency.

2.3 Wave reflection on an arbitrary impedance boundary

The aim of this section is to derive a relationship between the reflected and incident wave amplitudes when a transverse wave is incident upon an arbitrary impedance attached to the end of a semi-infinite string. Figure 2 depicts this situation.

The transverse motion of the string consists of two waves, the incident wave of amplitude, A_i and the reflected wave of amplitude A_r. It is described by the equation,

$$Y(x) = A_i e^{-jk_s x} + A_r e^{jk_s x} \tag{2.9}$$

where harmonic excitation is assumed but the $e^{j\omega t}$ time dependency is suppressed for simplicity. Because the impedance is rigidly attached to the string, the following boundary condition applies,

$$F_1 = -F_2 \tag{2.10}$$

F_1 can be calculated using equation (2.3). Substituting the spatial derivative of equation (2.9) evaluated at $x = 0$ into equation (2.3) results in,

$$F_1 = -jk_s T(A_i - A_r) \tag{2.11}$$

F_2 is related to the attached impedance and its velocity, and is given by,

$$F_2 = Z\dot{Y}(0) \tag{2.12}$$

Substituting for $\dot{Y}(0)$ derived from equation (2.9) evaluated at $x = 0$ into equation (2.12) gives,

$$F_2 = j\omega Z(A_i + A_r) \tag{2.13}$$

Combining equations (2.11), (2.12) and (2.13) gives the relationship between the amplitude of the reflected wave and that of the incident wave,

$$\frac{A_r}{A_i} = \frac{1 - \dfrac{Z}{Z_s}}{1 + \dfrac{Z}{Z_s}} \tag{2.14}$$

where Z_s is the string impedance given by equation (2.8). To check the above expression three tests can be performed. Letting the attached impedance become infinite should result in complete reflection, i.e. the incident to reflected wave amplitude ratio should be unity. The second test involves setting the attached impedance to match that of the string so that there will be no reflected wave and hence the ratio of the amplitudes should be zero. The third test involves setting the attached impedance to zero and again there should be a reflected wave amplitude ratio of unity

As $\dfrac{Z}{Z_s} \rightarrow \infty$

$$\frac{A_r}{A_i} = \frac{1 - \infty}{1 + \infty} \approx -1$$

The minus sign shows that the reflected wave has a phase difference of $180°$. The magnitude of the reflected wave is equal to that of the reflected wave.

If $Z = Z_s = \rho_L c_S$

$$\frac{A_r}{A_i} = \frac{1-1}{1+1} = 0$$

No reflection occurs and so complete transmission takes place.

As $\frac{Z}{Z_s} \to 0$

$$\frac{A_r}{A_i} \approx 1$$

Note that again, the amplitude of the reflected wave equals that of the incident wave, but in this case, there is no phase change at the boundary.

2.4 Simulations and examples

Assigning the properties of mass, stiffness and damping individually to the arbitrary impedance is useful in that when a beam is attached (which is discussed later in this report); the results may be compared to give physical insight into the dynamic behaviour of the connection.

2.4.1 Mass-like impedance

Figure 3 shows a mass, M, attached to the semi-infinite string. F_1 and F_2 are the internal forces between the mass and string. A_i is incident upon the mass resulting in a reflected wave A_r.
Now the impedance of mass is given by, [11]

$$Z_M = j\omega M \tag{2.15}$$

Substituting this impedance for Z in equation (2.14) gives,

$$\frac{A_r}{A_i} = \frac{1 - \dfrac{j\omega M}{\rho_s c_s S_s}}{1 + \dfrac{j\omega M}{\rho_s c_s S_s}} \tag{2.16}$$

This can be re-written in terms of non-dimensional frequency, $\Omega_M = \dfrac{\omega}{\omega_M}$, where $\omega_M = \dfrac{\rho_s c_s S_s}{M}$ which is the frequency at which the magnitude of the impedance of the mass is equal to the impedance of the string, to give

$$\frac{A_r}{A_i} = \frac{1 - j\Omega_M}{1 + j\Omega_M} \tag{2.17}$$

Figure 4 shows the modulus and phase of the amplitude ratio $\frac{A_r}{A_i}$ plotted against the non-dimensional frequency Ω_M.

It can be seen from Figure 4(a) that the modulus of $\frac{A_r}{A_i}$ is constant with frequency, which means that the magnitude of the reflected wave is equal to the magnitude of the incident wave for all frequencies. This result occurs because a mass cannot dissipate energy and hence all of the incident wave energy is reflected.

It can be seen from Figure 4(b) that the incident and reflected wave are in-phase when Ω_M is zero. This is because at low frequencies the mass does not constrain the displacements significantly and the result is similar to a free end. When ω is equal to ω_M, there is a phase difference of $-90°$. This is the point when the impedance of the string is equal to the magnitude of that of the mass. As Ω_M increases, the phase difference approaches $-180°$, which is consistent with the infinite impedance boundary condition. In this limit, the mass has a very large impedance and constrains the displacement to zero, resulting in a similar effect that arises from a string with a clamped end.

2.4.2 Stiffness-like impedance

Figure 5 shows a stiffness, K attached to the semi-infinite string. The internal forces F_1 and F_2 exist between the string and the stiffness.

The impedance of a stiffness is given by, [11]

$$Z_K = \frac{K}{j\omega} \tag{2.18}$$

Substituting this impedance for Z into equation (2.14) gives:

$$\frac{A_r}{A_i} = \frac{1 - \dfrac{K}{j\omega\rho_L c_S S_s}}{1 + \dfrac{K}{j\omega\rho_L c_S S_s}} \tag{2.19}$$

This can also be re-written in terms of a non-dimensional frequency, $\Omega_K = \frac{\omega}{\omega_K}$, where $\omega_K = \frac{K}{\rho_L c_S S_s}$, which is the frequency at which the magnitude of the impedance of the stiffness is equal to that of the string, to give

$$\frac{A_r}{A_i} = \frac{1 + \frac{j}{\Omega_K}}{1 - \frac{j}{\Omega_K}}, \tag{2.20}$$

Figure 6 shows the graphs of modulus and phase of $\frac{A_r}{A_i}$ are plotted against Ω_K.

The modulus of $\frac{A_r}{A_i}$ is unity and thus independent of frequency. This is because stiffness cannot dissipate any energy causing all the energy in the wave to be reflected. The phase difference is 180°, when Ω_K is zero. When ω is equal to ω_K there is a 90° phase difference, which occurs when the string's impedance matches that of the magnitude of the stiffness impedance. As Ω_K approaches infinity the phase approaches zero. This behaviour is in direct contrast to that of the mass, i.e. a spring provides a pinned end at low frequencies, which becomes free as frequency increases.

2.4.3 Damping-like impedance

A diagram of a viscous damper attached to the semi-infinite string is shown in Figure 7. The internal forces F_1 and F_2 exist between the string and damper.

The impedance of a damper is given by, [11]

$$Z_C = C \tag{2.21}$$

where C is the viscous damping coefficient.

It can be seen that the impedance is entirely real in contrast with the impedance of mass and stiffness, which are imaginary.

Substituting this impedance for Z in equation (2.14) results in:

$$\frac{A_r}{A_i} = \frac{1 - \overline{Z}_C}{1 + \overline{Z}_C} \tag{2.22}$$

where $\overline{Z}_C$ is the ratio of the impedance of a damper to that of the string, i.e. $\overline{Z}_C = \frac{Z_C}{Z_s}$.

Unlike the two previous results, this expression is independent of frequency. However, in this case the modulus and phase can be plotted against the ratio of the impedances $\overline{Z}_C$, as shown in Figure 8, which shows how $\frac{A_r}{A_i}$ varies for different values of the attached impedance.

The modulus of $\frac{A_r}{A_i}$ is unity when $\overline{Z}_C$ is zero. As $\overline{Z}_C$ increases to unity, the amplitude of the reflected wave goes to zero. At this point, the impedance of the damper is equal to that of the string and the damper dissipates all of the incident energy resulting in no reflected wave. This is consistent with the result found earlier, that when the impedances match no reflection occurs. As $\overline{Z}_C$ increases above unity the modulus of the reflected wave amplitude increases. When $\overline{Z}_C$ is infinite the modulus will be unity as the damper effectively pins the end of the string resulting in zero displacement and no energy is dissipated.

There is no phase difference between the reflected and incident wave when $\overline{Z}_C$ is equal to zero, i.e. as if the end were free. When $\overline{Z}_C$ is equal to unity, there is a 180° phase change, and further increases in $\overline{Z}_C$ do not alter the phase difference, which remains constant at 180°.

2.5 Conclusions

In this section, the impedance of a semi-infinite string has been derived and shown to be dependent upon the tension force and the phase velocity and independent of frequency. When an impedance is attached at one end of the string, causing reflected waves to be present in the string, the amplitude of the reflected wave compared to that of the incident wave is a function of the ratio of the attached impedance to that of the string impedance. Applying a mass-like impedance results in no energy being dissipated, i.e. the amplitude of the reflected wave is always equal to the incident wave. The phase angle between the waves varies with frequency. A stiffness-like impedance does not dissipate energy and has a phase change that is dependent upon frequency. A damping-like impedance is independent of frequency, and thus the impedance of the damper can match the string impedance exactly at one particular frequency. Hence at this frequency all the energy in the incident wave is dissipated by damper and there is no reflected wave in the string.

In this section, the attachment of an arbitrary impedance to the string has been explored. The next section considers the case of a beam in flexure attached to the string and by direct comparison with the mass, stiffness and damping impedances. The behaviour of the string and beam junction is investigated.

3. A string connected in-line to a beam

3.1 Introduction

This section considers the case of a beam attached in-line with a string. A wave in the string is incident upon the beam, exciting flexural waves in the beam, and a reflected wave in the string. Firstly, the impedance of the beam is derived and used to obtain the reflection and transmission coefficients, which give an indication of the wave filtering properties of the junction.

3.2 The flexural impedance of a beam

The ultimate aim of this section is to explore the case where the beam is joined in-line to a string and the string excited harmonically. Obtaining the impedance of a beam in flexure and substituting it for the general impedance used in equation (2.14) will give the reflection coefficient. The results obtained in section 2.4 are used for comparison with the results for the beam, and hence the mass, stiffness or damping -like qualities of the beam are determined.

Consider a semi-infinite beam subject to a harmonic, transverse force, at frequency ω, at the left, fixed end, defined as $x = 0$, as shown in Figure 9. As a result two flexural waves are generated, namely an exponentially decaying wave (evanescent wave), A_2, and a propagating wave, A_4. The lateral motion of the beam is given by the following expression, (see Appendix B)

$$y(x,t) = \left(A_2 e^{-k_f x} + A_4 e^{-jk_f x}\right) e^{j\omega t} \tag{3.1}$$

where k_f is the flexural wavenumber of the beam, given by

$$k_f = \left(\frac{\rho_b S_b}{E_b I_b}\right)^{1/4} \omega^{1/2} \tag{3.2}$$

where ρ_b is the density, S_b is the cross-sectional area, E_b is the Young's modulus of elasticity and I_b is the second moment of area of the beam. The subscript f denotes the flexural wavetype and the subscript b denotes the beam.

To obtain expressions for A_2 and A_4, the boundary conditions are applied and the resulting equations solved. The end of the beam is a free end, so there is no bending moment present, and the applied force is set equal to the shear force.

Boundary condition 1:

$$m = E_b I_b \frac{\partial^2 y(0)}{\partial x^2} = 0 \quad (3.3)$$

Boundary Condition 2:

$$Q = E_b I_b \frac{\partial^3 y(0)}{\partial x^3} = F_b \quad (3.4)$$

Taking the spatial derivative of equation (3.2) and substituting this into equations (3.3) and (3.4) gives:

$$E_b I_b \left(k_f^2 A_2 - k_f^2 A_4\right) = 0,$$

which implies that $A_2 = A_4$ (3.5)

and $\quad E_b I_b \left(-k_f^3 A_2 + j k_f^3 A_4\right) = F_b$ (3.6)

Substituting equation (3.5) into equation (3.6) and rearranging gives an expression for the wave amplitude in terms of the applied force:

$$A_4 = -\frac{F_b(1+j)}{2E_b I_b k_f^3} \quad (3.7)$$

Combining equations (3.2), (3.5) and (3.7) gives the transverse displacement of the beam

$$y(x,t) = -\frac{F_b(1+j)}{2E_b I_b k_f^3}\left(e^{-k_f x} + e^{-jk_f x}\right)e^{j\omega t} \quad (3.8)$$

To find the impedance of the beam, its velocity at $x = 0$ is required. Hence, equation (3.8) is differentiated with respect to time and evaluated at $x = 0$ so that the impedance $Z_f = \frac{F_b}{\dot{Y}(0)}$, where $\dot{Y}(0)$ is the velocity at $x = 0$, can be determined. This gives

$$Z_f = \frac{E_b I_b k_f^3}{2\omega}(1+j) \quad (3.9)$$

To compare this with the impedances of mass, stiffness and damping given in Section 2, the real and imaginary parts of the expression are examined. Equation (3.9) has a positive real part and a positive imaginary part. The positive real part corresponds to that of equivalent damping and the positive imaginary part corresponds to that of equivalent mass. Hence, the wave in the string is incident upon the beam and experiences an equivalent damper $\frac{E_b I_b k_f^3}{2\omega}$, which is proportional to $\omega^{1/2}$, in parallel with an equivalent mass of $\frac{E_b I_b k_f^3}{2\omega^2}$ which is proportional to $\omega^{-1/2}$.

So far the impedances of a string and a beam have been derived, and the wave motion present in the string when the string is attached to an arbitrary impedance has been described. Combining these enables the reflected wave in the string to be determined when the wave in the string is incident upon the beam. Comparison between this reflected wave and the waves that are reflected from mass, stiffness and damping elements leads to an understanding of the dynamic characteristics of a beam connected to a string.

3.3 A string connected to a beam

A semi-infinite string, as described in section 2.2, is joined to a semi-infinite beam, as described in section 3.2, and this is shown in Figure 10. They join at $x = 0$, and the string is excited harmonically. It is assumed that no significant in-plane motion of the string and beam occurs. The displacements of the string and beam at the join are $Y_s(0)$ and $Y_b(0)$ respectively. A_i is the amplitude of the wave in the string incident upon the junction and A_r is the reflected wave in the string. An evanescent wave with amplitude A_2 and propagating wave with amplitude A_4 are excited in the beam. The aim of this section is to derive the ratio of the transmitted wave, A_4 in the beam to the incident wave in the string and the ratio of the reflected wave, A_r, to the incident wave in the string as they describe the wave filtering properties of the junction.

When the beam is attached to the string, the following boundary conditions apply: At $x = 0$, there is no net force and the displacements of the string and beam are equal, therefore:

$$F_s = -F_b \qquad (3.10)$$

and

$$Y_s(0) = Y_b(0) \qquad (3.11)$$

Both boundary conditions are used to derive the ratio of the transmitted wave amplitude to that of the incident wave. The string displacement at $x = 0$ is given by

$$Y_s(0) = A_i + A_r \qquad (3.12)$$

and the displacement of the beam is given by

$$Y_b(0) = A_2 + A_4 \qquad (3.13)$$

Combining equations (3.5), (3.11), (3.12) and (3.13) gives:

$$A_i + A_r = A_2 + A_4 = 2A_4 \qquad (3.14)$$

Which can be written as

$$1 + \frac{A_r}{A_i} = 2\frac{A_4}{A_i} \qquad (3.15)$$

Substituting the expression for $\frac{A_r}{A_i}$ given in equation (2.14) and setting $Z = Z_f$, and rearranging, gives the ratio of the transmitted wave, A_4 to the incident wave

$$\frac{A_4}{A_i} = \frac{1}{1+\frac{Z_f}{Z_s}} \qquad (3.16)$$

This expression can be further developed by substituting for Z_s in (2.10) and Z_f in (3.9) to give:

$$\frac{A_4}{A_i} = \frac{1}{1+\frac{E_b I_b k_f^3 (1+j)}{2k_s T}} \qquad (3.17)$$

Substituting for k_f from equation (3.2) into equation (3.17) gives an expression in terms of a non-dimensional frequency, $\Omega_b = \frac{\omega}{\omega_b}$ where $\omega_b^{1/2} = \frac{2T}{c_s (E_b I_b)^{1/4} (\rho_b S_b)^{3/4}}$.

$$\frac{A_4}{A_i} = \frac{1}{1+\Omega_b^{1/2}(1+j)} \qquad (3.18)$$

When ω is equal to ω_b the impedance of the string is equal to the real part of the flexural impedance of the beam.

To obtain the amplitude of the reflected wave, the impedance of the beam and the impedance of the string are substituted into equation (2.14) to give

$$\frac{A_r}{A_i} = \frac{1-\frac{E_b I_b k_f^3 (1+j)}{2k_s T}}{1+\frac{E_b I_b k_f^3 (1+j)}{2k_s T}} \qquad (3.19)$$

which can also be written as

$$\frac{A_r}{A_i} = \frac{1-\Omega_b^{1/2}(1+j)}{1+\Omega_b^{1/2}(1+j)} \qquad (3.20)$$

Figure 11 shows the modulus and phase graphs of both $\frac{A_r}{A_i}$ and $\frac{A_4}{A_i}$ plotted against Ω_b. The moduli of both the reflected and transmitted waves are equal to that of the incident wave when Ω_b is zero. As the phase of the reflected wave is equal to that of the incident wave, it suggests that the end behaves as if it were free. This is because the equivalent mass of the beam does not restrict the displacements at low frequencies. The reflected wave amplitude is a minimum when $\Omega_b = 1/2$, which means that at this frequency $Z_s = \sqrt{2}\,\mathrm{Re}(Z_f)$. This corresponds to a phase difference of 90° between the incident and reflected wave in the string and is caused by the equivalent damping in the beam. The fact that the reflected wave amplitude does not go to zero is due to the mismatch of impedance between the string and beam, as the beam impedance has an imaginary part. At $\omega = \omega_b$, the frequency at which the string impedance is equal to the real part of the beam impedance, $\left|\frac{A_r}{A_i}\right| = \left|\frac{A_4}{A_i}\right|$. As $\frac{\omega}{\omega_b}$ increases to infinity $\left|\frac{A_r}{A_i}\right|$ tends to unity, and the phase tends to $180°$, which suggests that the mass of the beam is effectively clamping the end of the string.

3.4 Power flow through the junction

It is useful to consider the transmitted and reflected waves in terms of power flow through the junction between a string and beam. Assuming that the junction between the string and beam is conservative, the power balance is

power incident = power reflected +power transmitted.

The incident wave in the string contains the input power. When this is incident upon the junction some of the power will reflect back along the string in the reflected wave and some will be transmitted to the beam in the form of the flexural propagating wave in the beam. The evanescent wave does not carry any energy away from the junction so long as it does not interact with any other evanescent wave [12]. The input time-averaged power for any structure is given by, [11]

$$\text{Power} = \frac{1}{2}\text{Re}\left\{F.V^*\right\} \tag{3.21}$$

where, F is the complex harmonic force applied to the structure and V is the complex velocity of the structure at the point where the force acts. Re{} denotes the real part of the expression and * denotes the complex conjugate.

3.4.1 Power in the string

Because the string is attached to the beam, the expressions for the force and velocity are taken from Section 2.3. The equations for the force and velocity of the semi-infinite string with an arbitrary impedance attached to the end, equation (2.11), and the time derivative of (2.9) are substituted into equation (3.21) and rearranged to give the net power flow in the string,

$$p_s = p_i - p_r = \frac{1}{2}\omega k_s T\left(|A_i|^2 - |A_r|^2\right) \tag{3.22}$$

where p_s is the total power in the string, p_i is the power in the incident wave and p_r is the power in the reflected wave. The total power in the string is equal to the power that is transmitted to the beam, as can be seen from the power balance, transmitted power = incident power- reflected power.

3.4.2 Power in the beam

The relationship between the applied force and the resulting A_4 propagating wave on a semi-infinite beam can be determined from equation (3.7) to give

$$F_b = -E_b I_b k_f^3 (1-j) A_4 \tag{3.23}$$

The relationship between the velocity at $x = 0$ and the A_4 propagating wave is given by

$$\dot{Y}(0) = j\omega 2 A_4 \tag{3.24}$$

Substituting equations (3.23) and (3.24) into equation (3.21) and rearranging gives the power that is transmitted into the beam,

$$p_b = |A_4|^2 E_b I_b k_f^3 \omega \tag{3.25}$$

where p_b is the power in the beam.

Hence, from (3.22) and (3.25) an expression for the power balance condition is obtained, where power in = power reflected + power transmitted

i.e. $$\frac{1}{2} k_s T\omega |A_i|^2 = \frac{1}{2} k_s T\omega |A_r|^2 + E_b I_b k_f^3 \omega |A_4|^2 \tag{3.26}$$

Dividing through by $\frac{1}{2} k_s T\omega |A_i|^2$ gives,

$$1 - \left|\frac{A_r}{A_i}\right|^2 = \frac{2 E_b I_b k_f^3}{k_s T} \left|\frac{A_4}{A_i}\right|^2 \tag{3.27}$$

This can be rewritten in terms of the non-dimensional frequency $\Omega_b = \dfrac{\omega}{\omega_b}$

where ${\omega_b}^{1/2} = \dfrac{2T}{c_S (E_b I_b)^{1/4} (\rho_b S_b)^{3/4}}$ to give

$$\left|\frac{A_r}{A_i}\right|^2 + 4(\Omega_b)^{1/2} \left|\frac{A_4}{A_i}\right|^2 = 1 \tag{3.28}$$

Using the expressions for $\dfrac{A_r}{A_i}$ and $\dfrac{A_4}{A_i}$ from equations (3.18) and (3.20) respectively, graphs of power reflection and transmission coefficients are plotted against non-dimensional frequency, in Figure 12. The minimum reflected power occurs when $\Omega_b = 1/2$, and this corresponds to a maximum power transmission in the beam. In the limits as Ω_b tends to zero and infinity the reflected power equals the incident power. At low and frequency limits, the transmitted power is very small compared with the incident power. The reflected power equals the transmitted power when $\Omega_b = 4 \pm \frac{3\sqrt{7}}{2}$.

3.5 Conclusions

In a tensegrity structure, junctions occur between strings and beams. In this section, the case where a semi-infinite string is attached in-line to a semi-infinite beam has been examined. When the transverse wave in the string is incident upon the beam, the beam is excited into flexure. The impedance of the beam in flexure has been derived and shown to be inversely proportional to the square root of frequency. The real and imaginary parts of the beam impedance were compared with the results from the mass, stiffness and damping boundaries used in section 2 and it was shown that a beam in flexure is equivalent to a mass and a damper in parallel. Two waves are produced in the beam; an evanescent wave and a flexural propagating wave. Both the reflection and transmission coefficients of the string-beam junction have been described in terms of the ratio of the flexural beam impedance to the string impedance, which was expressed in terms of a non-dimensional frequency. Graphs plotted of the transmission and reflection coefficients show that reflection always occurs, as total impedance matching cannot take place due to the imaginary part of the beam impedance. The reflected wave amplitude and power is a minimum when the impedance of the string is equal to the square root of two times the real part of the beam impedance. The transmitted and reflected waves have equal amplitude when the impedance of the string is equal to the real part of the beam impedance. At low frequencies, the boundary acts as though it were free, at high frequencies the boundary acts as though clamped. The power transmission coefficient has a maximum at the frequency that corresponds to the impedance of the string being equal to the square root of two times the real part of the beam impedance.

This section has considered one configuration of the string attached to the beam; that of the beam and string in-line. The next section considers the string attached perpendicular to the beam and hence longitudinal wave motion will be excited in the beam.

4. A string connected perpendicular to a beam

4.1 Introduction

The aim of this section is to investigate the wave filtering properties of a junction consisting of a string connected perpendicular to a beam. When a transverse wave in the string is incident upon the beam, it will excite longitudinal wave motion in the beam. Firstly, the longitudinal wave propagation along a semi-infinite beam is considered and the longitudinal impedance of the beam is derived. The transmission and reflection coefficients of the joint are derived to give an indication of its wave filtering properties.

4.2 The longitudinal impedance of a beam

A beam of semi-infinite length, as shown in Figure 13, is subject to an in-plane harmonic force, F_b, acting at the left end, defined as $x = 0$, resulting in the generation of a longitudinal wave of amplitude A_l. The in-plane displacement in the beam is denoted by u. As the wave is travelling to the right the displacement of the beam is given by

$$u(x,t) = A_l e^{j(\omega t - k_l x)} \tag{4.1}$$

Where k_l is the longitudinal wave number, given by $\frac{\omega}{c_l}$, where c_l is the longitudinal wave speed in the beam, given by $c_l = \sqrt{\frac{E}{\rho}}$ and ω is the frequency of the harmonic force. The equation of longitudinal wave motion of a beam is given in Appendix C. The subscript "l" denotes longitudinal wavetype.

The force at the point $x = 0$ is equal to the stress in the direction of the force multiplied by the cross-sectional area that the force is acting over, i.e.

$$S_b \sigma_x = -F_b \tag{4.2}$$

where σ_x is the stress in the x-direction and S_b is the cross-sectional area of the beam. The convention that tension is denoted as a positive force and compression as a negative force has been adopted. The stress in the x-direction is given as

$$\sigma_x = E_b \varepsilon_x \tag{4.3}$$

where E_b is the Young's modulus of the beam and ε_x is the strain in the x-direction, which is given by

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{4.4}$$

Combining equations (4.2), (4.3) and (4.4) results in the expression for the force acting at $x = 0$

$$f_b = -S_b E_b \frac{\partial u}{\partial x} \tag{4.5}$$

Substituting the spatial derivative of equation (4.1), evaluated at $x = 0$, into equation (4.5) results in,

$$f_b = F_b e^{j\omega t} = jk_l A_l S_b E_b e^{j\omega t} \tag{4.6}$$

As the impedance is given by $Z_l = \frac{F_b}{\dot{U}(0)}$, dividing equation (4.6) by the time derivative of equation (4.1) evaluated at $x = 0$, results in the longitudinal impedance of the beam, i.e.,

$$Z_l = \frac{S_b E_b}{c_l}, \tag{4.7}$$

where $c_l = \sqrt{\frac{E_b}{\rho_b}}$ is the speed of wave propagation in the beam. Examining the real and imaginary parts of the impedance enables comparison with the mass, stiffness and damping impedances as discussed in section 2.2. As equation (4.7) is real and positive, the beam acts as though it were a damper.

4.3 A string connected to a beam

The combination of the semi-infinite string and the semi-infinite beam is shown in Figure 14. The point of intersection is at $x = 0$, and the string and beam displacements are given by $Y_s(0)$ and $U(0)$ and forces are F_s and F_b. A transverse wave of amplitude A_i in the string is incident upon the beam, which results in a reflected wave with amplitude A_r in the string, and a longitudinal wave with amplitude A_l transmitted in the beam. The aim of this section is to derive the reflection and transmission coefficients as they indicate the wave filtering properties of the junction.

At the boundary between the string and beam, the displacement of the string is equal to that of the beam at $x = 0$,

$$Y_s(0) = U(0) \tag{4.8}$$

and force equilibrium gives,

$$F_s = -F_b \tag{4.9}$$

Substituting equations (3.6) for Y_s and (4.1) for U into equation (4.8), gives

$$A_i + A_r = A_l \tag{4.10}$$

Dividing equation (4.10) by A_i and combining equations (2.14), (4.7) and (4.10) results in the transmission coefficient,

$$\frac{A_l}{A_i} = \frac{2}{1 + \frac{Z_l}{Z_s}} \tag{4.11}$$

where $$\frac{Z_l}{Z_s} = \sqrt{\frac{{S_b}^2 \rho_b E_b}{S_s \rho_s T}} \tag{4.12}$$

The reflection coefficient is found by substituting Z_l, from equation (4.7) for Z in equation (2.14) to give,

$$\frac{A_r}{A_i} = \frac{1 - \frac{Z_l}{Z_s}}{1 + \frac{Z_l}{Z_s}} \tag{4.13}$$

Figure 15 shows the reflection and transmission coefficients plotted against $\frac{Z_l}{Z_s}$. When $\frac{Z_l}{Z_s}$ is small the modulus of $\frac{A_r}{A_i}$ approaches unity and the reflected wave is in-phase with the incident wave in the string, i.e. the string behaves as if it had a free end. Hence, at the junction when $\frac{Z_l}{Z_s}$ tends to zero the amplitude of the transmitted wave tends to twice that of the incident wave, due to the doubling effect of the reflected and incident wave in the string. As $\frac{Z_l}{Z_s}$ approaches infinity $\left|\frac{A_r}{A_i}\right|$ approaches unity, however the reflected wave has a phase difference of 180°, i.e. the string behaves as if it had a fixed end. Hence at $x = 0$, there is a cancellation effect that causes the amplitude of $\frac{A_l}{A_i}$ to approach zero. When the impedance of the beam matches that of the string, there is no reflection and the modulus of the transmitted wave is equal to that of the incident wave. This is because there is no impedance difference between the string and beam and so the wave is transmitted as if the system were continuous. The phase of the reflected wave at this point changes from being completely in-phase to being completely out of phase. The transmitted wave remains constantly in-phase with the incident wave as the impedance ratio alters. The point at which $\left|\frac{A_r}{A_i}\right| = \left|\frac{A_l}{A_i}\right|$ occurs when $\frac{Z_l}{Z_s} = 3$.

The characteristics of the reflected wave are identical to that when a damper was attached to the string (see section 2.2), because of the damping-like properties of a beam with longitudinal motion.

4.4 Power flow through a junction

The aim of this section is to describe the power flow through the junction between the string connected perpendicularly to the beam. The motivation for this is two-fold: firstly to provide insight into the power that is transmitted and reflected, and secondly as a means of validating the previous results. Assuming that the junction between the string and beam is conservative, the power incident upon the junction will be equal to the power leaving the junction. The expression for power associated with the incident and reflected wave in the string is given in section 3.4.1.

4.4.1 Power in the beam

Substituting the equations for the force and the resulting velocity at $x=0$, equation (4.6) and the time derivative of equation (4.1) respectively, into equation (3.21) gives the power in the beam being carried by a longitudinal wave,

$$p_l = \frac{1}{2}\omega \mathrm{k_l S_b E_b} |A_l|^2 \tag{4.14}$$

where p_l is the power in the longitudinal wave in the beam. Using the power balance equation; power in = power reflected + power transmitted, and substituting equation (3.16) and (4.14) gives

$$\frac{1}{2}k_s T\omega |A_i|^2 = \frac{1}{2}k_s T\omega |A_r|^2 + \frac{1}{2}\omega k_l S_b E_b |A_l|^2 \tag{4.15}$$

Dividing through by the incident wave amplitude and rearranging results in:

$$\left|\frac{A_r}{A_i}\right|^2 + \frac{Z_l}{Z_s}\left|\frac{A_l}{A_i}\right|^2 = 1 \tag{4.16}$$

By substituting the expressions for $\frac{A_r}{A_i}$, equation (4.13) and $\frac{A_l}{A_i}$, equation (4.11) into equation (4.16), it enables the power coefficients to be plotted against the impedance ratio. Figure 16 shows the reflected and transmitted powers coefficients plotted against $\frac{Z_l}{Z_s}$. When the impedance ratio is unity, i.e. the longitudinal impedance of the beam matches the impedance of the string the reflected power is zero and the transmitted power is equal to the incident power. At the low and high frequency limits the reflected power approaches equality with the incident power and the transmitted power approaches zero. The reflection and transmission power coefficients are equal at $\frac{Z_l}{Z_s} = 3 \pm 2\sqrt{2}$.

4.5 Conclusions

The aim of this work is to investigate the wave filtering properties of a string and beam junction. This section has been concerned with a particular configuration; that of the string attached perpendicular to the beam. When the string is excited harmonically, a longitudinal wave is produced in the beam, and hence the longitudinal beam impedance was derived. It was shown to be independent of frequency and that it acts in the same way as a damper when connected to a string. Reflection and transmission coefficients were obtained and shown to be dependent only on the impedance ratio between the beam and the string, and hence independent of frequency. When the beam impedance is much less than that of the string, the junction acts as a free end, and when it is much greater than that of the string, the junction acts as a fixed end. A complete impedance match between the string and beam is possible and when this occurs there is no reflected wave generated in the string.

The next section considers the configuration of a beam joined to a string at an arbitrary angle, when both flexural and longitudinal waves are produced in the beam.

5. A string connected to a beam at an arbitrary angle

5.1 Introduction

In practice, the strings and beams in tensegrity structures are neither connected in-line nor at right angles. Hence, in this section a semi-infinite beam joined at an arbitrary angle, θ, to a semi-infinite string is considered. A propagating transverse wave of amplitude, A_i travels in the string and is incident upon the beam generating both longitudinal, A_l, and flexural propagating waves, A_4, and a decaying wave, A_2, in the beam and a reflected wave, A_r in the string. The aim this section is to describe each of these waves relative to the incident wave and so gain understanding of the reflection and transmission properties of the junction. Following this introduction, the next section derives the reflection and transmission coefficients using force equilibrium and displacement continuity at the junction. In the subsequent section, the power flow through the junction is derived to give an indication of the power carried in the transmitted and reflected waves.

5.2 String and beam joined at an arbitrary angle

Figure 17 shows a semi-infinite string attached to a semi-infinite beam at an angle of θ. A sliding boundary condition is assumed, i.e. there is no horizontal displacement at the attachment point .The displacement of the beam at $x = 0$, in the direction of the force is $W(0)$, where the force on the beam is F_b. This acts in the same direction as the force in the string, F_s. The boundary conditions at $x = 0$are equilibrium of forces and continuity of displacements, i.e.

$$F_b = -F_s \tag{5.1}$$

$$W(0) = Y_s(0) \tag{5.2}$$

By resolving the displacement of the beam, $W(0)$, and the force F_b into components that are in-line with and perpendicular to the axis of the beam, gives the following relationships

$$W(0) = Y_b(0)\ \cos\theta + U(0)\sin\theta \tag{5.3}$$

$$F_b \cos\theta = Z_f \dot{Y}_b(0) \tag{5.4}$$

$$F_b \sin\theta = Z_l \dot{U}(0) \tag{5.5}$$

Substituting for $\dot{Y}_b(0)$ from equation (5.4) and $\dot{U}(0)$ from equation (5.5) into the time derivative of equation (5.3) yields

$$\dot{W}(0) = \frac{F_b \cos^2\theta}{Z_f} + \frac{F_b \sin^2\theta}{Z_l} \tag{5.6}$$

Rearranging equation (5.6) gives

$$F_b = \dot{W}(0)\left(\frac{Z_f Z_l}{Z_l \cos^2\theta + Z_f \sin^2\theta}\right) \tag{5.7}$$

The term inside the brackets is the equivalent impedance of the beam, Z_{eq}, when the beam is attached to the string at an angle θ as shown in Figure 18, i.e.

$$Z_{eq} = \frac{Z_f Z_l}{Z_l \cos^2\theta + Z_f \sin^2\theta} \tag{5.8}$$

This impedance can be substituted for Z in equation (2.14) resulting in the reflection coefficient

$$\frac{A_r}{A_i} = \frac{1 - \dfrac{Z_{eq}}{Z_s}}{1 + \dfrac{Z_{eq}}{Z_s}} \tag{5.9}$$

Equation (5.9) can be expressed in non-dimensional form by noting that

$$\frac{Z_{eq}}{Z_s} = \frac{Z_l/Z_s}{\dfrac{Z_l/Z_s}{\Omega_b^{1/2}(1+j)}\cos^2\theta + \sin^2\theta} \tag{5.10}$$

where $\Omega_b = \dfrac{\omega}{\omega_b}$ where $\omega_b^{1/2} = \dfrac{2T}{c_S (E_b I_b)^{1/4} (\rho_b S_b)^{3/4}}$

The transmission coefficients are obtained by combining equations (5.4) and (5.5) and by applying the boundary condition given by equation (5.2).

Substituting for the flexural impedance of the beam, equation (3.9) and for the out of plane velocity, obtained from equations (3.13) and (3.14), into equation (5.4) gives

$$F_b \cos\theta = E_b I_b k_f^{\,3}(1+j) j A_4 \tag{5.11}$$

Substituting for the longitudinal impedance of the beam, equation (4.7) and for the in-plane velocity, obtained from equation (4.1), into equation (5.5) gives

$$F_b \sin\theta = j k_l E_b S_b A_l \tag{5.12}$$

Combining equations (5.11) and (5.12) leads to a relationship between the flexural and longitudinal waves in the beam

$$\frac{E_b I_b k_f^{\,3}(1+j) A_4}{\cos\theta} = \frac{E_b S_b k_l A_t}{\sin\theta} \tag{5.13}$$

Rearranging and simplifying equation (5.13) leads to the ratio of the longitudinal wave amplitude to that of the flexural wave amplitude

$$\frac{A_l}{A_4} = 2\tan\theta \frac{Z_f}{Z_l} \tag{5.14}$$

Substituting for W, equation (5.3), $Y_s(0)$ from equation (2.9), $Y_b(0)$ obtained from equations (3.13) and (3.36), and $U(0)$ obtained from equation (4.1), into equation (5.2) gives

$$A_i + A_r = 2A_4 \cos\theta - A_l \sin\theta \tag{5.15}$$

This expression can be arranged to give the flexural transmission coefficient and the longitudinal transmission coefficient, i.e.

$$\frac{A_4}{A_i} = \frac{(1+\alpha)\cos\theta}{2\cos^2\theta - \beta\sin^2\theta} \tag{5.16}$$

$$\frac{A_l}{A_i} = \frac{(1+\alpha)\sin\theta}{\dfrac{2\cos^2\theta}{\beta} - \sin^2\theta} \tag{5.17}$$

where $\alpha = \dfrac{A_r}{A_i} = \dfrac{1 - Z_{eq}/Z_s}{1 + Z_{eq}/Z_s}$ and $\beta = 2\dfrac{Z_f}{Z_l}$.

In Figure 19 the reflection and transmission coefficients are plotted for various angles between the string and beam. The graphs of modulus and phase are plotted against Ω_b for the angles $\theta = 30°, 45°, 90°$, using as an example the case when the longitudinal impedance of the beam is ten times greater than that of the string impedance. As θ increases, there are trends of behaviour in both the modulus and phase plots. The minimum value of the modulus of the reflected wave shifts to the left (decreases in frequency) with increasing θ. The amplitude of the flexural wave increases at low frequencies and the rate of roll-off increases with increasing θ. The longitudinal wave amplitude increases gradually with increasing θ. As θ increases all the phase shifts occurring in all wavetypes occurs at a lower frequency.

5.3 Power flow through the junction

The aim of this section is to examine how power flows through the junction between a semi-infinite string and a semi-infinite beam joined at an arbitrary angle. In section 2.7.1 and section 2.7.2 the power associated with the incident and reflected waves in the string and the power associated with the flexural wave in the beam were derived. In section 3.4.1 the power associated with the longitudinal wave in the beam was derived. As all wavetypes mentioned occur when the string is joined to the beam at an arbitrary angle and assuming the junction is conservative, the power balance equation is as follows

Power in = Power reflected + Power in transmitted flexural wave + Power in transmitted longitudinal wave

Substituting from equations (3.22), (3.25) and (4.14) gives

$$\frac{1}{2}k_sT\omega|A_i|^2=\frac{1}{2}k_sT\omega|A_r|^2+\frac{1}{2}\omega k_lS_bE_b|A_l|^2+\omega E_bI_bk_f^3|A_4|^2 \quad (5.18)$$

Expressing equation (5.18) relative to the input power in a non-dimensional form gives

$$\left|\frac{A_r}{A_i}\right|^2+4(\Omega_b)^{1/2}\left|\frac{A_4}{A_i}\right|^2+\frac{Z_l}{Z_s}\left|\frac{A_l}{A_i}\right|^2=1 \quad (5.19)$$

where $\Omega_b=\frac{\omega}{\omega_b}$ and $\omega_b{}^{1/2}=\frac{2T}{c_S(E_bI_b)^{1/4}(\rho_bS_b)^{3/4}}$.

Substituting $\frac{A_r}{A_i}$, equation (5.9), $\frac{A_4}{A_i}$, equation (5.16) and $\frac{A_l}{A_i}$, equation (5.17) into equation (5.19) enables the power coefficients to be plotted against Ω_b. Figure 20 shows graphs of the reflected and transmitted power coefficients for three angles, 30°, 45° and 60°, and setting $\frac{Z_l}{Z_s}=10$ as an example. As the angle between the string and beam increases, the maximum power transmitted in the flexural wave and the minimum power reflected occurs at a lower frequency and the magnitude of the power decreases. The magnitude of power transmitted in the longitudinal wave increases with angle and frequency. Hence, at high frequencies and large angle (still acute) of intersection between the string and beam the flexural wave is less significant than the longitudinal wave in the beam.

5.4 Conclusions

In this section, a semi-infinite string attached at an arbitrary angle to a semi-infinite beam has been examined. A transverse wave in the string incident upon the beam causes wave reflection and transmission. By assuming only transverse waves are present in the string and flexural and longitudinal wavetypes in the beam, the reflection and transmission coefficients for wave amplitudes and power have been derived. The effect of varying the angle between the string and beam was examined and it was found that as the angle increased, the frequency at which the reflection coefficient is a minimum decreases and the phase changes occur at lower frequencies in each wavetype. The flexural transmission power coefficient is dominant over the reflection power coefficient and the longitudinal transmission power coefficient for approximately one decade of frequency. The frequency at which this occurs decreases, as angle increases. The longitudinal power transmission coefficient is greater than the flexural power transmission coefficient at high frequencies.

6. Conclusions

6.1 Conclusions

In this report, a junction between a semi-infinite string and beam in three configurations has been considered. A transverse propagating wave in the string is incident upon the beam and the transmission and reflection coefficients and power transmission and reflection coefficients have been derived for each configuration. For the case when the string is joined in-line with the beam, a flexural propagating wave and a decaying wave are present in the beam. Maximum values of both the transmission and power transmission coefficient occur when the impedance of the string is equal to the square root of two times the real part of the flexural impedance of the beam. When the beam and string are joined transversely, a longitudinal propagating wave is present in the beam. Maximum transmission occurs when the impedance of the string equals that of the longitudinal beam impedance. The final configuration considered was that of the string and beam joined at an arbitrary angle. In this case, both flexural and longitudinal wavetypes are present in the beam. Transmission and reflection coefficients are a function of the angle. At low frequencies and small angle, flexural transmission is dominant, and at high frequencies and large angle, longitudinal transmission is dominant.

The ultimate aim of the work is to develop an analytical model of the dynamic behaviour of a tensegrity structure. The work completed so far has considered a single simplified junction between a single tension and a compression member. A semi-infinite string was chosen to represent the tension member and a semi-infinite beam to represent the compression member. Three junction-types have been examined: the string and beam in-line, the string and beam joined at right angles and finally the string and beam attached at an arbitrary angle. In each case, a transverse propagating wave in the string was incident upon the beam and reflection and transmission occurs. Using the boundary conditions, expressions for the transmitted and reflected waves have been derived in terms of the incident wave amplitude and the impedances of the string and beam.

Figures

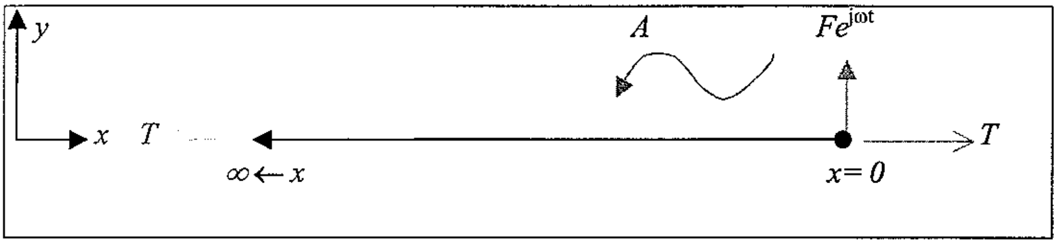

Figure 1 Transverse wave of amplitude A, generated by a harmonic force $Fe^{j\omega t}$, propagating along a semi-infinite string under a static tension force, T.

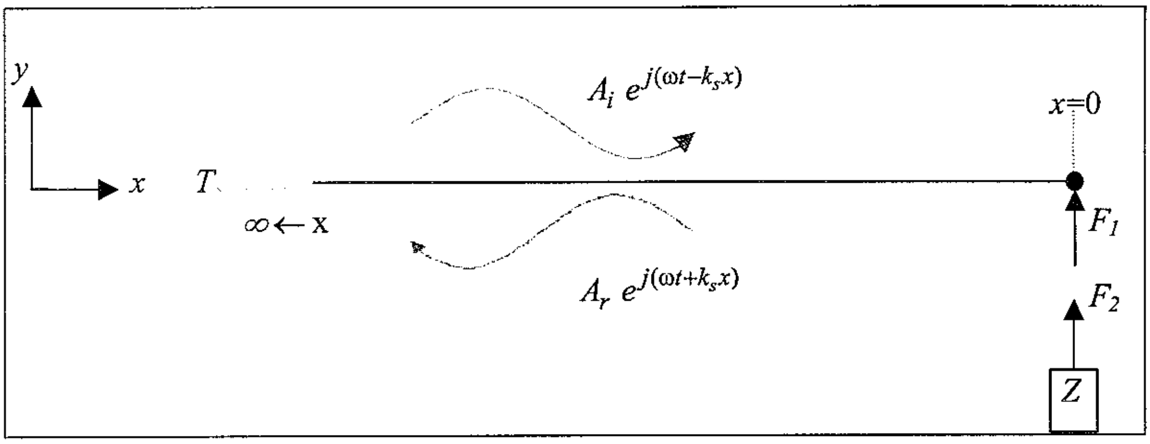

Figure 2 A semi-infinite string under tension *T* with an arbitrary impedance *Z* attached

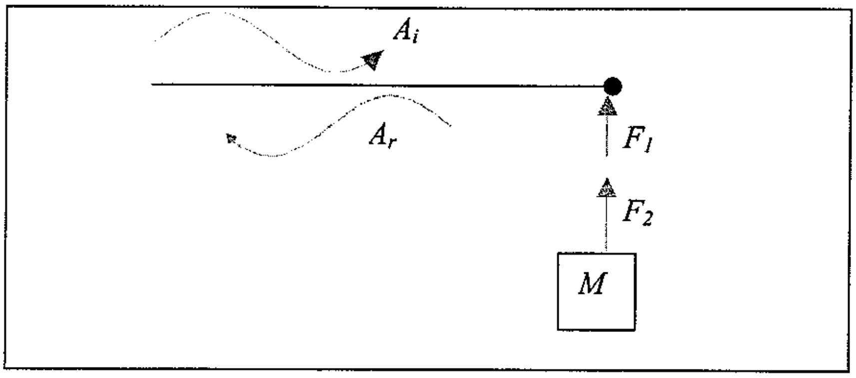

Figure 3 A mass attached to a semi-infinite string at $x = 0$. A_i is incident upon the mass M causing a reflected wave A_r.

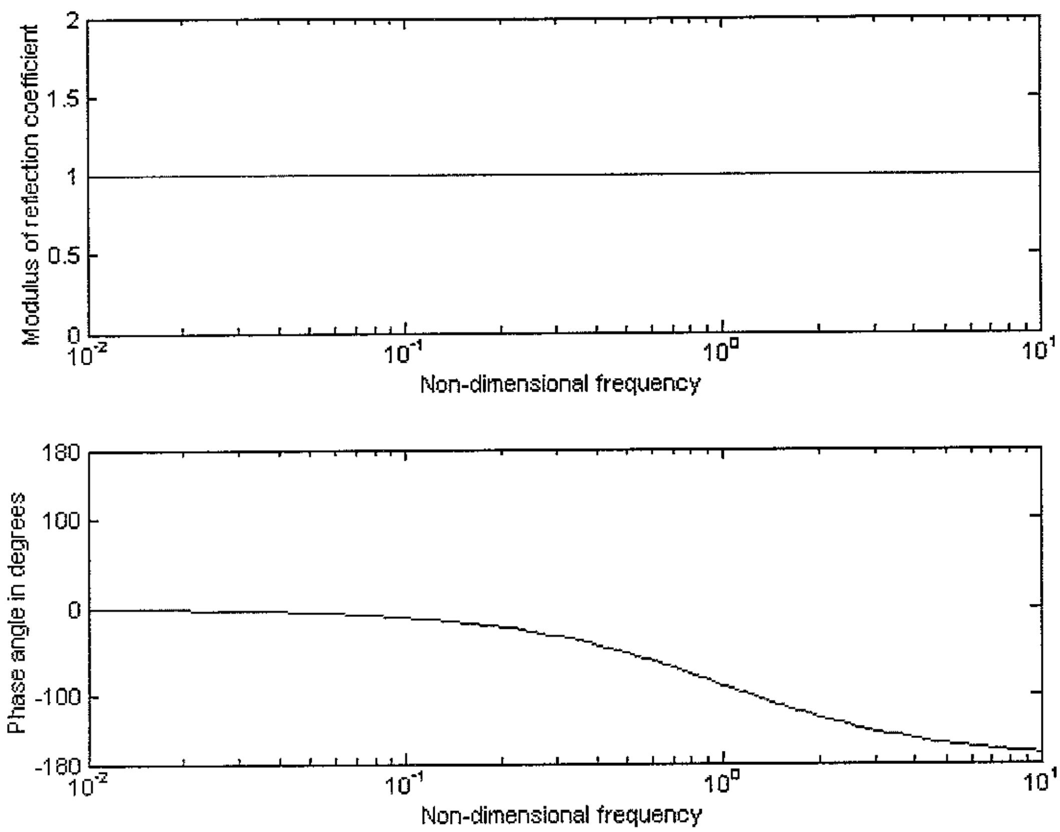

Figure 4 Modulus and phase of the reflection coefficient when incident upon mass-like impedance plotted against non-dimensional frequency, Ω_m.

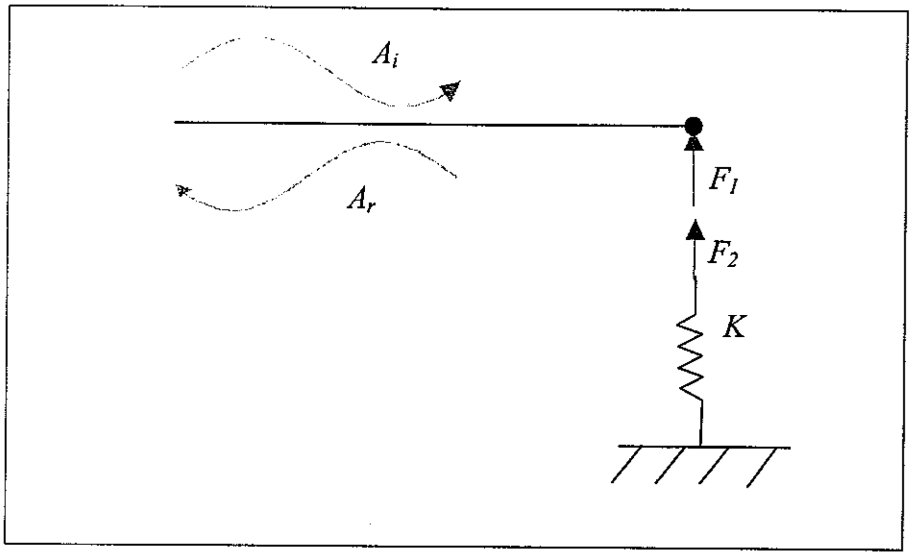

Figure 5 A semi-infinite string is attached to stiffness at $x = 0$, resulting in a reflected wave A_r.

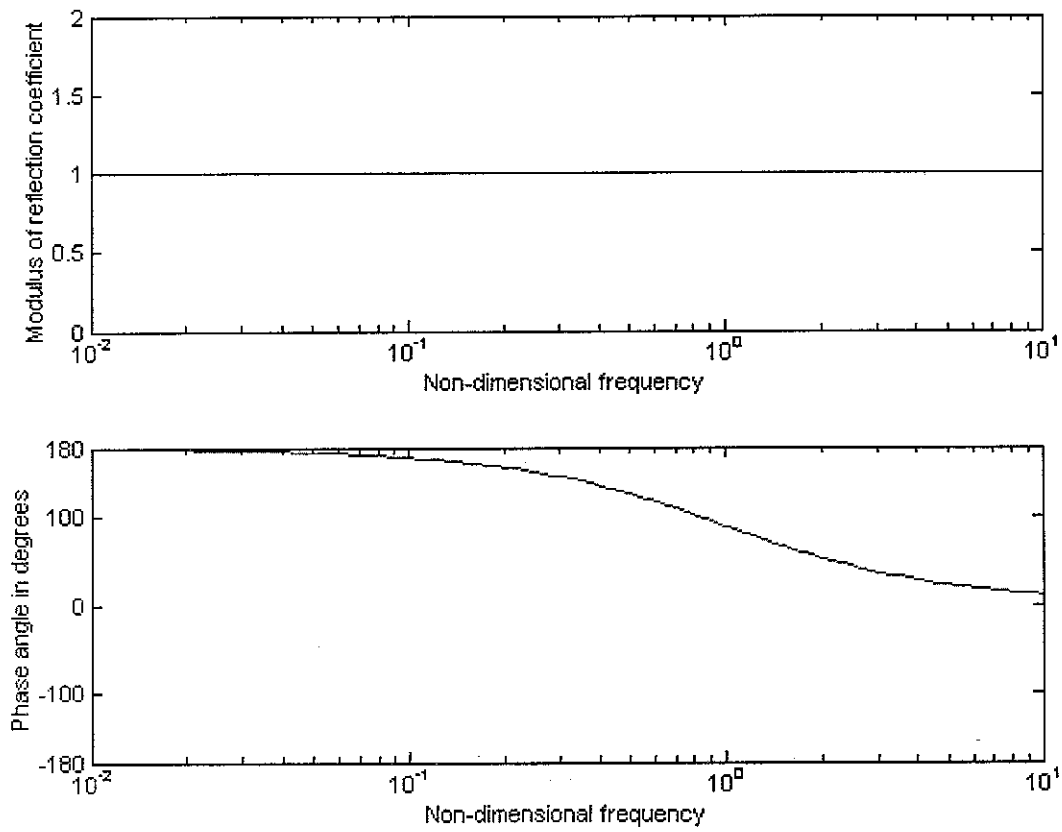

Figure 6 Modulus and phase of the reflection coefficient when incident upon stiffness-like impedance plotted against non-dimensional frequency, Ω_K.

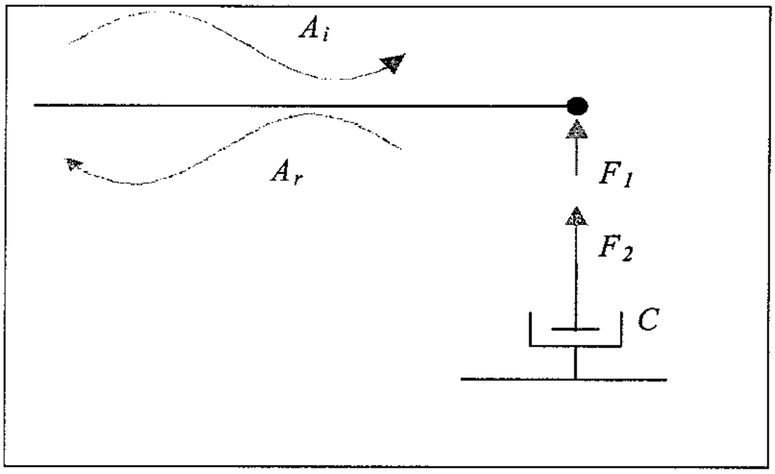

Figure 7 A semi-infinite string is attached to a viscous damper at $x = 0$, resulting in a reflected wave A_r.

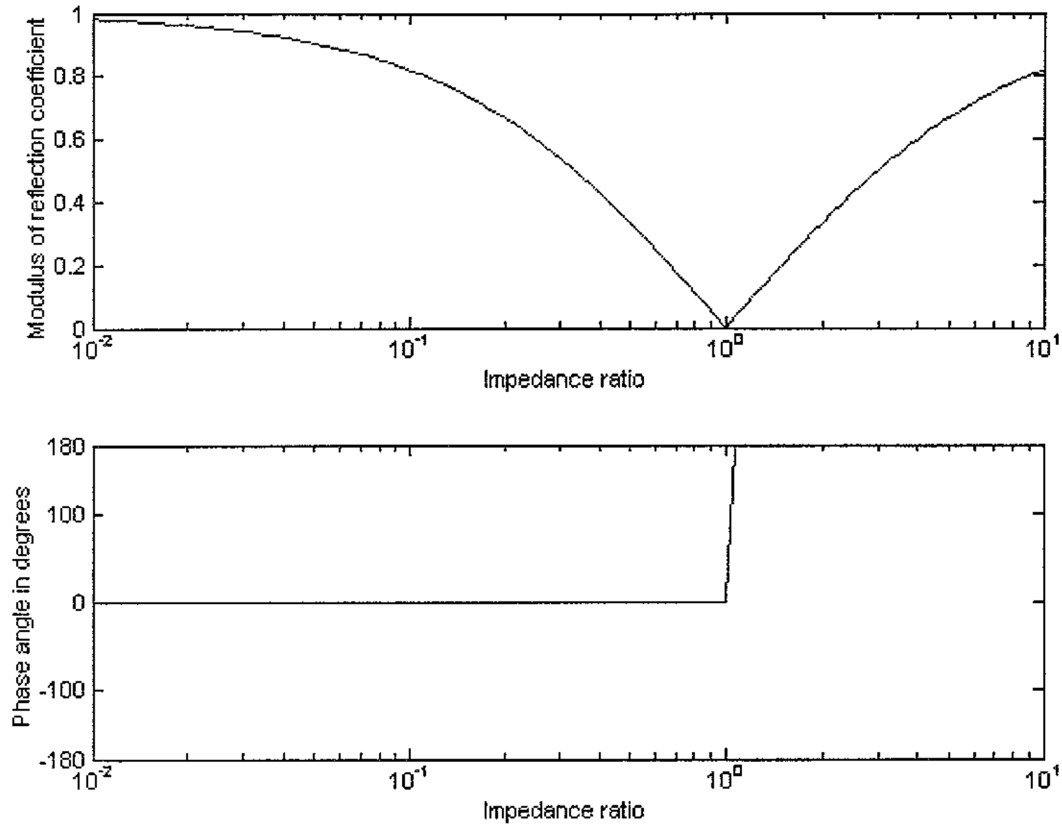

Figure 8 Modulus and phase of the reflection coefficient when incident upon a viscous damping impedance plotted against the impedance ratio $\overline{Z}_C$.

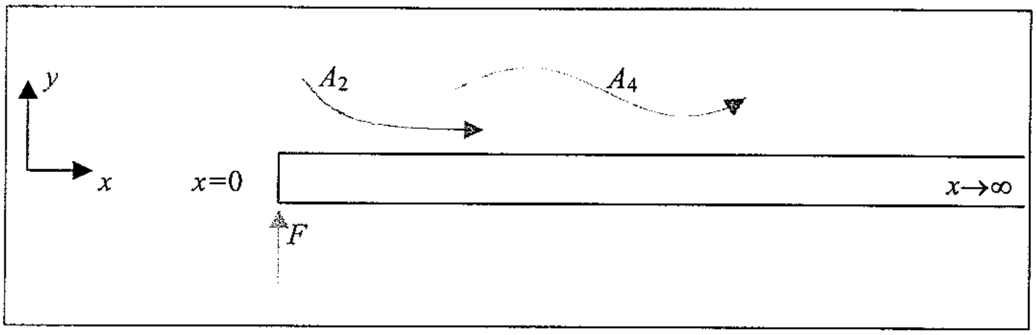

Figure 9 A semi-infinite beam with a transverse force, F, acting at $x = 0$ and A_2 is evanescent wave and A_4 is a flexural propagating wave.

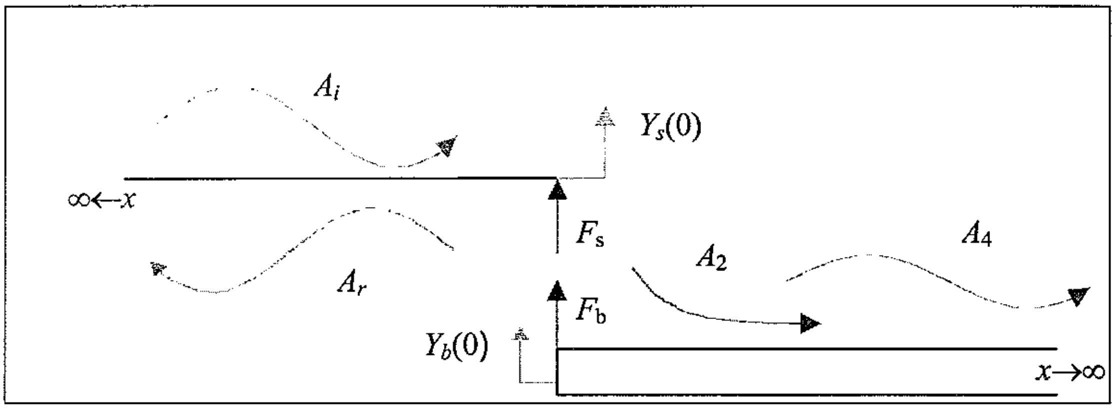

Figure 10 A semi-infinite string connected in-line to a semi-infinite beam. The forces and displacements of both the string and beam where they join are shown disjointed for clarity.

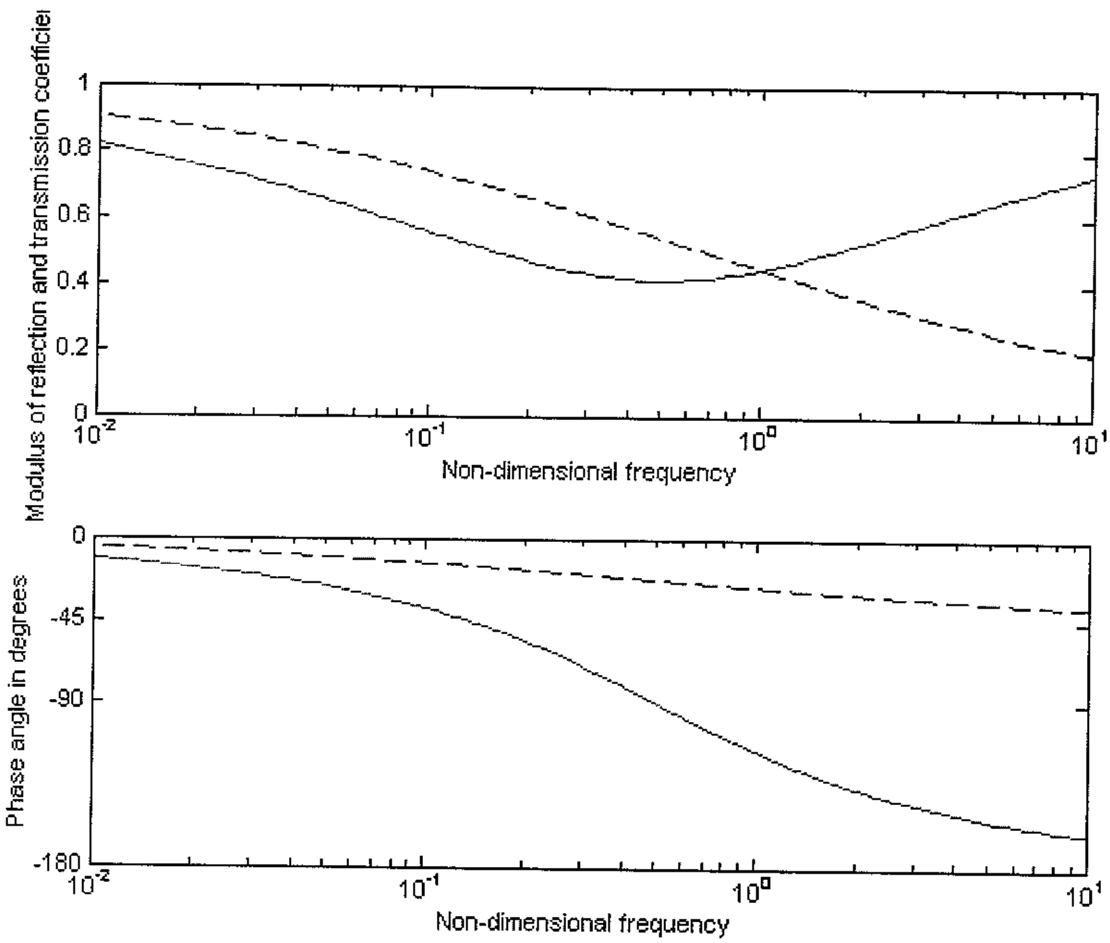

Figure 11 Modulus and phase of the reflection and transmission coefficients when incident upon a beam attached in-line plotted against non-dimensional frequency, Ω_b. The solid line represents the reflected wave and the dashed line represents the transmitted wave.

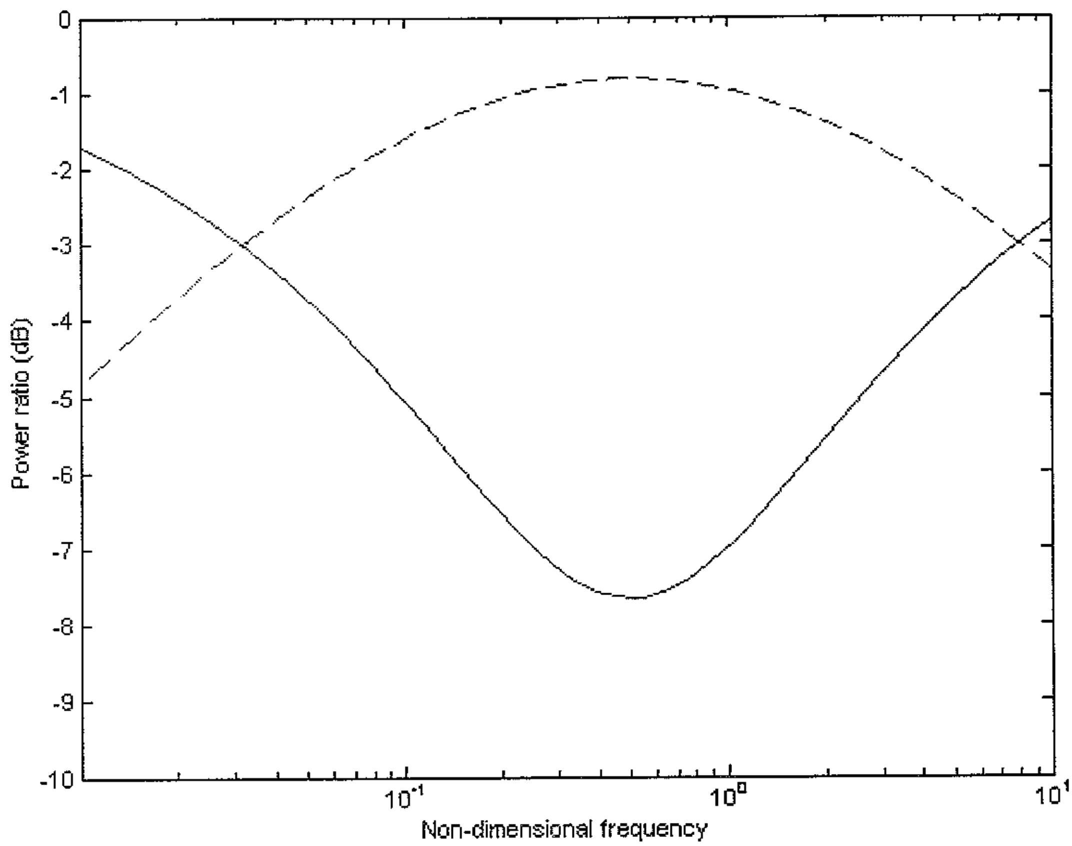

Figure 12 Power coefficients of the reflected and transmitted waves when the string is attached in-line with the beam. The solid line represents the reflected power coefficient and the dashed line represents the transmitted power coefficient.

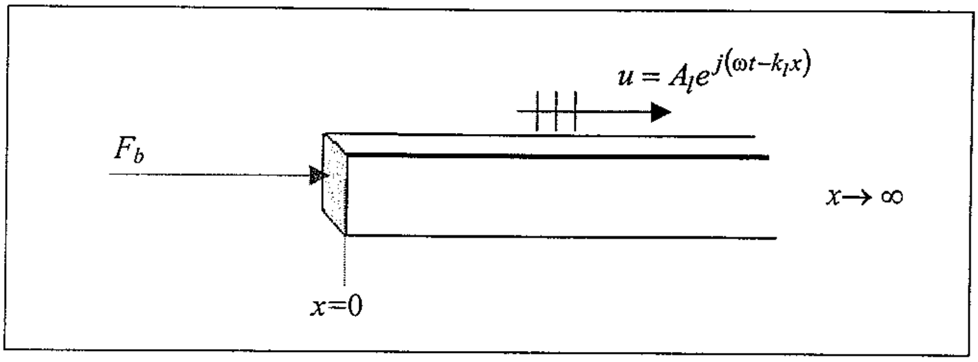

Figure 13 An in-plane force, F_b, produces a longitudinal wave, A_l, in the beam. The shaded area represents the cross-sectional area of the beam.

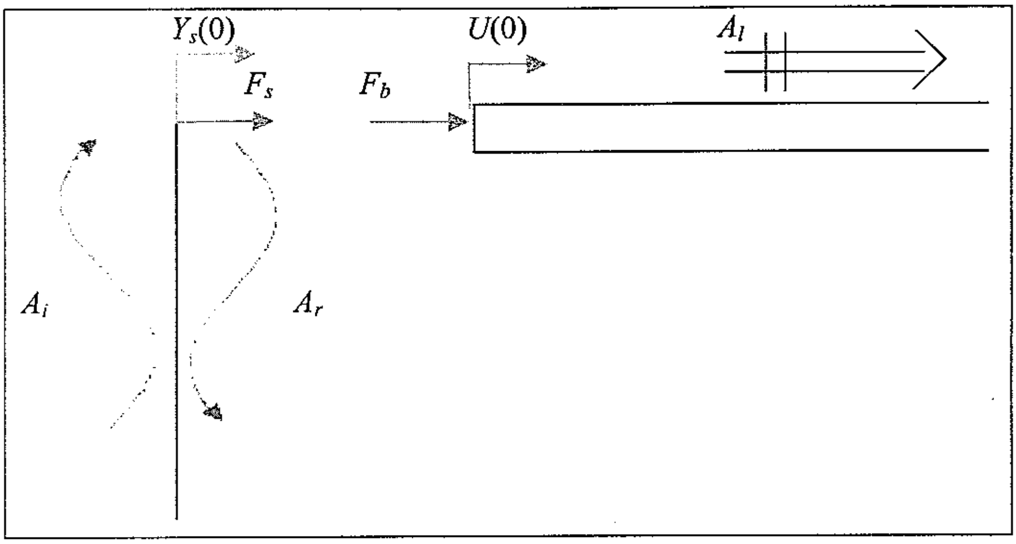

Figure 14 A semi-infinite string connected perpendicular to a semi-infinite beam. The forces and displacements of both the string and beam where they join are shown disjointed for clarity.

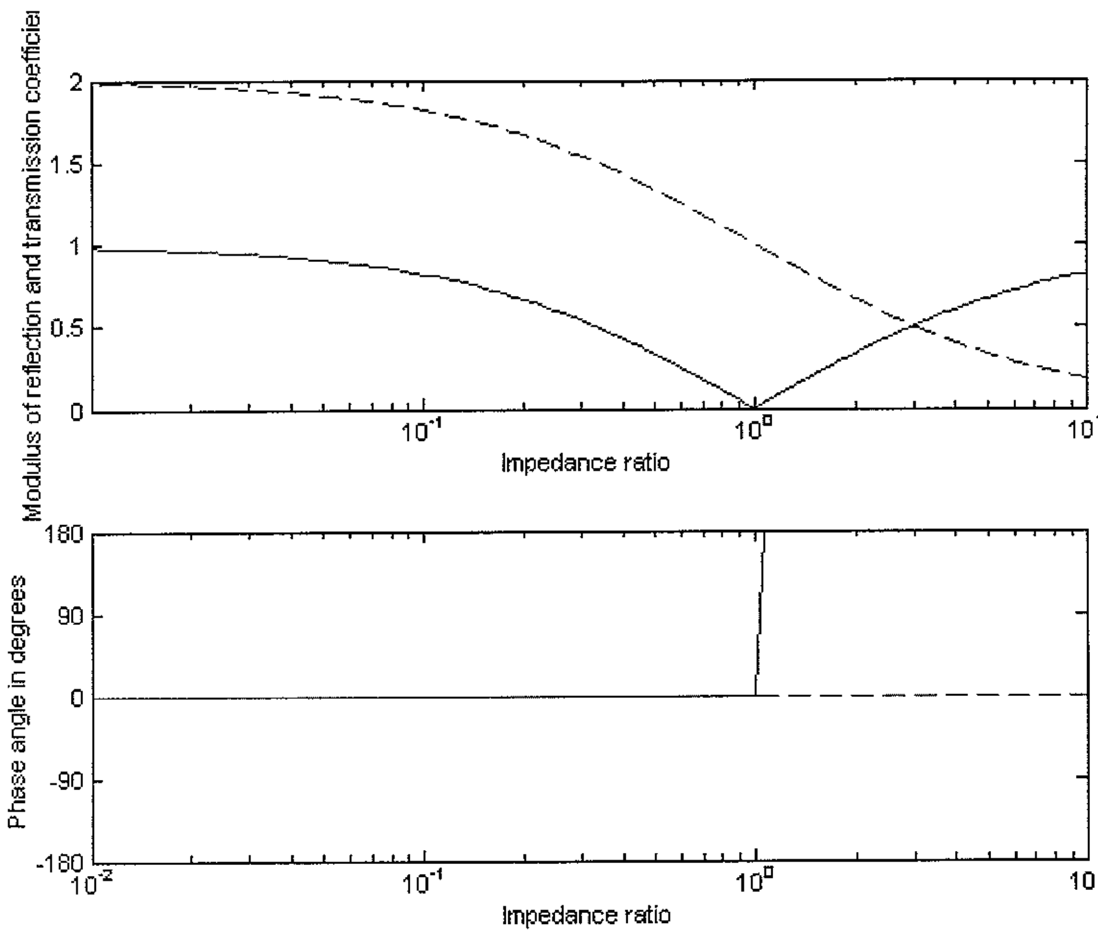

Figure 15 Modulus and phase of reflection and transmission coefficients plotted against $\frac{Z_l}{Z_s}$ when the string is attached perpendicular to the beam. Solid line denotes the reflected wave and the dashed line denotes the transmitted wave.

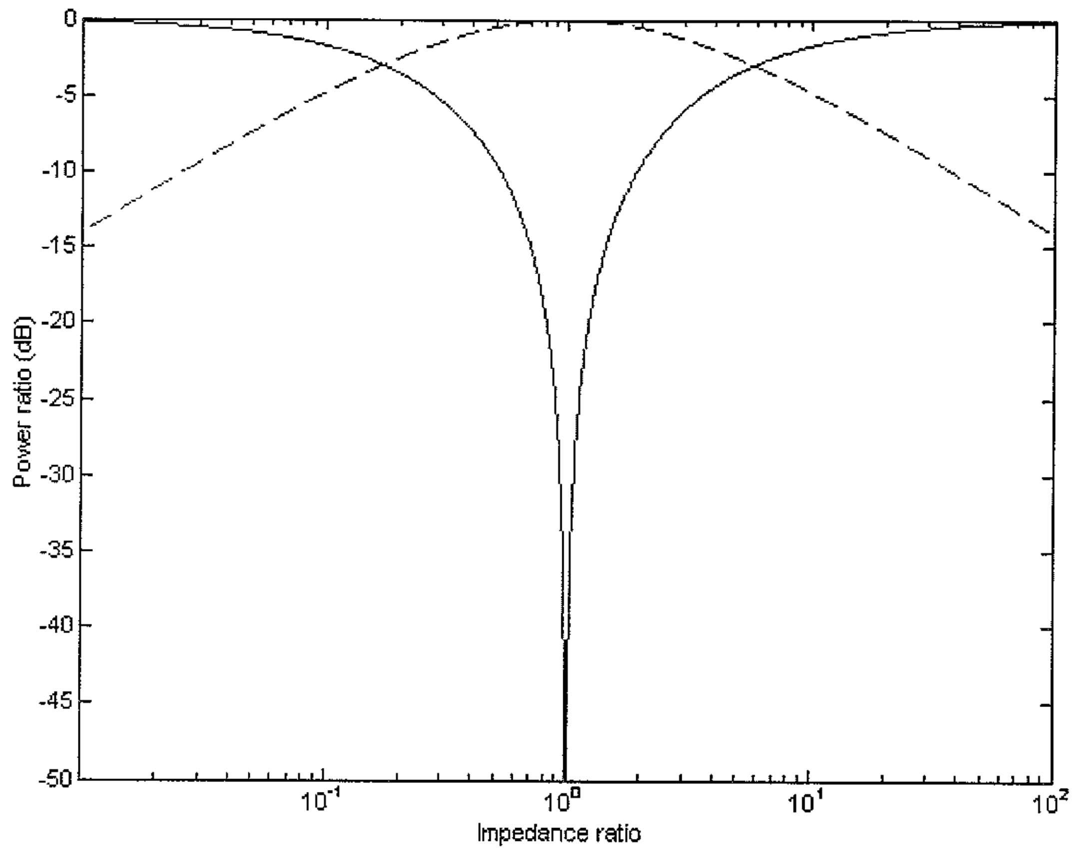

Figure 16 Power coefficients of the reflected and transmitted waves when the string is attached at 90 degrees to the beam plotted against $\frac{Z_l}{Z_s}$. The solid line represents the reflected power coefficient and the dashed line represents the transmitted power coefficient.

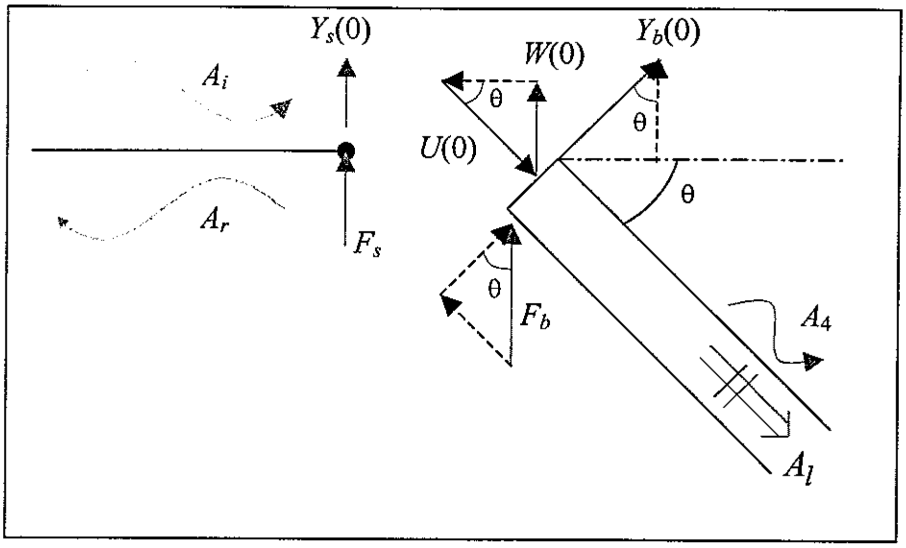

Figure 17 A semi-infinite string connected at an arbitrary angle θ to a semi-infinite beam.

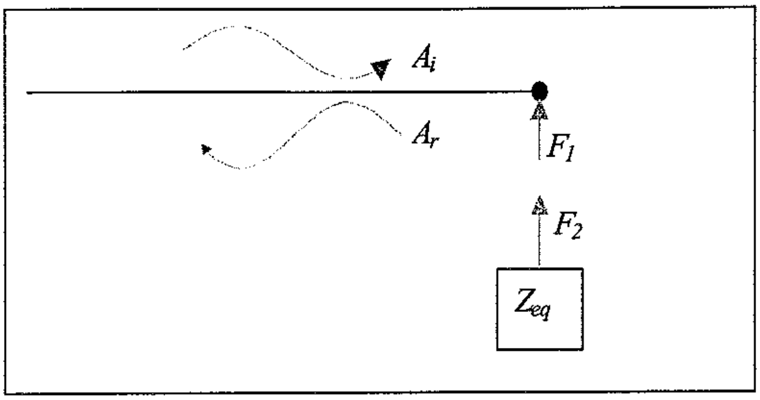

Figure 18 A semi-infinite string is attached to an equivalent impedance at $x = 0$, resulting in a reflected wave A_r.

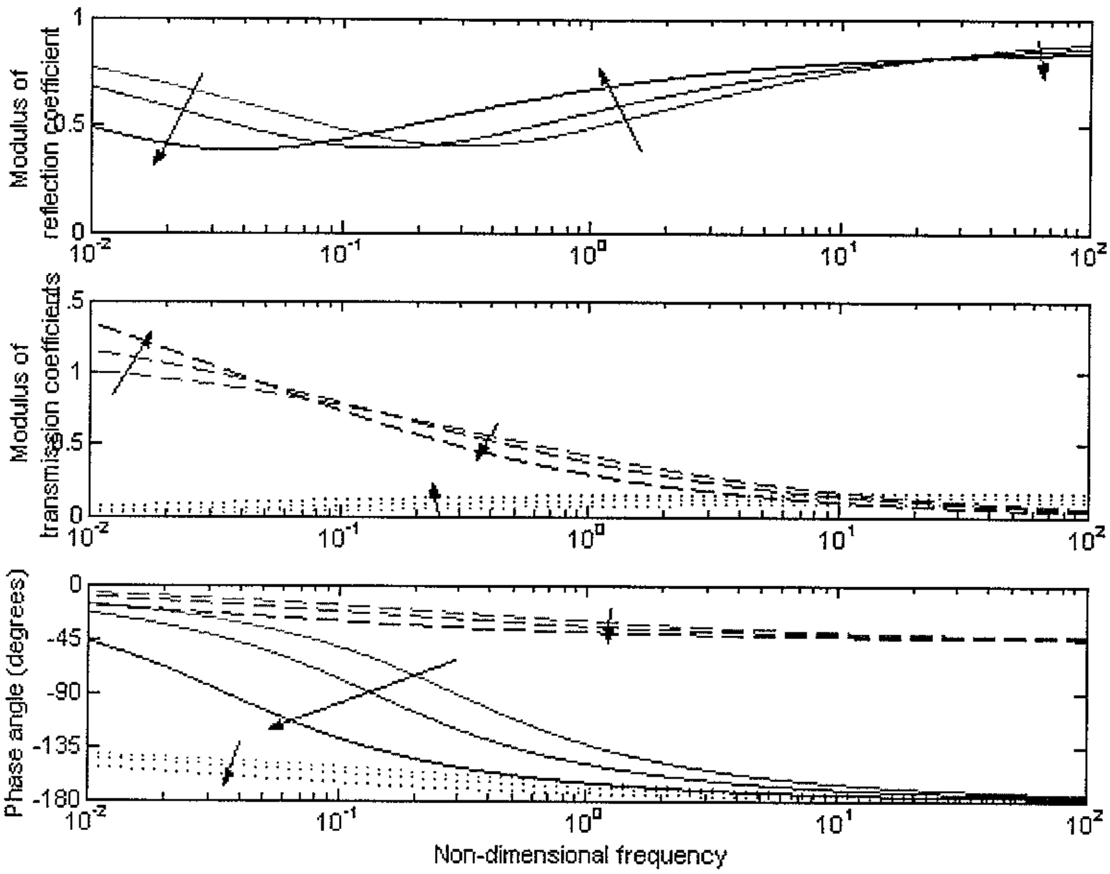

Figure 19 Modulus and phase of reflected and transmitted waves plotted against Ω_b for three different angles between the string and beam; 30°, 45° and 60°. The arrows indicate increasing angle, the solid line depicts the reflected wave, the dashed line represents the flexural transmitted wave and the dotted line represents the longitudinal transmitted wave.

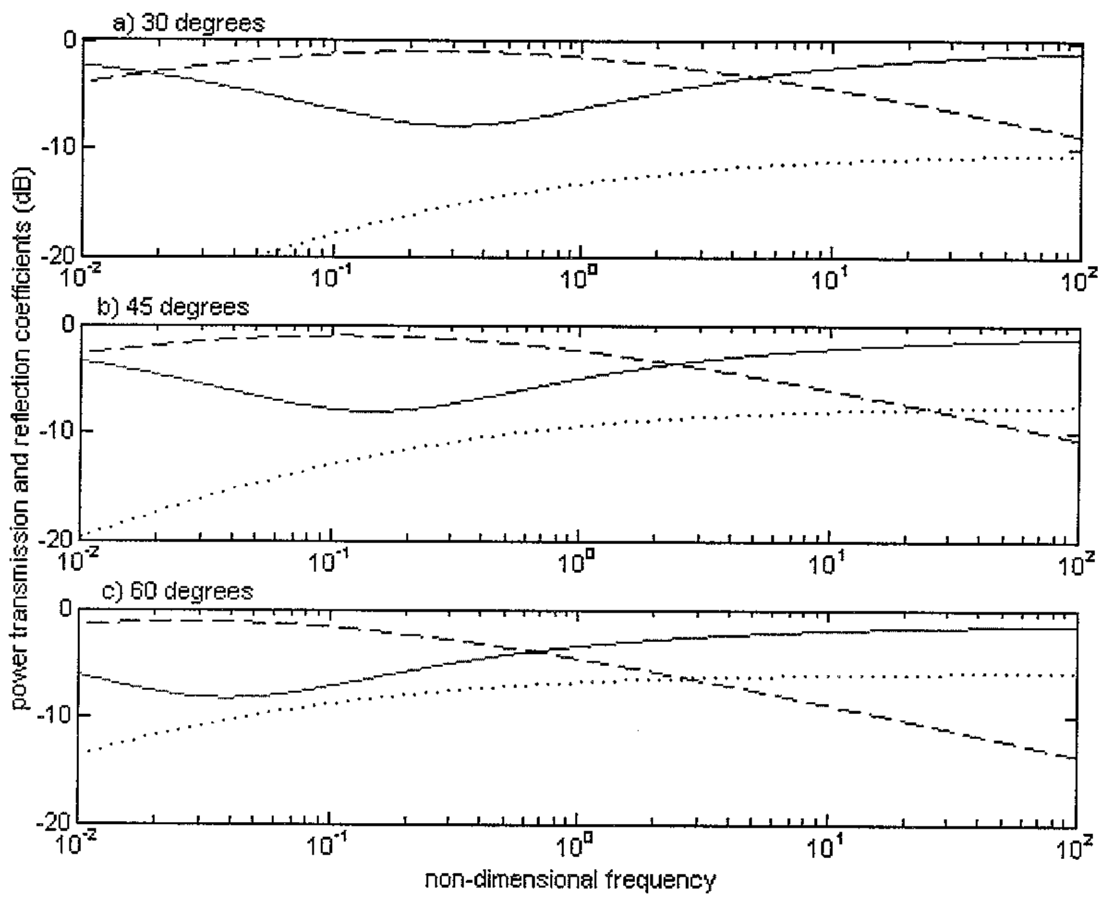

Figure 20 Power reflection and transmission coefficients plotted against Ω_b for various angles. The solid line represents the reflected wave, the dashed line represents the flexural transmitted wave and the dotted line represents the longitudinal transmitted wave.

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Appendix A

The aim of this appendix is to derive the transverse wave motion of a semi-infinite string under tension.

A.1 Equation of transverse motion for a string under tension

The transverse motion of a string under a tension force, T, is a function of the density, ρ_s, and the cross sectional area S_s of the string. The subscript "*s*" refers to the string. In the model used to describe the wave motion the following assumptions are made:

- The string has no bending stiffness
- The tension force is constant along the length of the string
- Transverse displacements (y) are small from the equilibrium position
- Out-of-plane motion is the only motion.

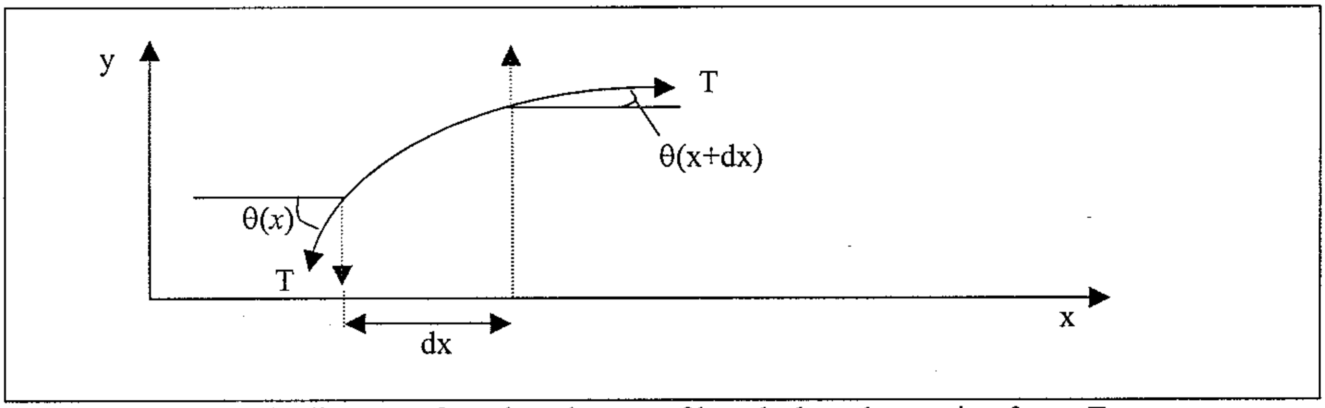

Figure A1 Free body diagram of a string element of length *dx* under tension force, T.

Figure A1 shows the forces on a section of string of length *dx*. There is a net transverse force, *dF*, which is the resultant of the two vertical forces, depicted by the vertical arrows in Figure A1. This force is caused by the change in angle θ, and hence the change in the direction of the tension force. It is given by,

$$dF = T\sin\theta(x+dx) - T\sin\theta(x) \tag{A1}$$

where the first term is $T\sin\theta$ evaluated at $x+dx$ and the second term is $T\sin\theta$ evaluated at x. Using the Taylor's series expansion on the first term and truncating the series gives

$$dF = \left[T\sin\theta(x) + \frac{\partial(T\sin\theta(x))}{\partial x}dx + \ldots\right] - T\sin\theta(x) = \frac{\partial(T\sin\theta(x))}{\partial x}dx \tag{A2}$$

Assuming that θ is small, $\sin\theta \approx \left(\frac{dy}{dx}\right)$, and hence

$$dF = T\frac{\partial^2 y}{\partial x^2}dx. \tag{A3}$$

This force causes the mass, $\rho_s Sdx$ to have an acceleration of $\frac{\partial^2 y}{\partial t^2}$. By equating the two expressions for the force, dF and dividing by dx, gives

$$T\frac{\partial^2 y}{\partial x^2} - \rho_s S\frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{(A4)}$$

which is the transverse wave equation of the string. It can be rewritten in terms of the phase velocity c_s.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c_s}\frac{\partial^2 y}{\partial t^2} = 0 \quad \text{, where } c_s = \sqrt{\frac{T}{\rho_s S}} \qquad \text{(A5)}$$

To obtain an expression for y, harmonic solutions with frequency ω are sought of the form

$$y(x,t) = Y(x)e^{j\omega t} \qquad \text{(A6)}$$

and substituted into the wave equation (A5) to give

$$\frac{d^2 Y}{dx^2} + k_s^{\,2} Y = 0 \text{, where } k_s = \frac{\omega}{c_s} \text{ is the wavenumber of the string.} \qquad \text{(A7)}$$

The solution is a sum of two complex exponentials that can be written as

$$Y(x) = A_1 e^{jk_s x} + A_2 e^{-jk_s x} \qquad \text{(A8)}$$

Substituting (A8) into (A6) gives the harmonic solution to the wave equation as a function of space and time

$$y(x,t) = A_1 e^{j(\omega t + k_s x)} + A_2 e^{j(\omega t - k_s x)} \qquad \text{(A9)}$$

The first term represents a wave varying harmonically in space and time, propagating to the left, the second term is a wave propagating to the right, and A_1 and A_2 are the respective wave amplitudes.

A.2 A semi-infinite string under tension.

A semi-infinite string has an end, and the string stretches to infinity from that end. This report considers the case where the end is on the right, defined as $x = 0$, and the string extends to $-\infty$ in the x-direction. Hence, when an excitation source is positioned at $x = 0$ only waves propagating to the left are present, therefore the motion of the string can be described as

$$y(x,t) = A_1 e^{j(\omega t + k_s x)} \qquad \text{(A10)}$$

Appendix B

The aim of this appendix is to derive the flexural wave motion of a semi-infinite beam. To investigate the effects of a compression force acting on the beam and the effects shear deformation and rotational inertia has on the motion of the beam.

B.1 Equation of flexural wave motion in an Euler-Bernoulli beam

A beam that undergoes flexural wave motion is described in terms of its cross-sectional area, S_b, the density ρ_b, the Young's modulus, E_b, and the second moment of area, I_b. The subscript "*b*" refers to the beam.

In the analysis in this appendix the following assumptions are made:

- The beam is uniform and elastic
- Motion only exists in the out-of-plane direction
- Plane cross-sections remain plane and perpendicular to the longitudinal axis

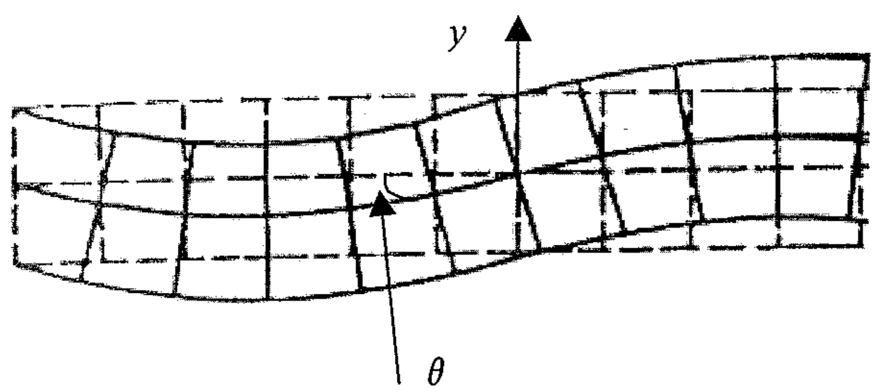

Figure B1 Flexural wave motion in beam causing rotation and vertical displacement, [11].

In addition to the vertical displacement of the beam and the rotation of the cross-section as shown in Figure B1, there are two other terms that contribute to the wave motion, namely that of the bending moments, $M(x)$ and vertical shear forces, $Q(x)$. By considering the effects of these on an element of length dx, as shown in Figure B2, four equations describing the motion are obtained.

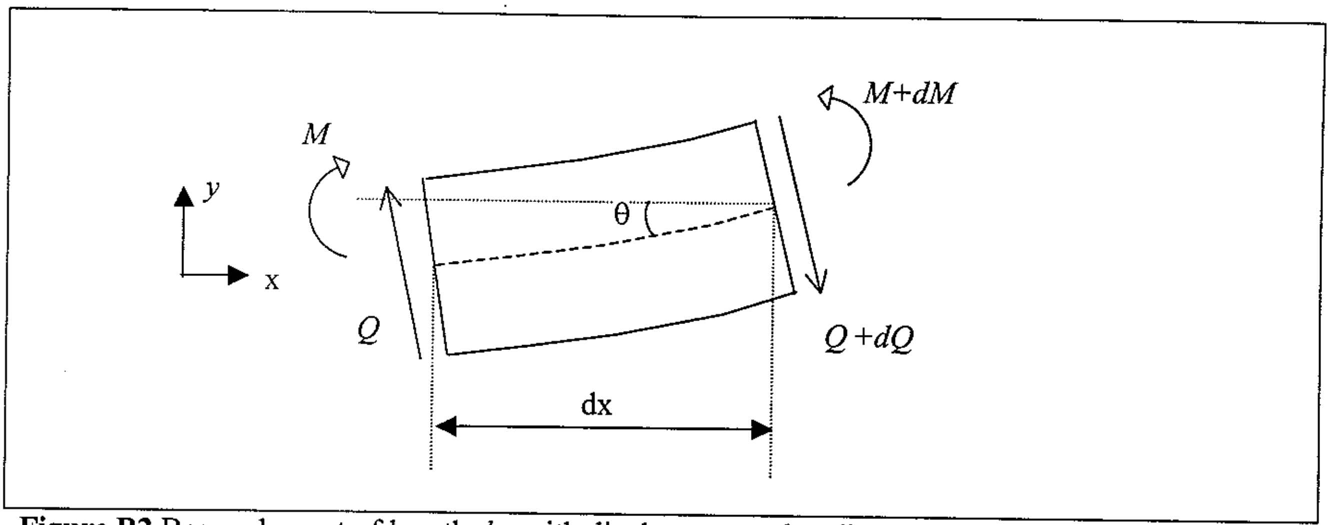

Figure B2 Beam element of length *dx*, with displacements, bending moments and shear forces shown.

By simple geometry and for small angles θ

$$\frac{dy}{dx} = \theta \tag{B1}$$

From Euler-Bernoulli theory $\frac{\partial\theta}{\partial x} = \frac{M}{E_b I_b}$ [11]. Taking the spatial derivative of equation B1 and substituting for $\frac{\partial\theta}{\partial x}$ gives,

$$M = E_b I_b \frac{\partial^2 y}{\partial x^2} \tag{B2}$$

The net moment on the element has to equal zero so $Qdx - dM = 0$. Rearranging this and combining with equation (B2) gives,

$$Q = \frac{dM}{dx} = E_b I_b \frac{\partial^3 y}{\partial x^3} \tag{B3}$$

The net force on the element is $-dQ$, which results in the mass, $\rho_b S_b dx$ having an acceleration of $\frac{\partial^2 y}{\partial t^2}$. Hence

$$\frac{dQ}{dx} = -\rho_b S_b \frac{\partial^2 y}{\partial t^2} \tag{B4}$$

Combining equations (B3) and (B4) gives the flexural wave equation of a beam.

$$E_b I_b \frac{\partial^4 y}{\partial x^4} + \rho_b S_b \frac{\partial^2 y}{\partial t^2} = 0 \tag{B5}$$

Assuming a harmonic solution of the form $y(x,t) = Ye^{j\omega t}$ and substituting for *y* in equation (B5) yields,

$$\frac{\partial^4 Y}{\partial x^4} - k_f{}^2 Y = 0, \tag{B6}$$

where $k_f = \left(\frac{\rho_b S_b}{E_b I_b}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}}$ is the flexural wavenumber and the phase velocity is given by

$c_f = \frac{\omega}{k_f} = \left(\frac{E_b I_b}{\rho_b S_b}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}}$, where the subscript "*f*" denotes the wave type (flexural).

The solution to equation (B6) is a sum of four exponentials that can be written as

$$y(x,t) = A_1 e^{+k_f x} + A_2 e^{-k_f x} + A_3 e^{+jk_f x} + A_4 e^{-jk_f x} \tag{B7}$$

Therefore the harmonic solution to the flexural wave equation of the beam varying in space and time is

$$y(x,t) = A_1 e^{(j\omega t + k_f x)} + A_2 e^{(j\omega t - k_f x)} + A_3 e^{j(\omega t + k_f x)} + A_4 e^{j(\omega t - k_f x)} \tag{B8}$$

The first two terms are harmonic non-propagating waves or evanescent waves. They decay away exponentially from discontinuities. The last two terms are waves varying harmonically in space and time. The A_1 and A_3 waves exist in the negative x-direction and the A_2 and A_4 waves exist in the positive x-direction.

B.2 Solution for a semi-infinite beam

A semi-infinite beam has an end, and the beam goes to infinity from that end. This report considers the case where the end is at the left, defined as $x = 0$, and the string extends to ∞ in the x-direction. Hence, when an excitation source is positioned at $x = 0$ only waves propagating to the right are present, therefore the motion of the string is given by

$$y(x,t) = A_2 e^{(j\omega t - k_f x)} + A_4 e^{j(\omega t - k_f x)} \tag{B9}$$

B.3 Equations of flexural wave motion in an Euler-Bernoulli beam under a static in-plane tension force.

Consider a small element of the beam with length dx that is subject to a tensile force, T, as shown in Figure B3

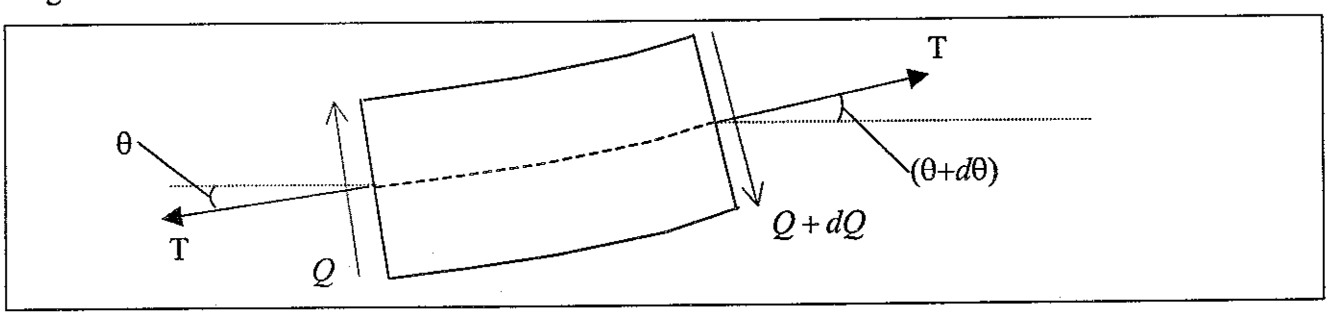

Figure B3 An element, of length dx, of a beam in flexure and subject to a tensile force, T

The net force on the element is given by

$$dF = Q - (Q + dQ) - T\theta + T(\theta + d\theta)$$

which simplifies to

$$dF = \left(-\frac{\partial Q}{\partial x} + T\frac{\partial \theta}{\partial x} \right) dx \qquad \text{(B10)}$$

Substituting for θ from equation (B1) and Q from equation (B3), gives

$$dF = -\left(E_b I_b \frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} \right) dx \qquad \text{(B11)}$$

This force results in the mass $\rho_b S_b dx$ having an acceleration of $\frac{\partial^2 y}{\partial t^2}$. Using Newton's second law and force equilibrium gives the equation of flexural wave motion in a beam under tension

$$E_b I_b \frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + \rho_b S_b \frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{(B12)}$$

Assuming a harmonic solution of the form

$$y(x,t) = Ye^{j\omega t} \qquad \text{(B13)}$$

where $Y = A_n e^{k_n x}$ and A_n and k_n are constants, and substituting for y into equation (B12) gives

$$E_b I_b k_n^4 - T k_n^2 - \omega^2 \rho_b S_b = 0 \qquad \text{(B14)}$$

If the tension force, T is set to zero the result is the dispersion relation for an Euler-Bernoulli beam as in equation (B9). If the flexural stiffness of the beam, i.e. the first term, is set to zero then the dispersion relation for a string is obtained as in equation (A6).

Solving (B14) as a quadratic in term of k_n^2 and introducing the non-dimensional wavenumber $\hat{k}_n = k_n r$, where r is the radius of gyration, leads to

$$\hat{k}_n^2 = \frac{Tr^2}{2E_b I_b} \pm \sqrt{\left(\frac{(-T)^2 r^4}{4(E_b I_b)^2} + \frac{\omega^2 \rho_b S_b r^4}{E_b I_b} \right)} \qquad \text{(B15)}$$

By noting that $\hat{k}_f^4 = \frac{\omega^2 \rho_b S_b r^4}{E_b I_b}$ and setting $\alpha = \frac{Tr^2}{2E_b I_b}$, equation (B15) becomes,

$$\hat{k}_n{}^2 = \alpha \pm \sqrt{\left(\alpha^2 + \hat{k}_f{}^4 \right)} \qquad \text{(B16)}$$

Hence there are four wavenumbers, $\pm \hat{k}_1$ and $\pm \hat{k}_2$, where

$$\hat{k}_1 = \sqrt{\alpha + \sqrt{\alpha^2 + \hat{k}_f{}^4}}\text{, and} \qquad \text{(B17)}$$

$$\hat{k}_2 = \sqrt{\alpha - \sqrt{\alpha^2 + \hat{k}_f^{\,4}}} \qquad \text{(B18)}$$

$\hat{k}_1$ is real for all positive values of α and $\hat{k}_f$ whereas $\hat{k}_2$ is imaginary for all positive values of α and $\hat{k}_f$. Hence, $\hat{k}_1$ is the wavenumber for the decaying waves and $\hat{k}_2$ is the wavenumber for the propagating waves. Figure B4 shows $+\hat{k}_1$ plotted against $\hat{k}_f$ and Figure B5 shows $+\hat{k}_2$ plotted against $\hat{k}_f$ for positive values of $\hat{k}_f$ and α. Figure B4 shows that the near-field wavenumber in the tension beam is greater than that for the Euler-Bernoulli beam. At low frequencies, (small values of $\hat{k}_f$) the effects of tension are the most significant. When $\hat{k}_f = 0$ the near-field wavenumber is given by $\hat{k}_1 = \sqrt{2\alpha}$. As frequency increases, the effects of the tension are reduced. For small α, the tension effects are minimal for $\hat{k}_f \geq 3$.

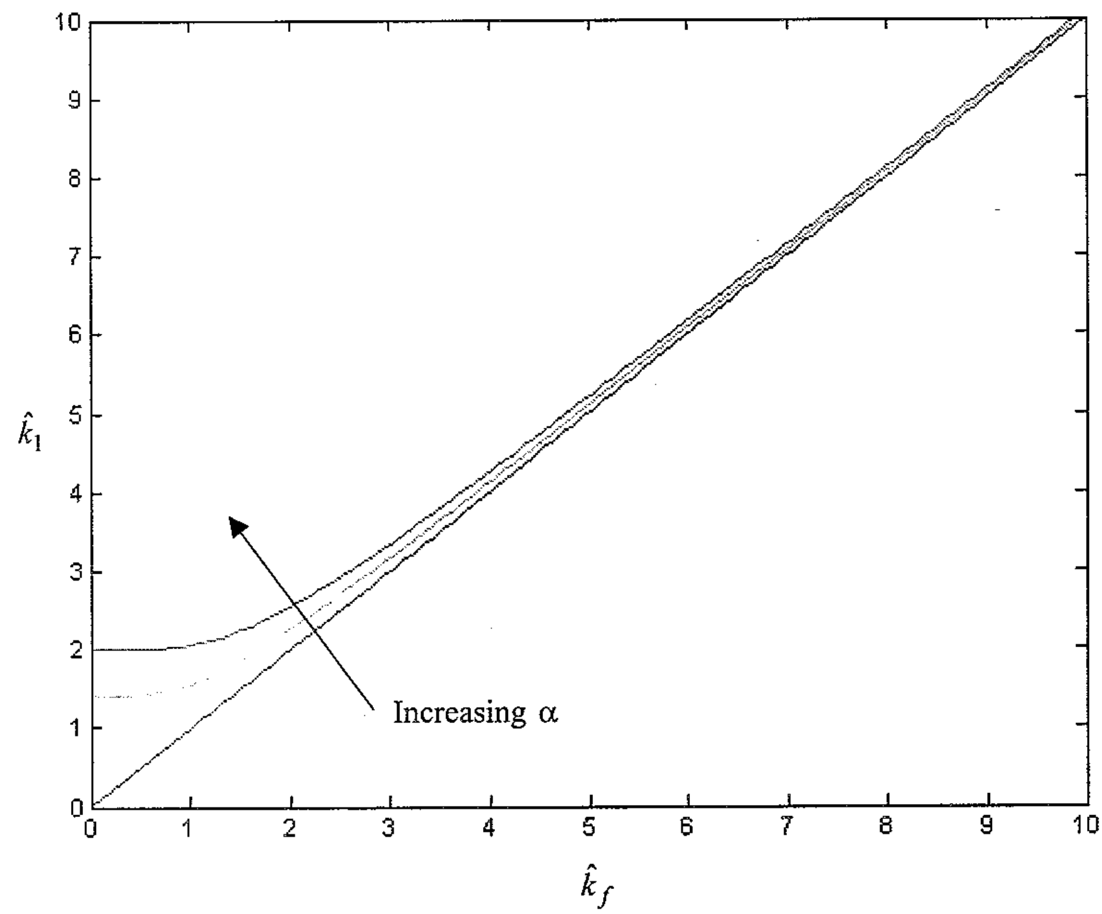

Figure B4 Non-dimensional tensioned beam wavenumber compared to a non-dimensional Euler-Bernoulli wavenumber for $\alpha = 0, 1, 2$ and 5.

Figure B5 shows that the propagating wavenumber for the tensioned beam is less than the Euler-Bernoulli beam, which means that when under tension the wavelengths are longer, again this is only significant at the lower frequencies. As $\hat{k}_f$ tends to zero, $\hat{k}_2$ also tends to zero, and as frequency increases the wavenumber tends to that of an Euler-Bernoulli beam.

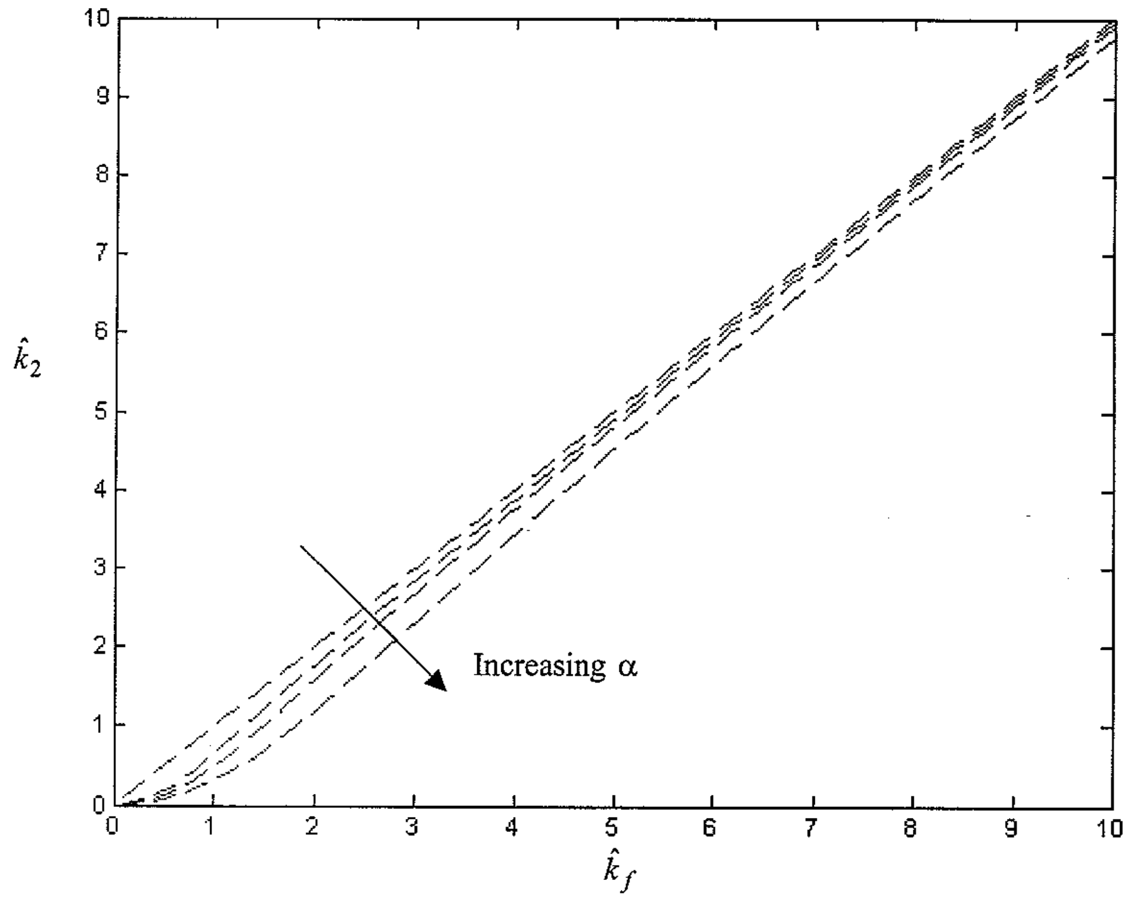

Figure B5 Non-dimensional tensioned beam wavenumber compared to a non-dimensional Euler-Bernoulli wavenumber for $\alpha = 0$, 1, 2 and 5. The dashed line represents a propagating wave.

B.4 A beam subject to an in-line compression force

In a tensegrity structure, the strings that are under tension hold the beams in place and hence the beams experience a compression force rather than tension. When a compression force is applied, i.e. $T < 0$ hence the second term in equation (B10) becomes negative, i.e

$$dF = \left(-\frac{\partial Q}{\partial x} - T\frac{\partial \theta}{\partial x} \right) dx \tag{B19}$$

This force causes the mass of the beam to accelerate, hence the force equilibrium equation becomes,

$$E_b I_b \frac{\partial^4 y}{\partial x^4} + T\frac{\partial^2 y}{\partial x^2} + \rho_b S_b \frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{(B20)}$$

This can be solved by assuming a harmonic solution $y(x,t) = Ye^{j\omega t}$, where $Y = A_n e^{k_n x}$,

$$E_b I_b k_n^4 + T k_n^2 - \omega^2 \rho_b S_b = 0 \qquad \text{(B21)}$$

Solving equation (B21) as a quadratic in terms of k_n^2 and introducing the non-dimensional wavenumber $\hat{k}_n = k_n r$, where r is the radius of gyration, gives

$$\hat{k}_n^2 = -\frac{Tr^2}{2E_b I_b} \pm \sqrt{\left(\frac{T^2 r^4}{4(E_b I_b)^2} + \frac{\omega^2 \rho_b S_b r^4}{E_b I_b}\right)} \qquad \text{(B22)}$$

Equation (B22) can be re-written in using the following observations, $\hat{k}_f^4 = \frac{\omega^2 \rho_b S_b r^4}{E_b I_b}$ and $\alpha = \frac{Tr^2}{2E_b I_b}$,

$$\hat{k}_n^{\,2} = -\alpha \pm \sqrt{\left(\alpha^2 + \hat{k}_f^{\,4}\right)} \qquad \text{(B23)}$$

Hence there are four wavenumbers, $\pm \hat{k}_1$ and $\pm \hat{k}_2$, where

$$\hat{k}_1 = \sqrt{-\alpha + \sqrt{\alpha^2 + \hat{k}_f^{\,4}}}\text{, and} \qquad \text{(B24)}$$

$$\hat{k}_2 = \sqrt{-\alpha - \sqrt{\alpha^2 + \hat{k}_f^{\,4}}} \qquad \text{(B25)}$$

As in the case of the compressive force, $\pm \hat{k}_1$ is the wavenumber for the decaying wave and $\pm \hat{k}_2$ is the wavenumber for the propagating wave. Figure B6 and B7 show $+\hat{k}_1$ and $+\hat{k}_2$ plotted against $\hat{k}_f$ for $\alpha = 0,1,2$ and 5, respectively.

Figure B6 shows that when the beam is under compression the near-field wavenumber is less than the Euler-Bernoulli wavenumber. The greatest effect of the compression is seen at low frequencies. As $\hat{k}_f$ tends to zero, $\hat{k}_1$ also tends to zero, and as frequency increases the wavenumber tends to that of an Euler-Bernoulli beam.

Figure B7 shows that by compressing the beam the wavenumber of the propagating wave is increasing compared to the Euler-Bernoulli wavenumber. The effects of compression are the most significant at low frequencies. When $\hat{k}_f = 0$ the near-field wavenumber is given by $\hat{k}_2 = \sqrt{-2\alpha}$. As frequency increases, the effects of the compression force are reduced. For small α, the compression effects are minimal for $\hat{k}_f \geq 3$.

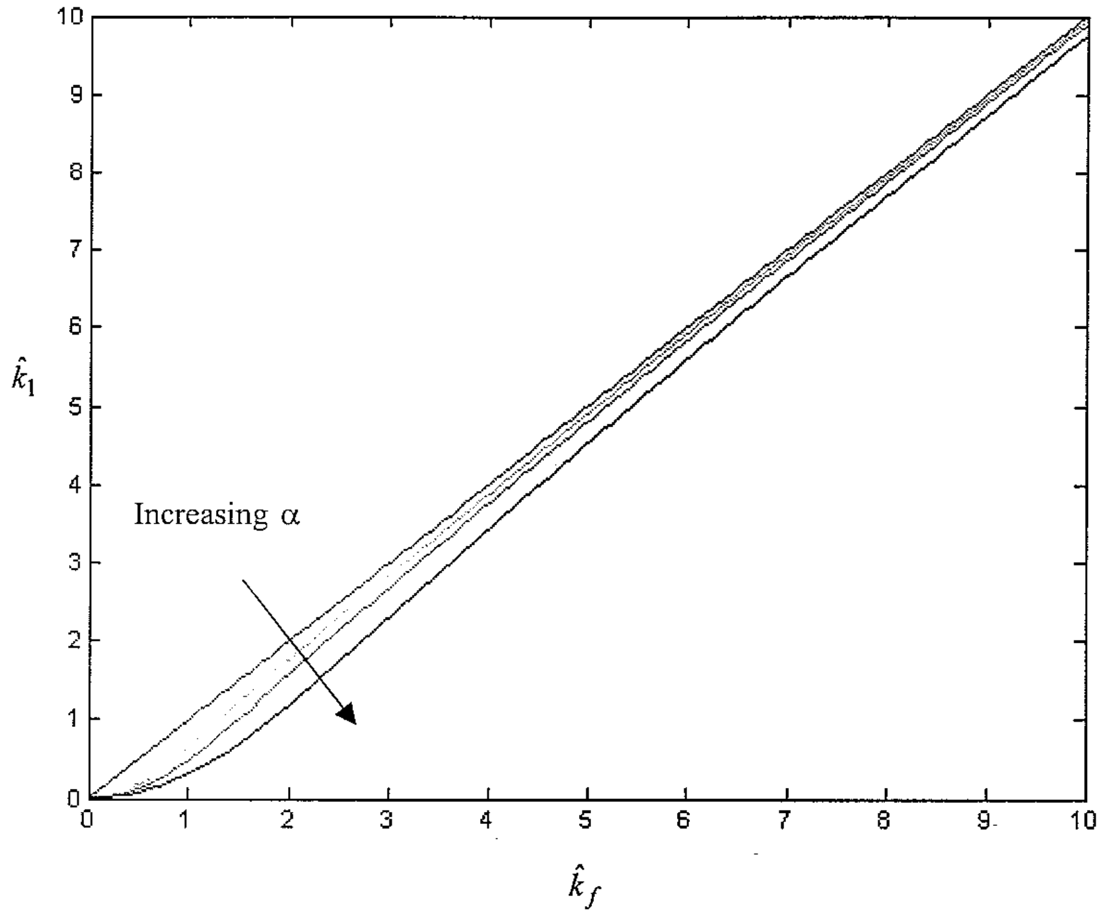

Figure B6 Non-dimensional compressed beam wavenumber compared to a non-dimensional Euler-Bernoulli wavenumber for $\alpha = 0, 1, 2$ and 5.

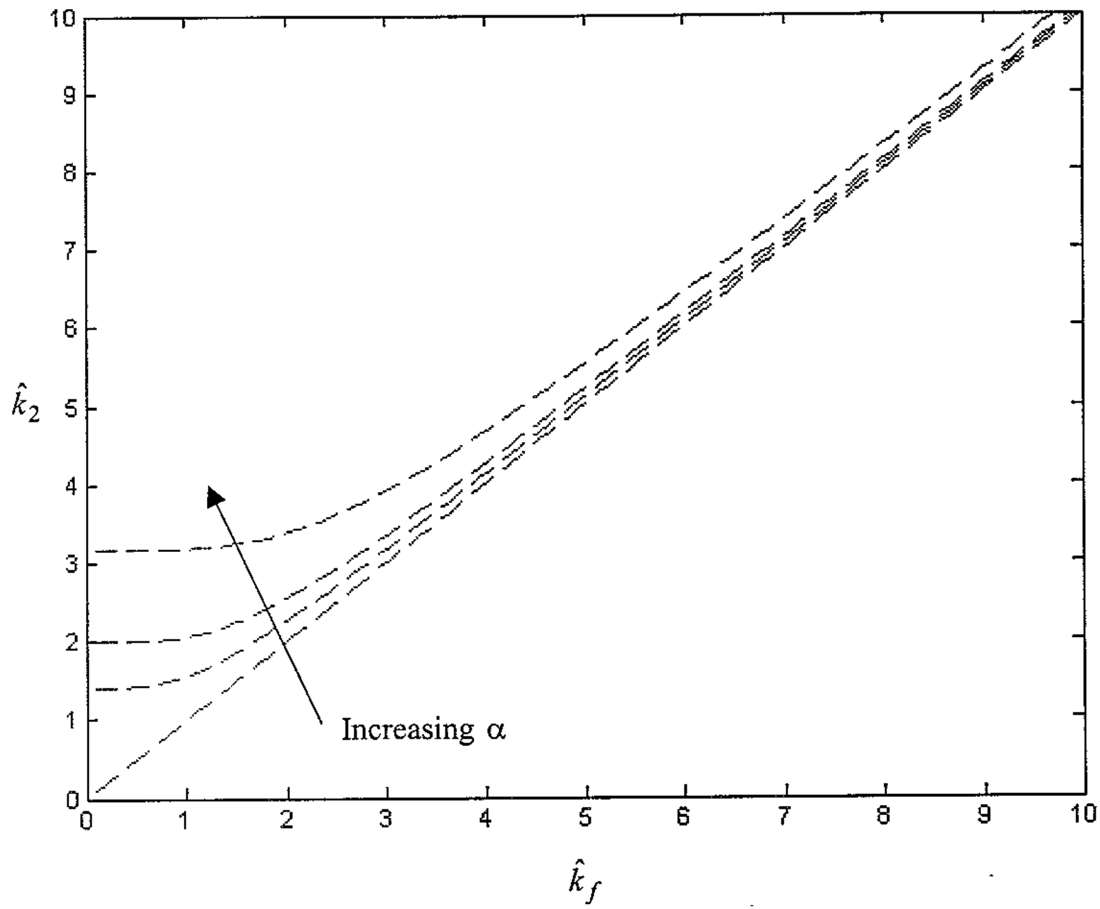

Figure B7 Non-dimensional compressed beam wavenumber compared to a non-dimensional Euler-Bernoulli wavenumber for $\alpha = 0$, 1, 2 and 5. The dashed line represents a propagating wave.

B.5 Wave equation of an infinite Timoshenko beam

A Timoshenko beam includes the effects of shear deformation and rotational inertia that are important at higher frequencies. The equation of motion is given by [11]

$$E_b I_b \frac{\partial^4 y}{\partial x^4} + \rho_b S_b \frac{\partial^2 y}{\partial t^2} - \rho_b I_b \left(1 + \frac{E_b}{G\kappa}\right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho_b{}^2 I_b}{G\kappa} \frac{\partial^4 y}{\partial t^4} = 0 \qquad \text{(B26)}$$

The shear modulus, G is given by $\frac{E_b}{2(1+\nu)}$, where ν is Poisson's ratio. The Timoshenko shear coefficient, κ, is a shape factor of the cross-section, which for a rectangular cross-section κ is equal to $5/6$ and for a circular cross-section it is $9/10$ [14].

For solutions of the form described by equation (B13), the equation of motion leads to the dispersion relation

$$E_b I_b k_n^4 + \rho_b I_b \left(1 + \frac{E_b}{G\kappa}\right) k_n^2 \omega^2 - \rho_b S_b \omega^2 + \frac{\rho_b{}^2 I_b}{G\kappa} \omega^4 = 0 \qquad \text{(B26)}$$

This can be defined in terms of a non-dimensional Euler-Bernoulli flexural wavenumber $\hat{k}_f$ and a non-dimensional flexural rigidity $\hat{s}^2 = \frac{E_b I_b}{\kappa S_b G r^2}$. Where r is the radius of gyration and $I_b = S_b r^2$.

$$\hat{k}_n^4 + \hat{k}_f^4\left(1+\hat{s}^2\right)\hat{k}_n^2 - \hat{k}_f^4\left(1-\hat{s}^2\hat{k}_f^4\right) = 0 \tag{B27}$$

This expression can be solved as a quadratic in terms of $\hat{k}_n^2$

$$\hat{k}_n^2 = \hat{k}_f^2\left[\frac{-\hat{k}_f^2}{2}\left(1+\hat{s}^2\right) \pm \sqrt{\frac{\hat{k}_f^4}{4}\left(\hat{s}^2-1\right)^2+1}\right] \tag{B28}$$

Hence, the solutions are the wavenumbers of the beam are given by,

$$\hat{k}_1 = \hat{k}_f\left[\frac{-\hat{k}_f^2}{2}\left(1+\hat{s}^2\right) + \sqrt{\frac{\hat{k}_f^4}{4}\left(\hat{s}^2-1\right)^2+1}\right]^{1/2} \tag{B29}$$

$$\hat{k}_2 = \hat{k}_f\left[\frac{-\hat{k}_f^2}{2}\left(1+\hat{s}^2\right) - \sqrt{\frac{\hat{k}_f^4}{4}\left(\hat{s}^2-1\right)^2+1}\right]^{1/2} \tag{B30}$$

Because equation (B30) will always be imaginary, it can be written as

$$\hat{k}_2 = \hat{k}_f\left[\frac{\hat{k}_f^2}{2}\left(1+\hat{s}^2\right) + \sqrt{\frac{\hat{k}_f^4}{4}\left(\hat{s}^2-1\right)^2+1}\right]^{1/2} \tag{B31}$$

Hence the wavenumbers of the beam are $\pm\hat{k}_1$ and $\pm j\hat{k}_2$. The positive wavenumbers are plotted in Figures B8 and B9 for various values of $\hat{s}$.

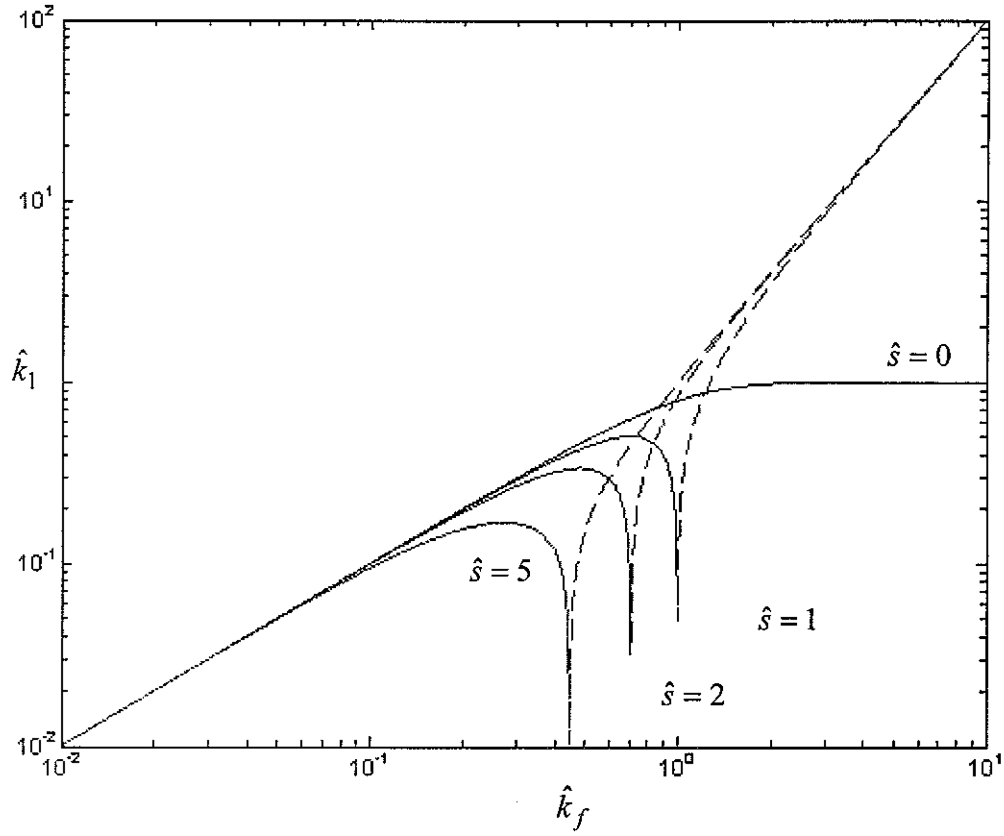

Figure B8 Non-dimensional wavenumber $\hat{k}_1$ plotted against non-dimensional Euler-Bernoulli wavenumber for $\hat{s} = 0, 1, 2$ and 5. A solid line denotes an evanescent wave, and a dashed line denotes a propagating wave.

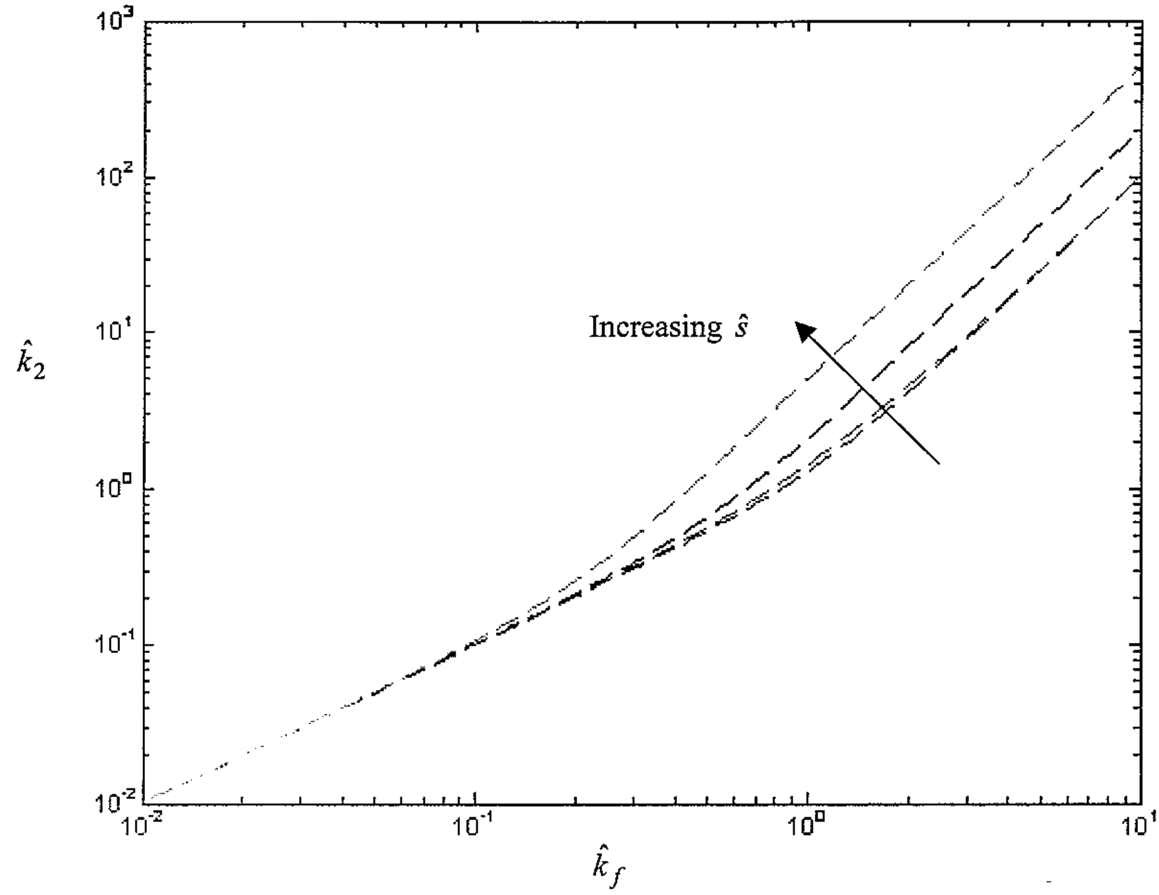

Figure B9 Non-dimensional wavenumber $\hat{k}_2$ plotted against non-dimensional Euler-Bernoulli wavenumber for $\hat{s} = 0, 1, 2$ and 5. A dashed line denotes a propagating wave.

Figure B8 shows that $\hat{k}_1$ is initially an evanescent wave and there is a frequency at which it turns into a non-dispersive propagating wave with speed of the longitudinal wave, [13] except for the case when $\hat{s} = 0$. This cut-on frequency is given by Brennan [13], to be

$$\omega_{cut-on} = \sqrt{\frac{GS_b \kappa}{\rho_b I_b}} \tag{B32}$$

Figure B9 shows that $\hat{k}_2$ is a dispersive propagating wave at low frequencies and change into non-dispersive propagating waves with speed similar to that of a shear wave.

Appendix C

The aim of this appendix is to derive the longitudinal wave motion of a semi-infinite beam.

C.1 Equation of longitudinal wave motion in a beam

The in-plane motion of a beam is characterised by its cross-sectional area, S_b, the Young's modulus, E_b and the density ρ_b. The subscript "*b*" refers to the beam. The following assumptions are made in the derivation of the equation of motion:

- The beam is uniform and elastic
- Longitudinal displacement *u(x)* is uniform over the normal cross-section (i.e. plane cross-sections remain plane and perpendicular to the longitudinal axis).
- The only motion is in the x-direction

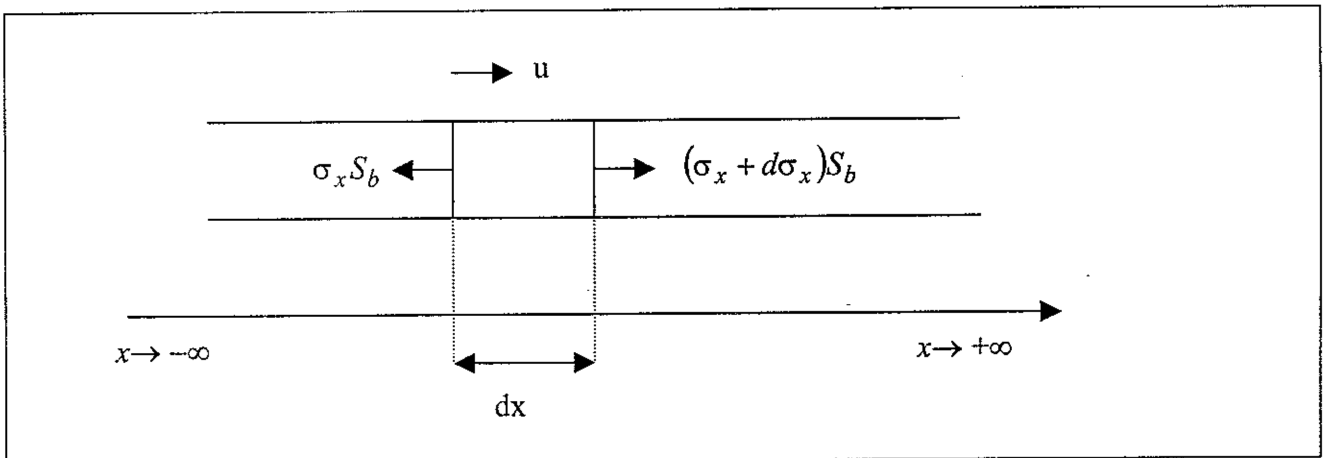

Figure C2 An element of beam, length *dx* with the forces and direction of motion shown

The net force on an element of length *dx* due to stress distribution as shown in Figure A2, is

$$dF = d\sigma_x S_b \tag{C1}$$

This force causes the mass of the element, $\rho_b S_b dx$, to have an acceleration of $\frac{\partial^2 u}{\partial t^2}$. Using Newton's second law and force equilibrium, and dividing by $S_b dx$ gives

$$\frac{d\sigma_x}{dx} = \rho_b \frac{\partial^2 u}{\partial t^2} \tag{C2}$$

When the stresses in the other two co-ordinate directions (y and z) are equal to zero, Hooke's Law applies, to give,

$$\sigma_x = E_b \varepsilon_x ,$$

where $\varepsilon_x = \frac{\partial u}{\partial x}$

Hence the stress-strain relationship is given by

$$\sigma_x = E_b \frac{\partial u}{\partial x} \tag{C3}$$

Substituting equation (C3) into equation (C2) gives the wave equation for in-plane motion

$$E_b \frac{\partial^2 u}{\partial x^2} - \rho_b \frac{\partial^2 u}{\partial t^2} = 0 . \tag{C4}$$

To obtain an expression for u, harmonic solutions with frequency ω are sought in the form of

$$u(x,t) = Ue^{j\omega t} \tag{C5}$$

Substituting this into the wave equation leads to the ordinary differential equation

$$\frac{d^2U}{dx^2} - k_l^2 U = 0 , \tag{C6}$$

where the wavenumber of the beam is $k_l = \frac{\omega}{c_l}$ and the phase velocity is $c_l = \sqrt{\frac{E_b}{\rho_b}}$.

The solution of equation (C6) is a sum of two complex exponentials that can be written as

$$U = A_1 e^{jk_l x} + A_2 e^{-jk_l x} \tag{C7}$$

Substituting (C7) into (C5) gives the harmonic solution to the wave equation as a function of space and time

$$u(x,t) = A_1 e^{j(\omega t - k_l x)} + A_2 e^{j(\omega t + k_l x)} \tag{C8}$$

The first term represents a wave varying harmonically with space and time, propagating to the left and the second term is a wave propagating to the right, where A_1 and A_2 are the wave amplitudes.

C.2 Solution for a semi-infinite beam

The beam described in this report is of semi-infinite length with the end at the left. Hence, when an excitation source is positioned at $x = 0$ only waves propagating to the left are present. Therefore, the motion of the string can be described as

$$u(x,t) = A_2 e^{j(\omega t + k_l x)} \tag{C9}$$

C.3 General Remarks

If the beam is subject to a constant in-plane compression or tension force, it does not contribute to the longitudinal wave equation.

The wave motion described is that of quasi-longitudinal waves, as described by Cremer et al [11]. Pure longitudinal waves exist only in solids whose dimensions are all much greater than that of the wavelength.

The same principles apply to a longitudinal wave in a string, only the wavenumber and phase velocity may differ due to the difference in material properties.