Primary Bjerknes forces

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Abstract. When a bubble in a liquid is subjected to a periodic sound field, the resulting bubble oscillations can interact with the sound field, giving rise to the primary Bjerknes force. A simple undergraduate-level derivation, and a graphical illustration of the underlying processes, are given.

Résumé. Quand une bulle dans un liquide est soumise à un champ acoustique périodique, les oscillations résultantes de cette bulle peuvent interagir avec le champ acoustique, donnant lieu aux forces de Bjerknes primaires. Nous présentons une dérivation simple, au niveau étudiant, de son expression ainsi qu'une illustration graphique du processus sous-jacent.

1. Introduction

The translational motions of bubbles within an acoustic field are of considerable significance in governing the effects of acoustic cavitation. An understanding of these motions is therefore important to disciplines such as material erosion, cavitation chemistry, and clinical therapy by ultrasound. Excluding buoyancy, the Bjerknes forces are the most potent driving forces for bubble translations in a non-flowing liquid.

When a gas bubble in liquid is subjected to an acoustic pressure field, it can undergo volume pulsations. If the acoustic pressure gradient is non-zero, then it can couple with the bubble oscillations to produce a translational force on the bubble. This is the primary Bjerknes force. Bubbles which are smaller than the size that is resonant with the sound field travel up a pressure gradient, and bubbles of a size larger than resonance travel down a pressure gradient. Therefore, in a planar standing-wave field, bubbles of smaller than resonance size collect at the pressure antinodes, whilst those larger than resonance aggregate at the pressure nodes (Leighton *et al* 1988).

The principle of the primary Bjerknes force was first formulated by Bjerknes (1906), though Blake (1949) gave the first satisfactory account of its origins. However there is no readily available formulation of the phenomenon in the literature, and so this paper attempts to provide one, as well as giving a graphical representation of the processes involved. The procedure is a good illustration of the effects of forced oscillation on a resonance system, and is based on the formulation of Walton and Reynolds (1984) which contains some inaccuracies.

2. Theory

A body of volume V in a pressure gradient ∇P experiences a force $-V\nabla P$. If this quantity varies in time, then the net force on the body is simply the time average of this:

$$F = -\langle V(t)\nabla P(r,t)\rangle . \qquad (1)$$

Now consider a bubble in a sound field. The pressure gradient oscillates, as does the bubble volume. It can be shown that the bubble radius, $R(t)$, varies as:

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[\left(P_0 + \frac{2\sigma}{R_0} - P_v\right)\left(\frac{R_0}{R}\right)^{3\kappa} - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R} - P_0 - P(t)\right] \qquad (2)$$

where R_0 is the equilibrium bubble radius, P_0 the hydrostatic pressure, P_v is the vapour pressure, and ρ, σ and μ are the liquid density, surface tension and viscosity respectively. The polytropic index of the gas within the bubble is given by κ, and $P(t)$ is the time-varying acoustic pressure. This equation is known as the RPNNP, or Rayleigh–Plesset, equation. A simple derivation has been given in the appendix.

An investigation into the small-amplitude limiting form of this equation provides useful approximations to the resonance behaviour of bubbles. Firstly, assume the time-varying pressure (which is superimposed on the constant hydrostatic pressure, P_0) has the form of a sinusoidal sound wave of amplitude P_A and circular frequency ω, i.e.

$$P(t) = -P_A \sin\omega t \qquad (3)$$

(it is conventional to use the negative, rather than the positive, sinusoid). If vapour pressure and viscosity

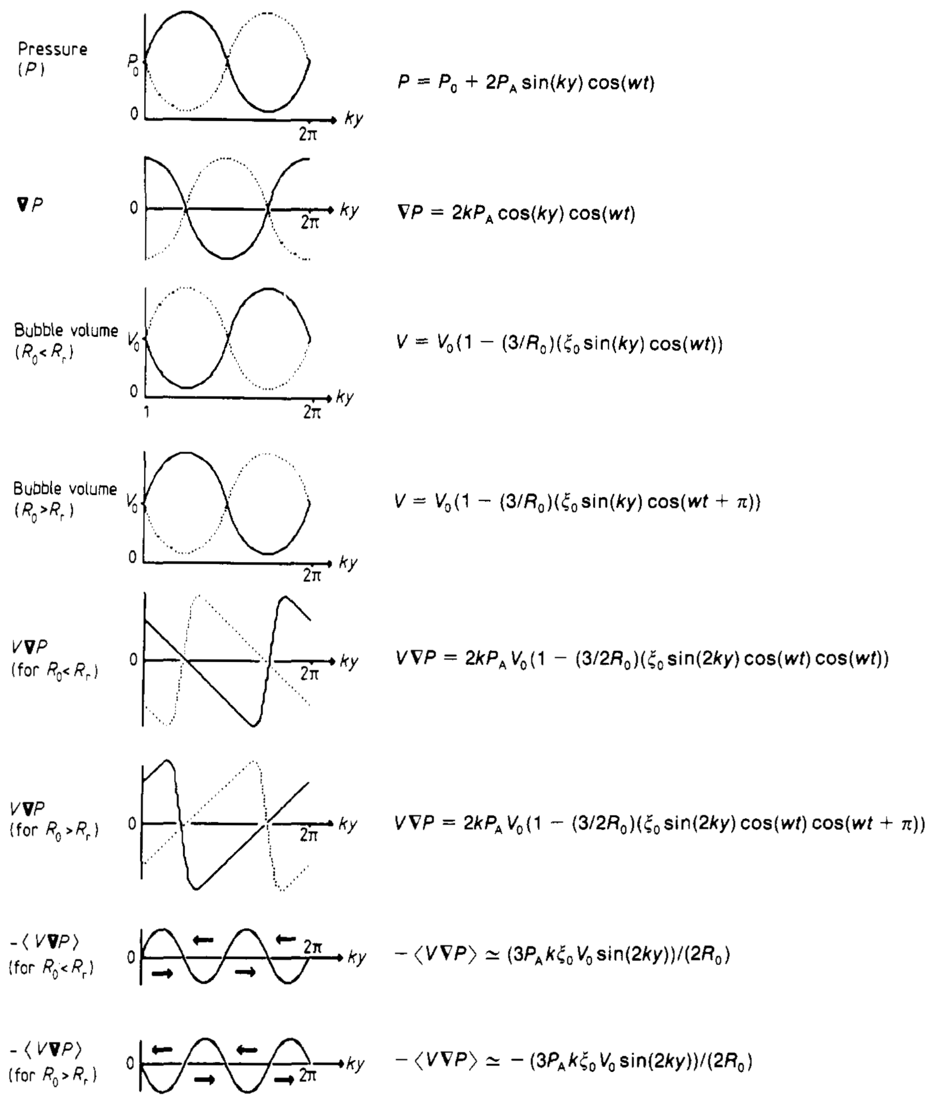

Figure 1. The key functions are plotted against a common spatial axis for two times (corresponding to $\omega t = 0$ (——) and $\omega t = \pi$ (····)). From the plot of P, ∇P can be deduced. The bubble volume V can be inferred from the theory of forced harmonic oscillation (see the text), for bubbles of less than, and greater than, resonance size (giving the third and fourth plots). Simple multiplication of the appropriate bubble volume graph with the ∇P graph gives the two $V\nabla P$ plots (the fifth and sixth graphs). The negative of the time average of these (the bottom two plots) provides the primary Bjerknes forces, illustrated by arrows in the bottom two plots. The force is to the left (indicated by ←) when $-\langle V\nabla P\rangle$ is negative, and to the right (indicated by →) when $-\langle V\nabla P\rangle$ is positive. By comparing these arrows with the plot of the pressure P in a standing-wave field (the first graph), it can be seen that bubbles of less than resonance size travel to the pressure antinodes, and bubbles of greater than resonance size travel to the pressure nodes.

are deemed negligible, and small-amplitude variations in bubble radius about the limiting value are assumed so that

$$R(t) = R_0 + r(t) \tag{4}$$

where $r \ll R_0$, then in an expansion to first order in powers of R_0^{-1}, equation (2) becomes

$$\ddot{r} + \omega_r^2 r = (P_A/\rho R_0)\sin\omega t \tag{5}$$

where ω_r is the resonant frequency of a bubble, and is given by

$$\omega_r^2 = \frac{1}{\rho R_0^2}\left[3\kappa\left(P_0 + \frac{2\sigma}{R_0}\right) - \frac{2\sigma}{R_0}\right]. \tag{6}$$

In the case of an air bubble in water at $P_0 = 10^5$ Pa, this reduces to the expression

$$\nu_r R_0 = 3\,\mathrm{Hz\,m^{-1}} \tag{7}$$

where ν_r is the linear resonance frequency (Minnaert 1933).

In the normal discussion of forced harmonic oscillation, the response of a given system to driving frequencies of above and below resonance is considered. However in this particular discussion we will consider the action of a fixed driving frequency on bubbles of greater than, and less than, resonant size (which, from equation (7), corresponds to bubbles with natural frequencies of respectively less than, and greater than, the driving frequency).

The bubble oscillations correspond to a familiar result from the theory of forced harmonic oscillation: a bubble of substantially less than resonant size oscillates in phase with the sound field, and bubbles larger than resonance oscillate π out of phase with the field. (This can be seen by solving equation (5) for the radial velocity of the bubble wall, and comparing this with the driving force. It should be noted that, since the positive driving pressure causes a reduction in volume, then the bubble volume will be a minimum when the pressure is a maximum if the two are oscillating in phase.) In this feature lies the root for the basic result of the primary Bjerknes force: the quantity $-\langle V\nabla P\rangle$ will be in one direction for bubbles with $R_0 < R_r$ and in the opposite direction for those with $R_0 > R_r$ (derived schematically in figure 1). Therefore, in a standing-wave field, bubbles of less than resonant size travel up a pressure gradient towards the pressure antinodes, and those larger than resonance travel down the gradient to the nodes.

To formalise the argument, it is necessary to consider a spatial dimension (given by y) in the standing-wave field

$$P(y, t) = P_0 + 2P_A \sin(ky)\cos(\omega t) \tag{8}$$

so that

$$\nabla P(y, t) = 2k\, P_A \cos(ky)\cos(\omega t) \tag{9}$$

where k is the wavevector and P_0 is assumed to be constant.

If a bubble is located at position y in this sound field, and if $2P_A \ll P_0$, then the bubble radius $R(t)$ will oscillate linearly as

$$R(t) = R_0 - \xi\cos(\omega t + \alpha) \tag{10}$$

where the phase term α equals zero for bubbles smaller than resonance, and equals π for bubbles larger than resonance. The negative sign is taken since a positive acoustic pressure causes a reduction in bubble volume when the two are in phase. It should be noted that the amplitude of the radial oscillation is

$$\xi = \xi_0 \sin(ky) \tag{11}$$

i.e. it varies sinusoidally with position in the sound field, in keeping with the acoustic pressure amplitude (ξ_0 being a constant, with $\xi_0 \ll R_0$). From the form taken by $R(t)$ in equation (10) the bubble volume $V(t) = 4\pi R(t)^3/3$ may consequently approximate to first order as

$$V(t) = V_0[1 - (3\xi_0/R_0)\sin(ky)\cos(\omega t + \alpha)] \tag{12}$$

when $V_0 = 4\pi R_0^3/3$, the equilibrium bubble volume. Substitution of equations (8) and (12) into equation (1) gives

$$F = [3P_A k\xi_0 V_0 \sin(2ky)]/(2R_0) \tag{13}$$

for bubbles smaller than resonance (i.e. when $\alpha = 0$), and

$$F = -[3P_A k\xi_0 V_0 \sin(2ky)]/(2R_0) \tag{14}$$

for bubbles larger than resonance (i.e. for $\alpha = \pi$).

3. Conclusion

By comparing the $\sin(2ky)$ and the $-\sin(2ky)$ forces in equations (13) and (14) with the $\sin(ky)$ variation of the pressure field in equation (7) (see figure 1) it can be seen that if $R_0 < R_r$ the bubble will move to the pressure antinode, and if $R_0 > R_r$ it will move to the node.

It should be remembered that the *primary Bjerknes forces* described above are active not just in standing wave fields, but in any field containing a pressure gradient. Therefore in a focused acoustic field, bubbles below resonance will travel to the focal pressure antinode, and those larger than resonance will travel away from the focus.

From a teaching perspective the behaviour of a bubble in a sound field provides a good example of the action of a forced damped oscillator.

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Appendix. The RPNNP equation

This equation describes the response of a spherical bubble to a time-varying pressure field in an incompressible liquid. It is also known as the Rayleigh–Plesset equation. The derivation given here is based upon that in Walton and Reynolds (1984).

At a time $t < 0$, a bubble of radius R_0 is at rest in an incompressible, viscous liquid. The hydrostatic pressure is P_0, a constant. At time $t > 0$, a pressure $P(t)$ which varies with time is superimposed on P_0, so that the liquid pressure at a point remote from the bubble is $P_\infty = P_0 + P(t)$. Consequently the bubble radius will change to some new value $R(t)$. During this process, the liquid will acquire a kinetic energy of

$$\tfrac{1}{2}\rho \int_R^\infty \dot{r}^2 4\pi r^2 \,\mathrm{d}r \tag{A1}$$

where r is the radial coordinate. Using the liquid incompressibility condition $(\mathrm{d}r/\mathrm{d}t)/(\mathrm{d}R/\mathrm{d}t) = R^2/r^2$ the expression in equation (A1) can be integrated to give $2\pi\rho R^3(\mathrm{d}R/\mathrm{d}t)^2$. Equating this to the difference between the work done remote from the bubble by P_∞ (the hydrostatic pressure there), and the work done by the hydrostatic pressure P_L in the liquid just outside the bubble walls gives

$$\int_{R_0}^{R} (P_\mathrm{L} - P_\infty) 4\pi r^2 \,\mathrm{d}r = 2\pi R^3 \dot{R}^2 \rho. \tag{A2}$$

If the hydrostatic pressure in a liquid of surface tension σ is P_0, then the internal pressure of a bubble (radius R_0) within that liquid is $P_0 + (2\sigma/R_0)$. Therefore the pressure of the gas (i.e. non-vapour) phase within the bubble is $P_0 + (2\sigma/R_0) - P_\mathrm{v}$ (where P_v is the vapour pressure). If the hydrostatic pressure then changes to P_∞, the bubble radius will change to R, and the gas pressure within the bubble will be $(P_0 + (2\sigma/R) - P_\mathrm{v})(R_0/R)^{3\kappa}$, assuming this gas is a perfect gas. κ is the polytropic index of the gas, which takes a value between unity and γ depending on whether the gas behaves isothermally, adiabatically, or with intermediate characteristics. Therefore the pressure in the liquid immediately beyond the bubble wall will be $(P_0 + (2\sigma/R_0) - P_\mathrm{v})(R_0/R)^{3\kappa} - 2\sigma/R$.

In fact the hydrostatic pressure in the layer of water adjacent to the bubble is,

$$P_\mathrm{L} = \left(P_0 + \frac{2\sigma}{R_0} - P_\mathrm{v}\right)\left(\frac{R_0}{R}\right)^{3\kappa} - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R}. \tag{A3}$$

Poritsky (1952) shows that the final term, containing the viscosity μ of the liquid, is required to ensure continuity of normal stress at the bubble wall. Substitution of equation (A3) into equation (A2), followed by differentiation with respect to R gives

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[\left(P_0 + \frac{2\sigma}{R_0} - P_\mathrm{v}\right)\left(\frac{R_0}{R}\right)^{3\kappa} - \frac{2\sigma}{R} - \frac{4\mu\dot{R}}{R} - P_0 - P(t)\right] \tag{A4}$$

Following the suggestion of Lauterborn (1976), this equation is commonly referred to as the RPNNP equation in tribute to the workers who contributed to its formulation: Rayleigh (1917), Plesset (1949), Noltingk and Neppiras (1950), Neppiras and Noltingk (1951) and Poritsky (1952). A more rigorous derivation is given by Neppiras (1980). This equation must in general be solved numerically.

The RPNNP equation gives the small-amplitude approximation to the resonant frequency of a bubble as

$$\omega_\mathrm{r} = \frac{1}{R_0\sqrt{\rho}}\left[3\kappa\left(P_0 + \frac{2\sigma}{R_0} - P_\mathrm{v}\right) - \frac{2\sigma}{R_0} - \frac{4\mu^2}{\rho R_0^2}\right]^{1/2}. \tag{A5}$$

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