Dynamics of a Tethered Bubble

Alexey O. Maksimov [1], Timothy G. Leighton[2], Peter R. Birkin [3]

[1]*Pacific Oceanological Institute, Far Eastern Branch of the Russian Academy of Sciences, 43 Baltic Street, Vladivostok 690041, Russia*
[2]*Institute of Sound and Vibration Research, University of Southampton, Highfield, Southampton SO17 1BJ, UK*
[3]*School of Chemistry, University of Southampton, Highfield, Southampton SO17 1BJ, UK*

Abstract. Small gas bubbles adhering to solids occur in a range of manufacturing processes, including printing, casting, coating and electroplating. The behavior of a gas bubbles tethered to a rigid plane boundary in an oscillatory pressure field is investigated by use conformal symmetry of the problem. The dynamics of the tethered bubble differ from those of the free bubble. The inertial (or added) mass depends on the contact angle and this variation is not monotonic. As a result, the natural frequency depends on the contact angle. Viscous damping of the tethered bubble is increased by more than two orders of magnitude, firstly, owing to the greater dissipation near rigid wall in comparison with free air/water bubble interface and secondly, because of a contact line dissipation effect.

INTRODUCTION

Small gas bubbles adhering to solids occur in a range of manufacturing processes, including printing, casting, coating and electroplating. These so-called ‘tethered’ bubbles generate a boundary between the solid and the liquid, preventing complete coverage. This is usually disadvantageous to production. In the pottery industry, the occurrence of bubbles in the liquid ‘casting slip’ which is injected into moulds results in loss of production time and raw materials, as the expansion of these bubbles in the kiln when the pottery is ‘fired’ generates unwanted holes and pitting in the finished product [1]. Similar problems can occur in the casting of metals. The importance of the tethered bubbles has also been apparent in the many studies of the tensile strengths of liquids. Micron-sized bubbles of contaminant gas form important weaknesses in the liquid. These bubbles persist for long periods, stabilized against dissolution or buoyant rise through their adhesion to crevices within the solid boundary. Unlike free microbubbles, only static thresholds of acoustic cavitation have been theoretically derived for the crevice model [2]. Sonoluminescence (SL) from an isolated bubble on a solid surface [3] differs from single-bubble SL. The forced oscillation of a gas bubble can have important biological effects. Rooney [4] showed that hemolysis of an erythrocryte was possible using an acoustically driven tethered gas bubble.

Clearly the forced oscillation of a tethered gas bubble is of significant technological importance for a thorough understanding of the problem to be necessary. Hence we present here a theoretical approach to this problem.

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BUBBLE DYNAMICS IN TOROIDAL COORDINATES

Consider an air bubble of radius curvature R_0 tethered to the rigid wall (see Figure 1) and driven by an acoustical wave of amplitude P_m and angular frequency ω. The basic bubble equilibrium shapes show a segment of the spherical profile with contact angle ϑ_c. The diameter of the ring of contact is denoted by L ($L = 2R_0 \sin\vartheta_c$). As the radius of the bubble is often much smaller than the acoustic wavelength λ, the pressure within the bubble is constant, when $R_0/\lambda << 1$ (homobaric bubble) and hence the bubble wall is an equipotential surface, if surface tension and inertial nonlinearities are neglected.

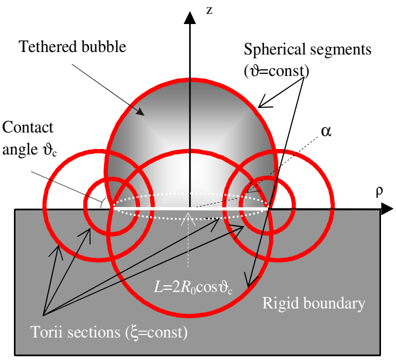

Fig. 1. Toroidal coordinates – natural coordinates for the tethered bubble.

The inversion method was utilized to obtain an exact solution for a gas bubble tethered to a rigid plane boundary in an oscillatory pressure field [5]. This method is based on the invariance of the Laplace equation to conformal transformations. It was also shown [6], by using the symmetry of the problem, that the toroidal coordinates are most suitable for analysis of the oscillations of the tethered bubble. Thus the dynamics of the tethered bubble in toroidal coordinates can be analyzed by using an analytical approach and by analogy to the dynamics of a free spherical bubble.

The solution can be expressed in terms of the Green's function of the Laplace's equation with a homogeneous (vanishing) condition on the boundary (bubble wall):

$$\nabla^2 G(\mathbf{r} \mid \mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \qquad (1)$$

The Green's function can be expressed in toroidal coordinates $z = (L/2)\sin\vartheta \times \left[\cosh\xi - \cos\vartheta\right]^{-1}$, $x = (L/2)\sinh\xi\cos\alpha\left[\cosh\xi - \cos\vartheta\right]^{-1}$, $y = (L/2)\sinh\xi\sin\alpha \times \left[\cosh\xi - \cos\vartheta\right]^{-1}$, where ξ goes from 0 to ∞, ϑ goes from $-\vartheta_c$ to ϑ_c and α (the azimuthal angle) goes from 0 to 2π. The surface $\xi = const$ is a torus (which is a cyclide) and the surface $\vartheta = const$ is a spherical bowl. It then takes the following form

$$G(\xi,\vartheta,\alpha \mid \xi',\vartheta',\alpha') = \sqrt{\cosh\xi - \cos\vartheta}\sqrt{\cosh\xi' - \cos\vartheta'}\,\frac{1}{\sqrt{2}\vartheta_c}\left(\frac{2}{L}\right)\int_s^{\infty}\frac{\sinh\left(\pi t/2\vartheta_c\right)dt}{\left(\cosh t - \cosh s\right)^{1/2}}\times$$

$$\times\left\{\cos\left[\frac{\pi}{2\vartheta_c}(\vartheta-\vartheta')\right]\left\{\cosh\left(\frac{\pi t}{\vartheta_c}\right)-\cos\left[\frac{\pi(\vartheta-\vartheta')}{\vartheta_c}\right]\right\}^{-1}+\cos\left[\frac{\pi}{2\vartheta_c}(\vartheta+\vartheta')\right]\left\{\cosh\left(\frac{\pi t}{\vartheta_c}\right)-\cos\left[\frac{\pi(\vartheta+\vartheta')}{\vartheta_c}\right]\right\}^{-1}\right\}, \qquad (2)$$

where $\cosh s \equiv \cosh\xi\cosh\xi' - \sinh\xi\sinh\xi'\cos(\alpha-\alpha')$. This is the main result of this section.

VOLUME OSCILLATIONS OF A TETHERED BUBBLE

As an application of the explicit form of the Green's function (2) we describe the volume oscillations of a tethered bubble. The modified Rayleigh equation for the volume pulsation $V = V_0 + \Delta V$ has the form:

$$\left[2R_0 \sin\vartheta_c C(\vartheta_c)\right]^{-1} d^2V/dt^2 + \left(\gamma P_0/\rho_0 V_0\right)\Delta V = -\left(P_m/\rho_0\right)\sin(\omega t)\,, \tag{3}$$

$$C(\vartheta_c) = \frac{\pi^2}{2\sqrt{2}\vartheta_c^2}\int_{\vartheta_c}^{\pi}\frac{\sin\vartheta_c \sin\kappa d\kappa}{\left(\cos\vartheta_c - \cos\kappa\right)^{3/2}}\int_{arch\left(\frac{1-\cos\vartheta_c\cos\kappa}{\cos\vartheta_c-\cos\kappa}\right)}^{\infty} d\xi\left(ch\xi - \frac{1-\cos\vartheta_c\cos\kappa}{\cos\vartheta_c-\cos\kappa}\right)^{-1/2}\frac{sh\left(\pi\xi/2\vartheta_c\right)}{ch^2\left(\pi\xi/2\vartheta_c\right)}\,,$$

where ρ_0 and P_0 are the equilibrium density and pressure, γ is the polytropic exponent. It follows directly from this equation that the fundamental frequency for the tethered bubble depends on the contact angle and this dependence is not monotonic

$$\Omega_0^2(R_0,\vartheta_c) = \Omega_*^2(R_0)\sin\vartheta_c\left(C(\vartheta_c)/2\pi\right)\left[1-\left(1-\cos\vartheta_c\right)^2\left(2+\cos\vartheta_c\right)/4\right] \tag{4}$$

here $\Omega_*^2(R_0) = \sqrt{3\gamma P_0/\rho_0}R_0^{-1}$ is the fundamental frequency of a free bubble. The expression for the natural frequency is given for a fixed radius of curvature R_0 of the spherical segment. However it is more interesting to trace the manner in which the natural frequency of a tethered bubble of fixed volume V_0 varies as the contact angle is changed. The dependence of $\Omega_0(V_0,\vartheta_c)/\Omega_*(V_0)$ on the contact angle ϑ_c is shown in Fig. 2 by solid line.

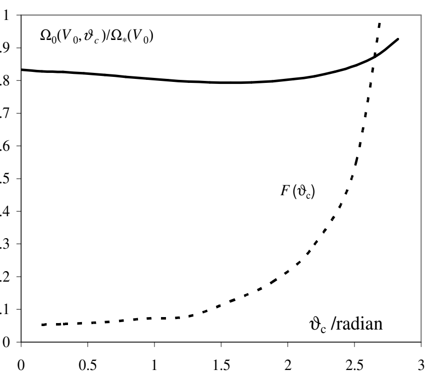

Fig. 2. Plot showing dependence of the normalized frequency and damping on the contact angle.

We have so far assumed the flow near the bubble and the rigid wall to be inviscid and irrotational. The viscous damping factor (for the kinematic viscosity ν) of the volume oscillations of the tethered bubble can be obtained by using the approach based on the total energy conservation in incompressible liquid when the time derivative of energy is balanced by a dissipative function – Longuet-Higgins [7] used this approach in finding the viscous damping of surface modes on the bubble wall. The relative decay of amplitude per cycle (known as the 'damping factor') is:

$$\gamma_0 = \left(2\nu/R_0^2\right)\sqrt{\omega R_0^2/2\nu}\,F(\vartheta_c)\,, \tag{5}$$

where we have separated out the co-factor $(2\nu/R_0^2)$ which describes damping of a free non-tethered bubble. The computed angular dependence of the normalized damping $F(\vartheta_c) = \gamma_0\left(2\nu/R_0^2\right)^{-1}(\omega R_0^2/2\nu)^{-1/2}$ is shown in Fig. 2 by the dotted line.

DISCUSSION AND CONCLUSIONS

This section discusses two examples for the application of this theory. The first follows from the development of a new electrochemical sensing technique [8, 9] in which millimetre sized bubbles were suspended under a glass rod. As this has proved to be the most precise technique for determining the resonance of a single bubble [1], it follows a comparison of the pressure threshold measurements for the onset of surface oscillations, and examination of the variation in this threshold due to the shift of the natural frequency of the tethered bubble (4) is thus possible.

The second example follows from the experiment of MacDonald *et al.* [10], where oscillations of a bubble at the tip of an optical fibers in liquid nitrogen have been studied. When a pulse of laser is delivered to the fiber tip, a bubble forms (with a radius of just a few micrometers), initially as a shallow meniscus, on the fiber core. It then grows toward the more spherical shape, and freely oscillates. The oscillations are fastest immediately after the pump pulse, when they have a frequency of ~ 17 MHz. The oscillation frequency then drop rapidly, settles at ~ 1.35 MHz within a few microseconds, and remains almost constant for 25-30 cycles.

The time history of these oscillations can be approximated by the Rayleigh equation (3) with varying inertial mass. Assuming adiabatic variation of this mass we can use the expressions (4) for natural frequency of the bubble with fixed volume and varying contact angle that is varying inertial mass. The initial state can be approximated by the spherical segment with contact angle $\vartheta_c \sim \pi$. The drop of oscillation frequency from 17 till 1.35 MHz has a natural explanation and can be described on the order of magnitude by the expressions derived for the natural frequency of the tethered bubble.

The dynamics of the tethered bubble differ from those of the free bubble mainly by the value and dependence of the inertial (or added) mass on the contact angle. This provides variation of the natural frequency with the contact angle and this variation is not monotonic. Viscous damping of the tethered bubble is increased by more than two orders of magnitude, firstly, owing to the greater dissipation near rigid wall in comparison with free air/water bubble interface and secondly, because of the contact line dissipation effect.

REFERENCES

1. T. G. Leighton, *International Journal of Modern Physics B*, **18**, 3267-3314 (2004).
2. A. Atchley, A. Prosperetti, *J. Acoust. Soc. Am.,* **86**, 1065-1084 (1989).
3. K. R. Weninger et al., *Phys. Rev. E.,* **56**, 6746-6749 (1997).
4. J. A. Rooney, *Science*, **169**, 869-871 (1970).
5. A. O. Maksimov, *J. Sound & Vibr.*, **283,** 915-926 (2005)
6. A. O. Maksimov, *Communications in Nonlinear Science and Numerical Simulation*, **9,** 83-92 (2004).
7. M. S. Longuet-Higgins, *J. Fluid Mech.* **201**, 543-565 (1989).
8. P. R. Birkin, Y. E. Watson, T. G. Leighton, K. L. Smith, *Langmuir Surfaces and Colloids*. **18**, 2135-2140 (2002).
9. Y. E. Watson, P.R. Birkin, T. G. Leighton, *Ultrasonics Sonochemistry*, **10**, 65–69 (2003).
10. K. F. MacDonald *et al*., *Phys. Rev. E*, **68**, 027301 (2003).