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Coupled Oscillators and Sets of Oscillators: Statistical Energy Analysis Revisited

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

Coupled Oscillators and Sets of Oscillators:

Statistical Energy Analysis Revisited

by

B.R. Mace and L. Ji

ISVR Technical Memorandum No: 961

April 2006

Authorised for issue by
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Group Chairman

ABSTRACT

The foundations of Statistical Energy Analysis (SEA) are based on the interaction of coupled oscillators and coupled sets of oscillators. The sets of oscillators might be subsystems modelled discretely, typically those produced by finite element analysis. This issue is revisited here. Various assumptions inherent in previous work are removed or relaxed.

First, the SEA of two coupled oscillators is considered. The oscillator properties are assumed to be random and ensemble averages found. Full account is taken of the correlation between the coupling parameters for the oscillators and their energies. Various observations are made about the qualitative and quantitative features of the behaviour under broadband excitation, some of which differ from those which are commonly assumed within SEA. It is seen that the coupling power or coupling loss factor can be written in terms of the "strength of connection" between the oscillators and a term involving their bandwidths and the separation of their natural frequencies. Equipartition of energy does not occur as damping tends to zero, except in the case where the uncoupled oscillators have identical natural frequencies.

Attention is focussed on spring-coupled oscillators, although other forms of conservative coupling are considered. These include general, conservative coupling and spring-like coupling which is characteristic of the form of coupling that arises from fixed interface component mode synthesis.

These results are then extended to two coupled sets of oscillators (i.e. coupled multi-modal subsystems). It is assumed that the interaction of each oscillator pair is not affected by the presence of a third oscillator. A second assumption concerns the coupling power between each pair. Two approaches are suggested: the first is that the coupling power between each oscillator pair is proportional to their actual energies; the second is that the coupling power between each oscillator pair is proportional to their "blocked" energies. Very similar, but different, expressions for the ensemble averages of coupling power, energy response and coupling loss factor (CLF) are found. The application of the coupled oscillator theory to coupled continuous subsystems is discussed.

The behaviour is seen to depend on both the strength of coupling and the strength of connection, and parameters describing the strength of coupling and the strength of connection are found. The coupling loss factor depends on damping: it is proportional to damping at low damping and is independent of damping in the high damping, weak coupling limit.

Applications and numerical examples are discussed. The examples of two spring-coupled rods and two spring-coupled plates are considered. Comparisons are made with the results of conventional SEA, an exact wave analysis for one-dimensional subsystems and numerical Monte Carlo simulations using an energy distribution approach.

Contents

1 INTRODUCTION

Statistical energy analysis (SEA) has become an established method for modelling the noise and vibration behaviour of complex, built-up structures at higher frequencies [1]. The response is described in terms of the flow of energy through the structure. It is recognized that the properties of the structure are uncertain, and hence a statistical description is required. In principle at least, the structure is assumed to be drawn from an ensemble of similar structures whose properties are random and estimates of the ensemble average response are required.

The earliest approaches to SEA and derivations of the SEA equations concerned systems comprising two coupled oscillators and two coupled sets of oscillators whose properties are random. In [2] it was seen that for broadband excitation the energy flow between two, conservatively coupled oscillators is proportional to the difference in their blocked energies, i.e. the energies when the other oscillator is fixed. This result was then applied to the coupling power between two specific modes of coupled, multi-modal subsystems, which can be regarded as comprising sets of oscillators. In [3] the results were extended to the case of coupled oscillator sets, various assumptions being made to show that the coupling power is proportional to the difference in the mean blocked modal energies of the subsystems. The coupling loss factor was introduced and estimated using wave approaches rather than from the modal approach itself. A more formal approach was developed in [4], which adopted a statistical description for the properties of the sets of oscillators, and ensemble averages of the coupling power found. Finally in [5] it was shown that the coupling power between two oscillators is also proportional to the difference in their actual energies. It is on this base that SEA is founded.

In this memorandum the SEA of coupled oscillators is revisited. The work departs somewhat from that of [1-5], in that various assumptions are removed or relaxed, and consequently some of the conclusions differ from those of [1-5].

Broadly, in what follows, the SEA of two coupled oscillators is considered first. The oscillator properties are assumed to be random and ensemble averages found. These results are then extended to two coupled sets of oscillators (i.e. coupled multi-modal subsystems). Attention is focussed on spring-coupled sets, although other forms of conservative coupling are considered. Results are compared with existing theories, including conventional SEA, and numerical examples are presented.

There are two main motivations. The first is to re-examine the underlying physics of the interaction of oscillator sets, since the conventional SEA approaches lead to conclusions which are contradictory to results found from wave and finite element analysis (FEA), especially for stronger coupling. These conclusions relate to the strength of coupling, and another factor, the strength of connection of the oscillators. The second motivation is to lay the basis of a technique for estimating SEA parameters from discrete, FEA subsystem models, without the need to solve the global eigenvalue problem. This work is on-going.

Regarding the underlying physics, in the analysis one major assumption is removed: full account is taken of the correlation between the coupling parameters for the oscillators or mode pairs and the specific energies of those modes. Various observations are made about the qualitative and quantitative features of the behaviour under broadband excitation, some of which differ from those which are commonly assumed within SEA. It is seen that the coupling power or coupling loss factor can be written in terms of the "strength of connection" between the oscillators and a term involving their bandwidths and the separation of their natural frequencies. Equipartition of energy does not occur as damping tends to zero, except in the case where the uncoupled oscillators have identical natural frequencies. The results are then extended to coupled sets of oscillators and ensemble averages. The coupling loss factor depends on damping: it is proportional to damping at low damping and is independent of damping in the high damping, weak coupling limit. A parameter which describes the strength of coupling is identified: this depends on both the damping and the strength of connection.

The subsequent aim is to be able to estimate SEA parameters from discrete, FEA subsystem models. Each subsystem can be modelled using FEA and its uncoupled modes determined. When two subsystems are joined these modal sets interact. Robust estimates of the coupling loss factors (CLFs) (i.e. estimates which do not depend on the exact subsystem properties chosen) can be determined by analytically ensemble-averaging along the lines described here. practical applications of this approach are likely to be aimed at weak coupling cases (for parameter estimation), as well as strong coupling cases. This work is on-going.

The work also forms the basis for estimating variability in the response of individual systems, since this variability results depends on the modal properties of the system, including the number of modes in the excitation band and the mode spacing statistics. This work is also on-going.

The next section concerns the case of two spring-coupled oscillators. Various expressions for powers and energies are derived and discussed. The oscillator properties are described statistically and ensemble averages taken, leading to expressions for the CLFs. The correlation of oscillator energies and coupling parameters is included: this is the cause of behaviour contrary to that predicted by conventional SEA. Other forms of conservative coupling are considered. These include general, conservative coupling and spring-like coupling which is characteristic of the form of coupling that arises from fixed interface component mode synthesis. These are described in Appendices C and D. The general conclusions still hold for these cases of coupling, but the expressions become quite lengthy.

The results are then extended to the case of two spring-coupled sets of oscillators in section 3. Ensemble averages are again taken, the ensembles being defined in terms of the statistics of the natural frequencies of the oscillators in each set. Expressions for the ensemble averages of coupling power, energy response and coupling loss factor (CLF) are derived in terms of the interaction of oscillator pairs. To find these expressions further assumptions are required and two are suggested: the first is that the coupling power between each oscillator pair is proportional to their actual energies; the second is that the coupling power between each oscillator pair is proportional to their "blocked" energies. Two similar but different expressions for CLF result. These are identical in the weak coupling limit. Parameters describing the strength of coupling and the strength of connection are found.

In section 4 applications and numerical examples are discussed. The application of the coupled oscillator theory to coupled continuous subsystems is discussed. The examples of two spring-coupled rods and two spring-coupled plates are considered. Comparisons are made with the results of conventional SEA, an exact wave analysis for one-dimensional subsystems and numerical Monte Carlo simulations using an energy distribution approach.

2 TWO SPRING-COUPLED OSCILLATORS

Consider the system comprising two spring-coupled oscillators shown in Figure 2.1, $m_{1,2}$, $k_{1,2}$ and $c_{1,2}$ being the mass, stiffness and damping of the oscillators, respectively, and k the stiffness of the coupling spring.

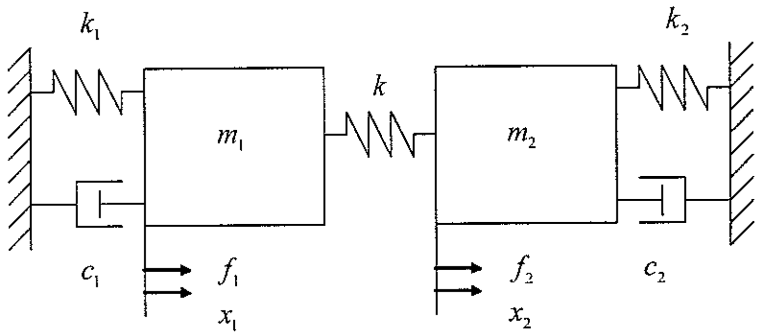

Figure 2.1 Two spring-coupled oscillators.

Suppose two time-harmonic forces $f_1(t) = F_1 e^{j\omega t}$ and $f_2(t) = F_2 e^{j\omega t}$ act on the oscillators. The response and the coupling power are derived in Appendix A, the expressions being somewhat cumbersome. This section concerns the average behaviour, being either frequency averages or ensemble averages taken over a distribution of properties of the system. In subsequent sections these results are applied to coupled sets of oscillators. Results are given elsewhere for more general cases of conservative coupling.

2.1 Broadband excitations

Now assume that the forces $f_1(t)$ and $f_2(t)$ are random, stationary, statistically independent, broadband white noise, with spectral densities S_{f_1} and S_{f_2} over some frequency band B. B is wide enough to contain both natural frequencies of the system.

The total input power, energy of each oscillator and coupling power between the two oscillators in the frequency band B are given by

$$P_{1,2} = \int_B P'_{1,2}(\omega)\,d\omega \tag{2.1}$$

$$P_{12} = \int_B P'_{12}(\omega)\,d\omega, \tag{2.2}$$

$$E_{1,2} = \int_B 2K'_{1,2}(\omega)\,d\omega \tag{2.3}$$

Here the discrete frequency kinetic energies $K'_1(\omega)$, $K'_2(\omega)$, input powers $P'_1(\omega)$, $P'_2(\omega)$ and coupling power $P'_{12}(\omega)$ are given in equations (A17-18), (A21-22) and (A29), respectively or equations (A39) to (A43) in terms of the excitation spectral densities. Note that in equation (2.3) the energy of each oscillator is taken to be twice the kinetic energy (there are no significant consequences of this). This is because the strain energy in the coupling spring cannot be unambiguously ascribed to one or other oscillator. Furthermore, the dissipated power is proportional to the kinetic energy. For other forms of conservative coupling there is similar but inconsequential ambiguity.

Since f_1 and f_2 are assumed to be uncorrelated and white, the integrals can be evaluated (making use of results from [1]) and equations (2.1)-(2.3) lead to

$$P_1 = \frac{\pi S_{f_1}}{m_1},\quad P_2 = \frac{\pi S_{f_2}}{m_2} \qquad (2.4), (2.5)$$

$$P_{12} = \frac{(\Delta_1 + \Delta_2)k^2}{m_1 m_2 Q}\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) \qquad (2.6)$$

$$E_1 = \frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{k^2(\Delta_1 + \Delta_2)}{m_1 m_2 Q\Delta_1}\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) \qquad (2.7)$$

$$E_2 = \frac{\pi S_{f_2}}{m_2\Delta_2} - \frac{k^2(\Delta_1 + \Delta_2)}{m_1 m_2 Q\Delta_2}\left(\frac{\pi S_{f_2}}{m_2\Delta_2} - \frac{\pi S_{f_1}}{m_1\Delta_1}\right) \qquad (2.8)$$

where $\Delta_{1,2}$ is the damping bandwidth of each oscillator and Q is a coupling parameter given by

$$Q = \left(\omega_1^2 - \omega_2^2\right)^2 + (\Delta_1 + \Delta_2)\left(\omega_2^2\Delta_1 + \omega_1^2\Delta_2\right) + \frac{(\Delta_1 + \Delta_2)^2 k^2}{m_1 m_2 \Delta_1 \Delta_2} \qquad (2.9)$$

Here $\omega_{1,2} = \sqrt{(k_{1,2} + k)/m_{1,2}}$, are the 'blocked' natural frequencies of the oscillators, i.e., the natural frequency of one oscillator when the other is held stationary. Note that the input powers (equations (2.4)-(2.5)) are independent of the natural frequencies ω_1 and ω_2. The transmitted power (from equations (2.6) and (2.9)), however, strongly depends on ω_1 and ω_2, and in particular the difference between them.

From equations (2.4)-(2.8), it follows that

$$P_1 + P_2 = \Delta_1 E_1 + \Delta_2 E_1 \qquad (2.10)$$

The above equation indicates that the time average power input to the system is equal to the power dissipated within the system, as is expected from conservation of energy.

The difference between the oscillators energies is found from equations (2.7)-(2.8) to be

$$E_1 - E_2 = \frac{\left(\omega_1^2 - \omega_2^2\right)^2 + (\Delta_1 + \Delta_2)\left(\omega_2^2\Delta_1 + \omega_1^2\Delta_2\right)}{Q}\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) \qquad (2.11)$$

This finally gives

$$P_{12} = \beta(E_1 - E_2) \qquad (2.12)$$

where the term β is given by

$$\beta = \left(\frac{k^2}{m_1 m_2} \right) \frac{(\Delta_1 + \Delta_2)}{\left(\omega_1^2 - \omega_2^2\right)^2 + (\Delta_1 + \Delta_2)\left(\omega_2^2 \Delta_1 + \omega_1^2 \Delta_2\right)} \tag{2.13}$$

Equation (2.13) is the familiar statement of coupling power proportionality, applied here to two, spring-coupled oscillators under broadband, random excitation. The term β is determined by the coupling stiffness k and the properties of the oscillators (m_1, $m_2, \omega_1, \omega_2, \Delta_1, \Delta_2$ etc) and is a constant, regardless of the oscillator energies. Note that β depends strongly on the difference of the subsystem uncoupled natural frequencies through the term $\left(\omega_1^2 - \omega_2^2\right)$.

Equation (2.13) was derived in [2] using a slightly different approach and formed the basis for the subsequent development of SEA. Similar analyses hold for other types of couplings, such as two oscillators connected via general, conservative coupling elements, or via an interface degree of freedom. Details are given in Appendices C and D. The expressions are somewhat complicated due to the number of parameters involved but the underlying physics is the same, especially the dependence of the coupling term on a group of parameters related to the coupling mechanism (here the term$\left(k^2/m_1 m_2\right)$) and a term depending on the oscillator properties which involves the difference of the natural frequencies.

It is worth noting that in the above derivations, the oscillator natural frequencies are taken as the blocked natural frequencies. This is one manner in which the oscillators can be uncoupled. The uncoupled system can equally be regarded as that in which the coupling spring $k = 0$, i.e., $\omega_{1,2}$ are defined in terms of the free natural frequencies $\omega_{f1} = \sqrt{k_1/m_1}$ and $\omega_{f2} = \sqrt{k_2/m_2}$. The power-energy relation holds equally for both definitions, although there are slight differences in the elements of the coupling term. In principle, therefore, it is not relevant whether the uncoupled natural frequencies $\omega_{1,2}$ are given by blocking (or clamping) the interface or by cutting (or freeing) it.

2.2 Ensemble averages

In SEA the system properties are usually not known exactly, but are random. Some joint probability density function $p\left(m_1, \omega_1, \Delta_1; m_2, \omega_2, \Delta_2; k; S_{f_1}, S_{f_2}\right)$ defines the statistics of the ensemble of two-oscillator systems. How p is defined in practice might be problematical. Here it is assumed that all parameters are known exactly except for the oscillator natural frequencies, which are both random and distributed over some frequency band Ω. If this band of uncertainty is fairly small compared to the bandwidth of excitation B, then it can equally be assumed that ω_1 is known and that the natural frequency spacing $\delta = (\omega_1 - \omega_2)$ is random and uniformly probable in some band Ω. As a result, the ensemble of two, spring-coupled oscillators is defined here by

$$p_\delta(\delta) = \begin{cases} 1/\Omega; & (-\Omega/2 \le \delta \le \Omega/2) \\ 0; & \text{otherwise} \end{cases} \tag{2.14}$$

Under these circumstances, if Ω is large compared to the half-power bandwidths but small compared to $\omega_{1,2}$, the ensemble averages of the input powers and subsystem energies (equations (2.4) and (2.7)) are given by

$$\overline{P}_1 = \frac{\pi S_{f_1}}{m_1}, \tag{2.15}$$

$$\overline{E}_1 = \frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{2k^2\Delta}{m_1 m_2 \Delta_1} \mathrm{E}\left[\frac{1}{Q}\right]\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right), \tag{2.16}$$

where both the expectation $\mathrm{E}[\bullet]$ and $\bar{\cdot}$ represent the ensemble average. Defining $\Delta = (\Delta_1 + \Delta_2)/2$, equation (2.9) gives

$$\mathrm{E}\left[\frac{1}{Q}\right] = \frac{1}{\Omega}\int_{\Omega}\frac{1}{Q}d\delta = \frac{\pi}{4\omega^2}\frac{1}{\Omega}\frac{1}{\sqrt{\Delta^2 + \kappa^2\left(\Delta^2/\Delta_1\Delta_2\right)}} \tag{2.17}$$

where ω is the centre frequency of band Ω and where

$$\kappa^2 = \frac{k^2}{m_1 m_2 \omega^2} \tag{2.18}$$

is a measure of the *strength of connection* between the oscillators.

Substituting equation (2.17) into equation (2.16), the ensemble average energy is given by

$$\overline{E}_1 = \frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{1}{\Omega}\left[\frac{\pi}{2}\frac{\kappa^2}{\sqrt{\Delta^2 + \kappa^2\left(\Delta^2/\Delta_1\Delta_2\right)}}\right]\left(\frac{\Delta}{\Delta_1}\right)\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right), \tag{2.19}$$

2.2.1 Coupling loss factors

From equations (2.15)-(2.19) the ensemble- and frequency-averaged energies and input powers are related by

$$\overline{\mathbf{E}} = \mathbf{A}\overline{\mathbf{P}}_{in} \tag{2.20}$$

where the elements of the energy influence coefficient (EIC) matrix $\mathbf{A}$ are given by

$$A_{11} = \frac{1}{\Delta_1}\left[1 - \frac{\alpha}{\Delta_1\Omega}\right],\quad A_{22} = \frac{1}{\Delta_2}\left[1 - \frac{\alpha}{\Delta_2\Omega}\right] \tag{2.21), (2.22}$$

$$A_{12} = A_{21} = \frac{1}{\Delta_1\Delta_2}\frac{\alpha}{\Omega} \tag{2.23}$$

where

$$\alpha = \frac{\pi}{2}\frac{\kappa^2}{\sqrt{1+\gamma^2}};\quad \gamma^2 = \frac{\kappa^2}{\Delta_1\Delta_2} \tag{2.24}$$

The coupling loss factor (CLF) matrix $\mathbf{L}$ of the two-oscillator system can be obtained by inverting the EIC matrix, i.e.

$$\overline{\mathbf{P}}_{in} = \mathbf{L}\overline{\mathbf{E}};\quad \mathbf{L} = \mathbf{A}^{-1} \tag{2.25}$$

and where

$$\mathbf{L} = \omega \begin{bmatrix} \eta_1 + \eta_{12} & -\eta_{21} \\ -\eta_{12} & \eta_2 + \eta_{21} \end{bmatrix} \tag{2.26}$$

From the above equations the coupling loss factors are found to be

$$\omega\eta_{12} = \omega\eta_{21} = \frac{\alpha/\Omega}{1-2(\alpha/\Omega)(\Delta/\Delta_1\Delta_2)} \tag{2.27}$$

In the above, γ is a parameter describing the *strength of coupling* while κ describes the *strength of connection*. The coupling loss factors in equation (2.27) thus relate the ensemble average input powers and energies.

2.3 Discussion

The above analysis reveals some results which are in contrast to those normally drawn in SEA: the coupling loss factor is a function of damping; there is a parameter γ that describes the strength of coupling between the two oscillators; equipartition does not occur in the limit $\Delta \to 0$ [6]. These are discussed below.

2.3.1 Equipartition of energy

In the very special case, and only in this case, where the uncoupled natural frequencies of the oscillators are identical ($\omega_1 = \omega_2$), then, in the limit $\Delta \to 0$, $\beta \to \infty$ and the oscillator energies become equal: this is equipartition of energy. However, if $\omega_1 \neq \omega_2$, then as $\Delta \to 0$, $\beta \to \kappa^2\Delta / \left[2(\omega_1 - \omega_2)^2\right]$ and the energies are not equal. For example, if only oscillator 1 is excited the energy ratio becomes

$$\frac{E_2}{E_1} = \frac{\kappa^2}{2\delta^2 + \kappa^2} \tag{2.28}$$

Thus we can conclude that equipartition of energy does *not* occur in the low-damping limit, except if the oscillator natural frequencies are identical: this is in contrast to what is commonly stated in the SEA community, but is consistent with behaviour that has been observed via wave [6] and system-mode approaches [8].

2.3.2 Coupling loss factors

Equation (2.27) indicates that $\omega\eta_{12}$ depends on both the strength of coupling (and hence on damping) and the strength of connection. Furthermore, at low damping η_{12} is proportional to η, while for high damping it asymptotes to the constant $\omega\eta_{12} = \pi\kappa^2/2\Omega$. Such behaviour has been noted before in terms of wave analysis [6], finite element analysis [13,14] and in terms of the modes of the system as a whole [8], so it is not surprising that it is also evident in the behaviour of two coupled oscillators.

The reason why these observations differ from those of previous work [1-5] is that the coupling loss factor was defined from the expected value of the coupling power, i.e.

$$\bar{P}_{12} = \mathrm{E}\left[\beta(E_1 - E_2)\right] \tag{2.29}$$

However, in the previous analyses, the further assumption was made that the terms on the right of equation (2.29) are statistically independent, so that

$$\bar{P}_{12} \approx \mathrm{E}[\beta]\mathrm{E}[E_1 - E_2] \tag{2.30}$$

If the coupling loss factor is estimated, in this manner, by ensemble averaging β then (Appendix B) it becomes

$$\omega\eta_{12} = \omega\eta_{21} = \frac{\pi\kappa^2}{2\Omega} \tag{2.31}$$

However, this is clearly an approximation: β and $(E_1 - E_2)$ are strongly correlated; if β is large then $(E_1 - E_2)$ is small, and vice versa.

2.4 Concluding remarks

In this section systems comprising two spring-coupled coupled oscillators were considered. Various observations were made about the behaviour under broadband excitation, some of which differ from those which are commonly assumed within SEA. It was seen that:

1. Equipartition of energy does not occur as damping tends to zero, except in the case where the uncoupled oscillators have identical natural frequencies.
2. The coupling power and coupling loss factor can be written in terms of the "strength of connection" between the oscillators and a term reflecting the "strength of coupling", these being two distinct parameters.
3. The strength of coupling, and hence the coupling loss factor, depends on damping, decreasing as the level of damping increases.
4. The coupling loss factor is proportional to damping at low damping and is independent of damping in the high damping, weak coupling limit.

The above analysis can be extended straightforwardly to two oscillators connected via general, conservative coupling elements (Appendix C), or via an interface degree of freedom (in Appendix D), except that the resulting expression are cumbersome.

3 TWO SETS OF SPRING-COUPLED OSCILLATORS

Consider now the case of two, spring-coupled subsystems a and b (Figure 3.1). As in conventional SEA, each subsystem is regarded as being a set of oscillators and, when coupled, each oscillator in one set shares energy with all the oscillators in the second set. Here, oscillators j and k in sets a and b are coupled by the spring k_{jk}^{ab}.

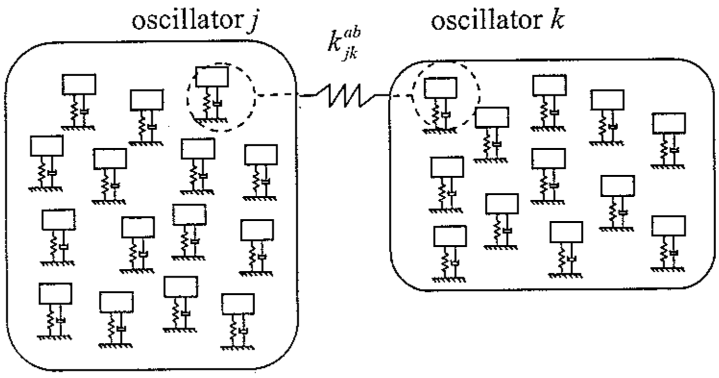

Figure 3.1. Two sets of spring-coupled oscillators.

In this section relations between the input powers, coupling powers and subsystem energies, and their ensemble averages are found. In principle one might proceed along the lines developed in section 2. The dynamic stiffness matrix of the system could be found in terms of the oscillator masses, natural frequencies and bandwidths and the coupling stiffnesses. The frequency response matrix **H** is the inverse of this dynamic stiffness matrix. White, uncorrelated excitation is then assumed to act on each oscillator and frequency integration gives the energies and powers. Simplifying approximations can be made by ignoring all out-of-band modes and extending the frequency range of integration to $[0,\infty]$ for all in-band, resonant modes. Ensemble averages could in principle be developed by taking many oscillators in each set, defining the statistics of their properties in some way and evaluating the multidimensional integral of the inverse of a large, random matrix. This approach is intractable. Instead, in what follows, simplifying assumptions are made and results for the two-oscillator case introduced.

In the next section the system power and energy relationships are stated. Following that, some comments are made concerning the ensemble and ensemble averages. Expressions for the input and dissipated powers are then found – this is fairly straightforward. The coupling powers are more problematical, however, and various assumptions must be made. Two different approximate approaches are developed, assuming different forms for the coupling power: first, that the coupling power between two oscillators is proportional to their energy difference, and secondly that it is proportional to the "blocked" energy difference. The latter can predict negative values for oscillator energies in the strong coupling limit, but seems to give more accurate predictions for moderate coupling strength. In any event, the two approaches give the same results for weak coupling or, for weak or strong coupling, if the strength of connection is weak.

3.1 Excitation, powers and energies: power balance relations

The subsystems are excited by white noise over some frequency band B, which contains N_a and N_b resonant modes of subsystems a and b, respectively. The modal densities $n_{a,b}$ are such that the expected number of modes

$$\mathrm{E}\left[N_{a,b}\right] = n_{a,b}B \tag{3.1}$$

Out-of-band, nonresonant modes are ignored.

Power balance for oscillator j in set a gives

$$P^a_{in,j} = P^a_{diss,j} + P^a_{coup,j} \tag{3.2}$$

where the terms represent the input, dissipated and net coupling powers respectively. In matrix form

$$\mathbf{P}^a_{in} = \mathbf{P}^a_{diss} + \mathbf{P}^a_{coup} \tag{3.3}$$

where the vectors $\mathbf{P}$ are now vectors of oscillator powers (throughout, similar expressions can be written for subsystem b).

The net coupling power for oscillator j,

$$P^a_{coup,j} = \sum_{\substack{\text{oscillators } k \\ \text{in subsystem } b}} P^{ab}_{jk} + \sum_{\substack{\text{oscillators } n \\ \text{in subsystem } a}} P^{aa}_{jn} \tag{3.4}$$

is here written explicitly as the sum of powers P^{ab}_{jk} exchanged between oscillators j and k in sets a and b respectively and powers P^{aa}_{jn} exchanged between oscillators j and n, both of which are in set a. This in-set contribution is unimportant, since it has zero contribution to the net coupling power between the subsystems.

We assume that the total power input to each subsystem is simply the sum of the power input to each oscillator, i.e.

$$P_{tot,a} = \sum_{j=1,N_a} P^a_{in,j} \,,\; P_{tot,b} = \sum_{k=1,N_b} P^b_{in,k} \tag{3.5}$$

where $P^a_{in,j}$ is the power input to oscillator j in set a. In matrix form

$$\mathbf{P}_{tot} = \begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} = \mathbf{U}^T \begin{Bmatrix} \mathbf{P}^a_{in} \\ \mathbf{P}^b_{in} \end{Bmatrix} \tag{3.6}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{1}_{N_a} & \mathbf{0}_{N_a} \\ \mathbf{0}_{N_b} & \mathbf{1}_{N_b} \end{bmatrix} \tag{3.7}$$

and where $\mathbf{1}_{N_a}$ and $\mathbf{0}_{N_a}$ are $N_a \times 1$ vectors of 1's and 0's respectively. Hence $\mathbf{U}$ is an $\left(N_a + N_b\right)\times 2$ matrix.

Similarly, the subsystem total energies are

$$E_{tot,a} = \sum_{j=1,N_a} E^a_j \,,\; E_{tot,b} = \sum_{k=1,N_b} E^b_k \tag{3.8}$$

where E_j^a is the energy of oscillator j in set a. Again, in matrix form

$$\mathbf{E}_{tot} = \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} = \mathbf{U}^T \begin{Bmatrix} \mathbf{E}^a \\ \mathbf{E}^b \end{Bmatrix} \tag{3.9}$$

In a similar manner, the subsystem dissipated powers and the net coupling power between the subsystems can be written as the sums of oscillator terms as

$$P_{diss,a} = \sum_{j=1,N_a} P^a_{diss,j}; \quad P_{diss,b} = \sum_{k=1,N_b} P^b_{diss,k} \tag{3.10}$$

$$P_{ab} = \sum_{j=1,N_a} \sum_{k=1,N_b} P^{ab}_{jk} \tag{3.11}$$

3.2 The ensemble and ensemble averages

The properties of the two sets of oscillators are random and the ensembles from which they are drawn defined by a joint probability distribution function p which depends on the oscillator masses, resonant bandwidths and natural frequencies, the coupling spring stiffnesses and the excitation spectral densities. Clearly the problem is extremely complicated, and estimation of ensemble statistics requires the evaluation of an integral of very high dimension.

As in Section 2, assumptions will be made about the ensemble. The properties of the two sets of oscillators, the coupling stiffnesses and the excitations are assumed statistically independent (in practice, for continuous subsystems, k^{ab}_{jk} will depend on the mode shapes of the respective modes which in turn may be correlated with the natural frequencies in some manner). For simplicity, each oscillator in set a is assumed to have the same mass m_a, resonant bandwidth Δ_a and ensemble average spectral density S_a, i.e.,

$$\Delta_j^a = \Delta_a, \; m_j^a = m_a, \; S_j^a = S_f^a \tag{3.12}$$

Similar assumptions are also made for set b. The joint probability density function that defines the ensemble then becomes

$$p_{\omega_a}\left(\omega_1^a, \omega_2^a, \ldots, \omega_{N_a}^a\right) p_{\omega_b}\left(\omega_1^b, \omega_2^b, \ldots, \omega_{N_b}^b\right) p_k\left(k_{11}^{ab}, k_{12}^{ab}, k_{21}^{ab}, \ldots, k_{N_aN_b}^{ab}\right)$$

Strictly, the discrete frequency problem should be solved, frequency averaged and then the results ensemble averaged. This is intractable. Instead, various approximations and assumptions are made, recalling results from section 2.

Subsequently it is assumed that each oscillator pair satisfies the assumptions made in section 2 which concerned two coupled oscillators, i.e. $\left(\omega_j^a - \omega_k^b\right)$ is random and uniformly probable within the band of excitation B, and statistically independent of the coupling stiffness.

3.3. Ensemble and frequency average input and dissipated powers

3.3.1. Input powers

The input power for oscillator j is taken to be (c.f. eq 2.4)

$$P^a_{in,j} = \frac{\pi S_j^a}{m_j^a} = \frac{\pi S_f^a}{m_a} \tag{3.13}$$

The oscillator input power vector then becomes

$$\begin{Bmatrix} \mathbf{P}_{in}^{a} \\ \mathbf{P}_{in}^{b} \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \dfrac{\pi S_f^a}{m_a} \\ \dfrac{\pi S_f^b}{m_b} \end{Bmatrix} \tag{3.14}$$

and hence the total power input to subsystem a is

$$P_{tot,a} = N_a \frac{\pi S_f^a}{m_a} \tag{3.15}$$

In matrix form

$$\mathbf{P}_{tot} = \mathbf{U}^T \begin{Bmatrix} \mathbf{P}_{in}^{a} \\ \mathbf{P}_{in}^{b} \end{Bmatrix} = \mathbf{U}^T\mathbf{U} \begin{Bmatrix} \dfrac{\pi S_f^a}{m_a} \\ \dfrac{\pi S_f^b}{m_b} \end{Bmatrix} = \begin{Bmatrix} N_a \dfrac{\pi S_f^a}{m_a} \\ N_b \dfrac{\pi S_f^b}{m_b} \end{Bmatrix} \tag{3.16}$$

Thus every oscillator in each set is assumed to be excited equally: this is "rain-on-the-roof" excitation.

Strictly, perhaps, ensemble averages should be found by taking the expectation of the sum of the individual oscillator input powers, i.e.

$$\overline{P}_a = \mathrm{E}\left[\sum_{j=1}^{N_a} P_{in,j}^a \right] \tag{3.17}$$

However, given the definition of the ensemble and the fact that broadband frequency averages are taken, there is no difference with regard to the input powers.

3.3.2. Dissipated powers

The dissipated power for oscillator j is taken to be

$$P_{diss,j}^a = \Delta_j^a; E_j^a = \Delta_a E_j^a \tag{3.18}$$

The vector of oscillator dissipated powers is thus related to the oscillator energies by

$$\begin{Bmatrix} \mathbf{P}_{diss}^{a} \\ \mathbf{P}_{diss}^{b} \end{Bmatrix} = \begin{bmatrix} diag(\Delta_a) & \mathbf{0} \\ \mathbf{0} & diag(\Delta_b) \end{bmatrix} \begin{Bmatrix} \mathbf{E}^a \\ \mathbf{E}^b \end{Bmatrix} \tag{3.19}$$

The total subsystem dissipated powers are therefore

$$\mathbf{P}_{diss,tot} = \begin{Bmatrix} P_{diss,a} \\ P_{diss,b} \end{Bmatrix} = \mathbf{U}^T \begin{Bmatrix} \mathbf{P}_{diss}^{a} \\ \mathbf{P}_{diss}^{b} \end{Bmatrix} = \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix} \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} \tag{3.20}$$

Again, strictly the ensemble average dissipated power for subsystem a should be taken as the expectation of the frequency average oscillator dissipated powers, i.e. for broadband frequency averages

$$\overline{P}_a = \mathrm{E}\left[\sum_{j=1}^{N_a} \Delta_j^a E_j^a \right] \tag{3.21}$$

However, given that all oscillator bandwidths are assumed equal there is no difference.

3.3.3. Power balance equations

To summarise, the power balance equations for individual oscillators (equations (3.2) or (3.3)) are, in matrix form

$$\begin{Bmatrix} \mathbf{P}^a_{in} \\ \mathbf{P}^b_{in} \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}^a_{diss} \\ \mathbf{P}^b_{diss} \end{Bmatrix} + \begin{Bmatrix} \mathbf{P}^a_{coup} \\ \mathbf{P}^b_{coup} \end{Bmatrix} \quad (3.22)$$

with the input and dissipated powers being given by equations (3.14) and (3.19). Premultiplying by $\mathbf{U}^T$ leads to (equations (3.16) and (3.20))

$$\begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} = \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix} \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} + \mathbf{U}^T \begin{Bmatrix} \mathbf{P}^a_{coup} \\ \mathbf{P}^b_{coup} \end{Bmatrix} \quad (3.23)$$

Finally, the last term becomes

$$\mathbf{U}^T \begin{Bmatrix} \mathbf{P}^a_{coup} \\ \mathbf{P}^b_{coup} \end{Bmatrix} = \begin{Bmatrix} P_{ab} \\ -P_{ab} \end{Bmatrix} \quad (3.24)$$

where

$$P_{ab} = \sum_{j=1,N_a} \sum_{k=1,N_b} P^{ab}_{jk} \quad (3.25)$$

Note that the in-set coupling powers do not contribute to the net coupling power: they sum to zero.

The problem thus remains of determining the coupling powers between the oscillator pairs and hence forming the SEA equations. Two possible approaches, in which results from section 2 are recalled, are described in the next two sections.

3.4. Coupling power proportional to difference of oscillator energies

Suppose that the inter-modal coupling power is assumed to be proportional to the difference in the actual oscillator energies. The result for the two oscillator case is given in equation (2.12) with the constant of proportionality β being given in equation (2.13). For the case of two sets of oscillators these equations become

$$P_{jk} = \beta^{ab}_{jk}\left(E^a_j - E^b_k\right);$$

$$\beta^{ab}_{jk} = \frac{\left(k^{ab}_{jk}\right)^2}{m_a m_b} \frac{\left(\Delta_a + \Delta_b\right)}{\left(\omega^{a^2}_j - \omega^{b^2}_k\right)^2 + \left(\Delta_a + \Delta_b\right)\left(\omega^{b^2}_k \Delta_a + \omega^{a^2}_j \Delta_b\right)} \quad (3.26)$$

Thus

$$P_{ab} = \sum_{j=1,N_a} \sum_{k=1,N_b} \beta^{ab}_{jk}\left(E^a_j - E^b_k\right) \quad (3.27)$$

It is now assumed that there are so many oscillators in each set that the mean value of the terms in this sum approximates the ensemble average. Under these circumstances

$$\overline{P_{ab}} = N_a N_b \overline{\beta^{ab}_{jk}\left(E^a_j - E^b_k\right)} \quad (3.28)$$

The analysis in section 2.2.1 is assumed to hold (i.e. the relations between ensemble average powers and energies leading to equation (2.27)), hence

$$\overline{P_{ab}} = \omega\eta_{ab}\overline{E_{tot,a}} - \omega\eta_{ba}\overline{E_{tot,b}} \tag{3.29}$$

where

$$\overline{E_j^a} = \frac{\overline{E_{tot,a}}}{N_a}; \quad \overline{E_k^b} = \frac{\overline{E_{tot,b}}}{N_b} \tag{3.30}$$

and where the coupling loss factors are

$$\begin{gathered} \omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \alpha n_a n_b; \\ \alpha = \frac{\pi\kappa^2}{2\sqrt{1+\gamma^2}}; \\ \gamma^2 = \frac{\kappa^2}{\Delta_a \Delta_b}; \quad \kappa^2 = \frac{\overline{\left(k_{jk}^{ab}\right)^2}}{m_a m_b \omega^2} \end{gathered} \tag{3.31}$$

Note that the conclusions made for the two-oscillator case hold here, and in particular that there is a measure γ of the strength of coupling, a measure κ of the strength of connection and the coupling loss factor depends on the damping and γ.

3.4.1. Conventional SEA

As discussed in section 2.2., in conventional SEA the average of the product is assumed to equal the product of the averages, i.e.

$$\overline{P_{ab}} = N_a N_b \overline{\beta_{jk}^{ab}}\,\overline{\left(E_j^a - E_k^b\right)} \tag{3.32}$$

These ensemble averages are

$$\begin{gathered} \overline{\beta_{jk}^{ab}} = \frac{\pi\kappa^2}{2\Omega}; \quad \kappa^2 = \frac{\overline{\left(k_{jk}^{ab}\right)^2}}{m_a m_b \omega^2} \\ \overline{E_j^a} = \frac{\overline{E_{tot,a}}}{N_a}; \quad \overline{E_k^b} = \frac{\overline{E_{tot,b}}}{N_b} \end{gathered} \tag{3.33}$$

The SEA equations, found by substituting equations (3.32) and (3.33) in (3.23), and putting $n_a = N_a/\Omega$, are

$$\begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} = \begin{bmatrix} \Delta_a + \omega\eta_{ab} & -\omega\eta_{ba} \\ -\omega\eta_{ab} & \Delta_b + \omega\eta_{ba} \end{bmatrix} \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} \tag{3.34}$$

where the coupling loss factors are

$$\omega n_a \eta_{ab}^{(c)} = \omega n_b \eta_{ba}^{(c)} = \frac{\pi\kappa^2}{2} n_a n_b \tag{3.35}$$

Note that the first of equations (3.31) can be written as

$$\omega n_a \eta_{ab}^{(a)} = \frac{1}{\sqrt{1+\gamma^2}} \omega n_b \eta_{ba}^{(c)} \tag{3.36}$$

3.4.2. *Comments*

Both analyses (leading to equations (3.31) and (3.35)) imply that the ensemble average oscillator energies are equal. They differ in the fact that, in the first analysis, the correlation between the constants β_{jk}^{ab} and the oscillator energy differences is included, as least as an approximation. Thus these constants are not identical over the ensemble. Note also that the analysis leading to equation (3.31) is approximate, in that strictly the energies themselves depend on the coupling powers so that the equations should be solved-then-averaged. Finally, the expressions for the coupling loss factors are equal in the weak coupling limit, $\gamma \to 0$.

3.5. Coupling power proportional to difference of "blocked" oscillator energies

An alternative approach is to note from equation (2.6) that, for two oscillators, the coupling power is also proportional to the difference of the "blocked" oscillator energies $(P_1/\Delta_1 - P_2/\Delta_2)$, i.e. the energies that each oscillator would have if the excitation remained the same but the other oscillator were held fixed. If this result is assumed to hold for two oscillators of the coupled sets then the coupling power between two such oscillators is

$$P_{jk}^{ab} = \frac{(\Delta_a + \Delta_b)}{m_a m_b} \frac{\left(k_{jk}^{ab}\right)^2}{Q_{jk}^{ab}} \left(\frac{1}{\Delta_a} \frac{\pi S_f^a}{m_a} - \frac{1}{\Delta_b} \frac{\pi S_f^b}{m_b} \right) \tag{3.37}$$

where, from equation (2.12)

$$Q_{jk}^{ab} = \left(\omega_j^{a^2} - \omega_k^{b^2}\right)^2 + (\Delta_a + \Delta_b)\left(\omega_k^{b^2}\Delta_a + \omega_j^{a^2}\Delta_b\right) + \frac{\left(k_{jk}^{ab}\right)^2 (\Delta_a + \Delta_b)^2}{m_a m_b \Delta_a \Delta_b} \tag{3.38}$$

The net coupling power P_{ab} is again found by summing the coupling powers between each oscillator pair. Again, if there are assumed to be so many modes that the sum can be approximated by the ensemble average, then

$$\overline{P_{ab}} = N_a N_b \overline{\left(k_{jk}^{ab}\right)^2} \, \overline{\frac{1}{Q_{jk}^{ab}}} \, \frac{(\Delta_a + \Delta_b)}{m_a m_b} \left(\frac{1}{\Delta_a} \frac{\pi S_f^a}{m_a} - \frac{1}{\Delta_b} \frac{\pi S_f^b}{m_b} \right) \tag{3.39}$$

Replacing the ensemble averages with the expressions from equation (2.17) and (3.31), noting that $P_{tot,a} = N_a \pi S_a / m_a$ and putting $n_a = N_a/\Omega$ once again leads to

$$\overline{P_{ab}} = \alpha \left(\frac{n_b}{\Delta_a} P_{tot,a} - \frac{n_a}{\Delta_b} P_{tot,b} \right) \tag{3.40}$$

where α, γ and κ are as defined in equation (3.31). Substituting the coupling power into the power balance equation (3.23) then gives

$$\begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} = \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix} \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} + \begin{bmatrix} \dfrac{n_b \alpha}{\Delta_a} & -\dfrac{n_a \alpha}{\Delta_b} \\ -\dfrac{n_b \alpha}{\Delta_a} & \dfrac{n_a \alpha}{\Delta_b} \end{bmatrix} \begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} \tag{3.41}$$

Hence the matrix of energy influence coefficients, which relates the ensemble averages of the input powers and energies is

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\Delta_a}\left(1-\frac{n_b\alpha}{\Delta_a}\right) & \frac{n_a\alpha}{\Delta_a\Delta_b} \\ \frac{n_b\alpha}{\Delta_a\Delta_b} & \frac{1}{\Delta_b}\left(1-\frac{n_a\alpha}{\Delta_b}\right) \end{bmatrix} \tag{3.42}$$

By inverting this matrix, the coupling loss factors are found to be

$$\omega n_a\eta_{ab}^{(b)} = \omega n_b\eta_{ab}^{(b)} = \frac{\alpha n_a n_b}{1-\left(\frac{n_a}{\Delta_b}+\frac{n_b}{\Delta_a}\right)\alpha} \tag{3.43}$$

This can be written as

$$\omega n_a\eta_{ab}^{(b)} = \frac{\alpha n_a n_b}{1-\left(\frac{1}{\mu_a}+\frac{1}{\mu_b}\right)\alpha n_a n_b} \tag{3.44}$$

where $\mu_{a,b} = n_{a,b}\Delta_{a,b}$ is the modal overlap based on the half-power bandwidth. This can be alternatively expressed in terms of the conventional SEA estimate as

$$\omega n_a\eta_{ab}^{(b)} = \frac{1}{\sqrt{1+\gamma^2}-\left(\frac{1}{\mu_a}+\frac{1}{\mu_b}\right)\omega n_a\eta_{ab}^{(c)}}\omega n_a\eta_{ab}^{(c)} \tag{3.45}$$

3.6 Variability

The response of a specific system excited over a finite frequency band will differ from the ensemble averages given above. This arises partly from the fact that only a finite sampling of oscillator pairs (and hence terms involving $\left|k_{jk}^{ab}\right|^2/Q_{jk}$) are taken and partly from the fact that the number of modes in the band is a random variable. Furthermore, natural frequency spacing statistics will affect the variance.

For a selected pair of oscillators $\delta_{jk} = \omega_j^a - \omega_k^b$ is typically a uniformly distributed random variable. However, for the case of coupled sets of oscillators, although $\left(\omega_j^a - \omega_k^b\right)$ might be uniformly probable, $\left(\omega_{j+1}^a - \omega_k^b\right)$ depends also on the natural frequency spacing statistics of subsystem a, which define the probability density function for $\left(\omega_{j+1}^a - \omega_k^b\right)$. In other words, $\left(\omega_{j+1}^a - \omega_k^b\right) = \left(\omega_j^a - \omega_k^b\right) + \left(\omega_{j+1}^a - \omega_j^a\right)$, and while $\left(\omega_j^a - \omega_k^b\right)$ may be uniformly probable, $\left(\omega_{j+1}^a - \omega_j^a\right)$ usually is not. The variance thus depends on these spacing statistics. Common spacing statistics include a Poisson distribution, which implies that ω_j^a and ω_{j+1}^a are statistically independent. This situation is characteristic of uniform, regular two- or three-dimensional subsystems (i.e., rectangular, or other shapes with a separable geometry). Examples include rectangular plates. Another distribution is GOE spacing statistics, which are held to be typical of subsystems with sufficiently large amounts of randomness. The natural frequency spacing is then Rayleigh distributed, but there are also statistics relating second-nearest neighbours, etc. Issues concerning variance are considered elsewhere.

3.7 Concluding remarks

To summarise, in the above, the power balance equations were written for each oscillator and oscillator set. Mild assumptions were made concerning the input and dissipated powers leading to (equations (3.23) to (3.25))

$$\begin{Bmatrix} P_{tot,a} \\ P_{tot,b} \end{Bmatrix} = \begin{bmatrix} \Delta_a & 0 \\ 0 & \Delta_b \end{bmatrix} \begin{Bmatrix} E_{tot,a} \\ E_{tot,b} \end{Bmatrix} + \begin{Bmatrix} \sum\limits_{j=1,N_a} \sum\limits_{k=1,N_b} P_{jk}^{ab} \\ -\sum\limits_{j=1,N_a} \sum\limits_{k=1,N_b} P_{jk}^{ab} \end{Bmatrix} \tag{3.46}$$

Solution to the full set of equations is intractable so assumptions were made concerning the coupling powers. In particular, if the coupling power is assumed to be proportional to the *actual* oscillator energy difference (equation (3.27)) then

$$\omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \alpha n_a n_b; \tag{3.47}$$

while if it assumed to be proportional to the *blocked* energy difference (equation (3.37)) then

$$\omega n_a \eta_{ab}^{(b)} = \omega n_b \eta_{ab}^{(b)} = \frac{\alpha n_a n_b}{1 - \left(\frac{n_a}{\Delta_b} + \frac{n_b}{\Delta_a} \right) \alpha} \tag{3.48}$$

In the above, the strength of connection is

$$\kappa^2 = \frac{\overline{\left(k_{jk}^{ab}\right)^2}}{m_a m_b \omega^2} \tag{3.49}$$

while the strength of coupling is

$$\gamma^2 = \frac{\kappa^2}{\Delta_a \Delta_b} \tag{3.50}$$

and

$$\alpha = \frac{\pi \kappa^2}{2\sqrt{1+\gamma^2}} \tag{3.51}$$

Finally, *conventional* SEA gives (equation (3.35))

$$\omega n_a \eta_{ab}^{(c)} = \omega n_b \eta_{ba}^{(c)} = \frac{\pi \kappa^2}{2} n_a n_b \tag{3.52}$$

The coupling loss factors can also be written in the following forms:

$$\eta_{ab}^{(a)} = \frac{1}{\sqrt{1+\gamma^2}} \eta_{ab}^{(c)} \tag{3.53}$$

$$\eta_{ab}^{(b)} = \left[\frac{1}{\sqrt{1+\gamma^2} - \left(\mu_a^{-1} + \mu_b^{-1}\right) \omega n_a \eta_{ab}^{(c)}} \right] \eta_{ab}^{(c)} \tag{3.54}$$

The 3 expressions for the coupling loss factor are identical in the weak coupling limit $\gamma \to 0$ (and hence $\mu_a \to \infty$). The conventional expression ignores all correlation between oscillator energy and the constant in the coupling power proportionality relation, overestimating the coupling powers (especially between oscillators with natural frequencies which are nearly identical) and hence overestimating the coupling loss factors when the coupling is strong. The two other expressions become proportional to damping as the damping is reduced. The "blocked" expressions, equation (3.48), can diverge, with the denominator becoming zero or negative, implying non-physical negative subsystem energies. This happens if κ is large enough, and the modal densities are large enough, such that (if $\Delta_a \approx \Delta_b$)

$$\left(n_a + n_b\right)\frac{\pi\kappa}{2} \geq 1 \tag{3.55}$$

In effect this means that each oscillator interacts sufficiently strongly with so many other oscillators that the "blocked" energies are reduced substantially by the energies lost to the other oscillators, and the assumption in equation (3.37) breaks down. It may be thought then that the actual and blocked assumptions give approximate lower and upper bounds for the ensemble average coupling loss factors.

3.7.1 Other forms of conservative coupling

The above analysis can be extended straightforwardly to other cases of conservative coupling, and in particular those considered in Appendices C and D. In all such cases the energies and coupling power for two coupled oscillators contain a term in whose denominator the difference in the oscillator natural frequencies occurs. Other terms appear in the numerator and denominator and these depends on the oscillator and coupling parameters.

In particular, comparing the results of section 2 with those derived in Appendices C and D, it is quite straightforward to apply the above equations to cases where two sets of oscillators are coupled either by general, conservative coupling elements or by a fixed spring at their interface, the strength of connection parameter (equation (3.49)) being modified accordingly.

In the case of general, conservative coupling (Appendix C), the jth and the kth oscillators of sets a and b are connected by a spring k_{jk}^{ab}, a mass m_{jk}^{ab}, and a gyroscopic constant g_{jk}^{ab}. From Appendix C, the strength of connection becomes

$$\kappa^2 = \mathrm{E}\left[\frac{\left(\mu_{jk}^{ab}\right)^2}{\left(\Delta_a+\Delta_b\right)\omega^2}\left(\Delta_a\left(\omega_k^b\right)^4+\Delta_b\left(\omega_j^a\right)^4+\Delta_a\Delta_b\left(\left(\omega_k^b\right)^2\Delta_a+\left(\omega_j^a\right)^2\Delta_b\right)\right)\right.$$
$$\left.+\left(\left(\hat{\gamma}_{jk}^{ab}\right)^2+2\mu_{jk}^{ab}\hat{\kappa}_{jk}^{ab}\right)\frac{\left(\left(\omega_k^b\right)^2\Delta_a+\left(\omega_j^a\right)^2\Delta_b\right)}{\left(\Delta_a+\Delta_b\right)\omega^2}+\frac{\left(\hat{\kappa}_{jk}^{ab}\right)^2}{\omega^2}\right] \tag{3.56}$$

In the above equations

$$\mu_{jk}^{ab} = \frac{m_{jk}^{ab}/4}{\sqrt{\left(m_a + m_{jk}^{ab}/4\right)\left(m_b + m_{jk}^{ab}/4\right)}} \tag{3.57}$$

$$\hat{\gamma}_{jk}^{ab} = \frac{g_{jk}^{ab}}{\sqrt{\left(m_a + m_{jk}^{ab}/4\right)\left(m_b + m_{jk}^{ab}/4\right)}} \tag{3.58}$$

$$\hat{\kappa}_{jk}^{ab} = \frac{k_{jk}^{ab}}{\sqrt{\left(m_a + m_{jk}^{ab}/4\right)\left(m_b + m_{jk}^{ab}/4\right)}} \tag{3.59}$$

Assuming $m_{jk}^{ab} << m_{a,b}$ and $\Omega << \omega$, equation (3.56) simplifies to

$$\kappa^2 \approx \frac{1}{m_a m_b} \mathrm{E}\left[\left(m_{jk}^{ab}/4\right)^2 \left(\omega^2 + \Delta_a \Delta_b\right) + \left(\left(g_{jk}^{ab}\right)^2 + 2\left(m_{jk}^{ab}/4\right) k_{jk}^{ab}\right) + \frac{\left(k_{jk}^{ab}\right)^2}{\omega^2} \right]$$
$$= \frac{1}{m_a m_b} \mathrm{E}\left[\left(\left(m_{jk}^{ab}/4\right)\omega + \frac{k_{jk}^{ab}}{\omega} \right)^2 + \Delta_a \Delta_b \left(m_{jk}^{ab}/4\right)^2 + \left(g_{jk}^{ab}\right)^2 \right] \quad (3.60)$$

Assuming the statistical independence of m_{jk}^{ab}, k_{jk}^{ab} and g_{jk}^{ab}, equation (3.60) further reduces to

$$\kappa^2 \approx \mathrm{E}\left[\frac{\left(m_{jk}^{ab}/4\right)^2}{m_a m_b} \left(\omega^2 + \Delta_a \Delta_b\right) \right] + \mathrm{E}\left[\frac{\left(g_{jk}^{ab}\right)^2}{m_a m_b} \right] + \mathrm{E}\left[\frac{\left(k_{jk}^{ab}\right)^2}{m_a m_b \omega^2} \right] \quad (3.61)$$

Usually, $\Delta_{a,b} << \omega$, and equation (3.61) finally reduces to

$$\kappa^2 \approx \overline{\mu^2}\omega^2 + \overline{\gamma^2} + \overline{\hat{\kappa}^2}/\omega^2 \quad (3.62)$$

where

$$\overline{\hat{\kappa}^2} = \mathrm{E}\left[\frac{\left(k_{jk}^{ab}\right)^2}{m_a m_b} \right] \quad (3.63)$$

$$\overline{\hat{\gamma}^2} = \mathrm{E}\left[\frac{\left(g_{jk}^{ab}\right)^2}{m_a m_b} \right] \quad (3.64)$$

$$\overline{\mu^2} = \mathrm{E}\left[\frac{\left(m_{jk}^{ab}/4\right)^2}{m_a m_b} \right] \quad (3.65)$$

Equation (3.62) indicates that the strength of connection mainly depends on the coupling mass terms at high frequencies and on the coupling stiffness terms at low frequencies.

Similarly, in the case of fixed-interface spring coupling (i.e., the jth and the kth oscillators of sets a and b are connected by a spring of stiffness k_{jk}^{ab}) from Appendix D, and assuming $\Delta_a \approx \Delta_b$, the strength of connection is given by

$$\kappa^2 = \mathrm{E}\left[\mu_j^a \mu_k^b \frac{k_j^a k_k^b}{m_a m_b \omega^2} \right] \quad (3.66)$$

where

$$\mu_j^a = \frac{k_j^a}{k_j^a + k_k^b + k_{jk}^{ab}} \quad (3.67)$$

$$\mu_k^b = \frac{k_k^b}{k_j^a + k_k^b + k_{jk}^{ab}} \quad (3.68)$$

Here k_j^a and k_k^b are respectively the stiffnesses of the jth and the kth oscillators of sets a and b. Equations (3.66)-(3.68) then yield

$$\kappa^2 = \frac{\overline{\left(k_{c,jk}^{ab}\right)^2}}{m_a m_b \omega^2} \tag{3.69}$$

where

$$\overline{\left(k_{c,jk}^{ab}\right)^2} = \mathrm{E}\left[\frac{1}{\left(\frac{1}{k_j^a} + \frac{1}{k_k^b} + \frac{k_{jk}^{ab}}{k_j^a k_k^b}\right)^2}\right] \tag{3.70}$$

Physically, $k_{c,jk}^{ab}$ is the coupling stiffness between the jth and the kth oscillators of sets a and b and is given by

$$k_{c,jk}^{ab} = \frac{1}{\frac{1}{k_j^a} + \frac{1}{k_k^b} + \frac{k_{jk}^{ab}}{k_j^a k_k^b}} \tag{3.71}$$

It is seen, therefore, for flexible coupling springs such that $k_{jk}^{ab} << \left(k_j^a, k_k^b\right)$, $k_{c,jk}^{ab} \to \frac{1}{1/k_j^a + 1/k_k^b}$, in which case the coupling stiffness terms are mainly determined by the internal stiffnesses of the oscillators. The two sets of oscillators are usually strongly connected. For stiff coupling springs such that $k_{jk}^{ab} >> \left(k_j^a, k_k^b\right)$, however, $k_{c,jk}^{ab} \to \frac{k_j^a k_k^b}{k_{jk}^{ab}}$. For the special case when $k_{jk}^{ab} \to \infty$, $k_{c,jk}^{ab} \to 0$, which corresponds to the case when the oscillators are uncoupled. In general, the coupling strength decreases as k_{jk}^{ab} increases.

4 EXAMPLES AND APPLICATIONS

In this section two examples are considered, namely two spring-coupled rods (as shown in Figure 4.1) and two spring-coupled plates (Figure 4.2). Various analytical expressions and numerical predictions are found using the present, coupled-oscillator theory and compared to those of conventional SEA [1] and other energy-based approaches.

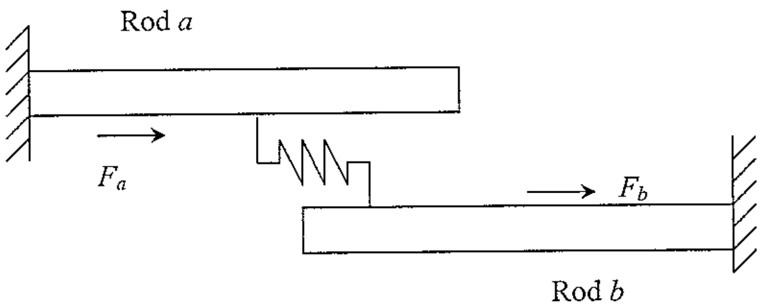

Figure 4.1 Two spring-coupled rods

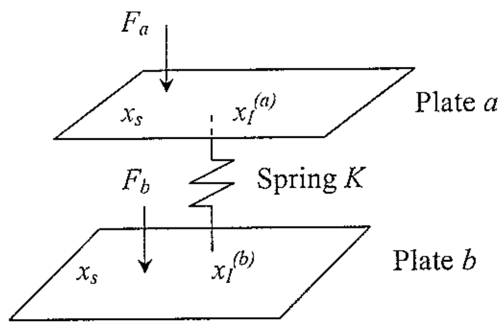

Figure 4.2 Two spring-coupled plates

In section 4.1 some results from the coupled oscillator theory are reviewed and expressions for the coupling loss factors for these systems given. In section 4.2 results of conventional SEA are developed. These are seen to be identical to those of the coupled oscillator theory for the case of weak coupling and weak connection. Section 4.3 presents various exact analyses, while section 4.4 describes an alternative approximate theory based on modal analysis. Finally in section 4.5 some numerical results are given, comparing the various estimates.

4.1 Coupling loss factors from the coupled oscillator theory

Section 3 concerned the power/energy relations for two spring-coupled sets of oscillators. Various assumptions and approximations were made, resulting in two different expressions for the coupling loss factor. In particular, if the coupling power is assumed to be proportional to the *actual* oscillator energy difference then

$$\omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \alpha n_a n_b; \tag{4.1}$$

while if it assumed to be proportional to the *blocked* energy difference then

$$\omega n_a \eta_{ab}^{(b)} = \omega n_b \eta_{ba}^{(b)} = \frac{\alpha n_a n_b}{1 - \left(\frac{n_a}{\Delta_b} + \frac{n_b}{\Delta_a} \right) \alpha} \tag{4.2}$$

In the above, the strength of connection is

$$\kappa^2 = \frac{\overline{\left(k_{jk}^{ab}\right)^2}}{m_a m_b \omega^2} \tag{4.3}$$

while the strength of coupling is

$$\gamma^2 = \frac{\kappa^2}{\Delta_a \Delta_b} \tag{4.4}$$

and

$$\alpha = \frac{\pi \kappa^2}{2\sqrt{1+\gamma^2}} \tag{4.5}$$

In the above equations, the term k_{jk}^{ab} is the coupling stiffness between the *j*th oscillator (in the context of this section the *j*th mode) of subsystem a and the *k*th oscillator (i.e. mode) of subsystem b.

In terms of the conventional SEA expressions, the coupling loss factors can also be written as

$$\eta_{ab}^{(a)} = \frac{1}{\sqrt{1+\gamma^2}} \eta_{ab}^{(c)} \tag{4.6}$$

$$\eta_{ab}^{(b)} = \left[\frac{1}{\sqrt{1+\gamma^2} - \left(\mu_a^{-1} + \mu_b^{-1}\right) \omega n_a \eta_{ab}^{(c)}} \right] \eta_{ab}^{(c)} \tag{4.7}$$

where $\mu_{a,b}$ are the modal overlap factors of subsystems a and b. The 3 expressions for the coupling loss factor are identical in the weak coupling limit $\gamma \to 0$.

4.1.1. Coupled continuous subsystems

In Appendix F is considered the case of two continuous subsystems connected by a spring. The intermodal coupling stiffness between modes *j* and *k* in subsystems *a* and *b*, coupled at points $x_l^{(a)}$ and $x_l^{(b)}$, is

$$k_{jk}^{(ab)} = K \phi_j^{(a)}\left(x_l^{(a)}\right) \phi_k^{(b)}\left(x_l^{(b)}\right) \tag{4.8}$$

where $\phi_j^{(a)}$ and $\phi_k^{(b)}$ are the *j*th and *k*th mode shapes of the subsystems in the absence of the coupling spring. The mode shapes are such that modal mass of the *j*th mode is given by

$$m_j^{(a)} = \int_{(a)} m^{(a)}(x) \phi_j^{(a)2}(x) dx \tag{4.9}$$

where $m^{(a)}(x)$ is the mass density. Hence the strength of connection κ between the mode pair is such that

$$\kappa_{jk}^2 = \frac{K^2 \phi_j^{(a)2}\left(x_I^{(a)}\right)\phi_k^{(b)2}\left(x_I^{(b)}\right)}{m_j^{(a)} m_k^{(b)} \omega^2} \tag{4.10}$$

This is the relevant value for κ^2 if the connection points are known (e.g. for end-coupled rods).

Now consider the case where, in the ensemble of oscillator sets, the coupling points are randomly located over the subsystems (with probability density functions proportional to the mass density). The strength of connection, averaged over all possible interface points in each subsystem, then becomes

$$\overline{\kappa_{jk}^2} = \frac{K^2}{M^{(a)} M^{(b)} \omega^2} \tag{4.11}$$

where

$$M^{(a)} = \int_{(a)} m^{(a)}(x)dx \tag{4.12}$$

is the total mass of subsystem a.

4.1.2. Two rods coupled at randomly selected points

The modal density of a rod is given by $n = L/\pi\sqrt{\rho/E} = L/(c\pi)$, where ρ, E and $c = \sqrt{E/\rho}$ are respectively the mass density, elastic modulus and wave speed of the rod material. If the rods are connected at randomly-selected points then the strength of connection is given in equation (4.11). The coupling loss factors, based on the assumption that the coupling power is proportional to the *actual* energy difference (equation (4.1)) then becomes

$$\omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \left(\frac{1}{\sqrt{1+\gamma^2}}\right)\left(\frac{K^2}{2\pi\omega^2}\frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)}\right) \tag{4.13}$$

where A is the cross-sectional area of the rod. (If the coupling power is assumed to be proportional to the *blocked* energy difference, the coupling loss factors are given by equation (4.2), with α being given by equation (4.5).) In the limit of weak coupling (i.e. $\gamma \to 0$), equation (4.13) becomes

$$\omega n_a \eta_{ab}^{(c)} = \omega n_b \eta_{ba}^{(c)} = \frac{K^2}{2\pi\omega^2}\frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)} \tag{4.14}$$

4.1.3. Two end-coupled rods

If, on the other hand, the two rods are end-coupled, then, at the ends of the rods, the squared mode shapes in equation (4.10) are equal to $2m_{j,k}^{(a,b)}/M^{(a,b)}$. The appropriate value for the strength of connection is now

$$\overline{\kappa_{jk}^2} = \frac{4K^2}{M^{(a)} M^{(b)} \omega^2} \tag{4.15}$$

and the coupling loss factors become

$$\omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \left(\frac{1}{\sqrt{1+\gamma^2}}\right)\left(\frac{2K^2}{\pi\omega^2}\frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)}\right) \tag{4.16}$$

In the limit of weak coupling

$$\omega n_a \eta_{ab}^{(c)} = \omega n_b \eta_{ba}^{(c)} = \frac{2K^2}{\pi\omega^2} \frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)} \tag{4.17}$$

4.1.4. Two spring-coupled plates

The modal density of a plate is

$$n = \frac{A}{4\pi}\sqrt{\frac{m}{B}} \tag{4.18}$$

where A, m and B are respectively the area, mass per unit area and bending stiffness of the plate. Consider the system comprising two spring-coupled plates shown in Figure 4.2. From equations (4.11) and (4.18), together with equations (4.1) to (4.5), it follows that

$$\omega n_a \eta_{ab}^{(a)} = \omega n_b \eta_{ba}^{(a)} = \left(\frac{1}{\sqrt{1+\gamma^2}}\right)\left(\frac{K^2}{32\pi\omega^2} \frac{1}{\sqrt{m^{(a)}B^{(a)}}\sqrt{m^{(b)}B^{(b)}}}\right) \tag{4.19}$$

and in the weak coupling limit

$$\omega n_a \eta_{ab}^{(c)} = \omega n_b \eta_{ba}^{(c)} = \frac{K^2}{32\pi\omega^2} \frac{1}{\sqrt{m^{(a)}B^{(a)}}\sqrt{m^{(b)}B^{(b)}}} \tag{4.20}$$

4.2 Conventional SEA

In Ref. [1], the coupling loss factor for two point-connected subsystems is given by

$$\omega n_a \eta_{ab}^{(c)} = \frac{1}{2\pi}\tau_{ab,\infty} \tag{4.21}$$

where $\tau_{ab,\infty}$ is infinite system power transmission coefficient, given by

$$\tau_{ab,\infty} = \frac{4R_{a,\infty}R_{b,\infty}}{\left|Z_{a,\infty} + Z_{b,\infty}\right|^2} \tag{4.22}$$

where $Z_{a,\infty}$ and $Z_{b,\infty}$ are the input impedance of the two subsystems, assuming that they extend uniformly to infinity, and $R_{a,\infty}$ and $R_{b,\infty}$ are the real parts of these impedances.

4.2.1. Two rods coupled at randomly selected points

For an infinite rod $Z_\infty = R_\infty = 2\rho Ac$. By defining subsystem a as the combination of rod a and the spring, the input impedance of subsystem a then becomes

$$Z_a = \left(-iR_{a,\infty} K/\omega\right)/\left(R_{a,\infty} - i K/\omega\right) \tag{4.23}$$

while that of subsystem b is $Z_b = Z_{b,\infty} = 2\rho_b c_b A_b$. In the case of weak connection, such that $(K/\omega)^2 << \left\{\left|R_{a,\infty}\right|^2, \left|R_{b,\infty}\right|^2\right\}$, equation (4.22) yields

$$\tau_{ab,\infty} \approx \frac{4\left(K^2/\omega^2\right)}{R_{a,\infty}R_{b,\infty}} = \frac{K^2}{\omega^2} \frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)} \tag{4.24}$$

Substituting equation (4.24) into (4.21), equation (4.14) is recovered.

4.2.2. End-coupled rods

In a similar way, the input impedance of a semi-infinite rod excited at it end is given by $Z_\infty = R_\infty = \rho c A$. The power transmission coefficient is

$$\tau_{ab,\infty} = \frac{4Z_{a,\infty}Z_{b,\infty}}{\left(Z_{a,\infty}\right)^2 + \left(\omega Z_{a,\infty}Z_{b,\infty}/K + 1\right)^2} \approx \frac{4K^2}{\omega^2}\frac{1}{\left(\rho_a A_a c_a\right)\left(\rho_b A_b c_b\right)} \tag{4.25}$$

The coupling loss factor given by the conventional SEA approach is then given by equation (4.17) and is, again, 4 times the value for rods coupled at internal points (equation (4.14)).

4.2.3. Two spring-coupled plates

For an infinite plate $Z_\infty = R_\infty = 8\sqrt{mB}$. Now define subsystem a as the combination of plate a and the spring. The input impedance of subsystem a is then given in equation (4.23). With $R_a = \mathrm{Re}\{Z_a\}$, $R_{b,\infty} = Z_{b,\infty} = 8\sqrt{m^{(b)}B^{(b)}}$, and assuming weak connection (i.e. $(K/\omega)^2 << \left\{\left(R_{a,\infty}\right)^2, \left(R_{b,\infty}\right)^2\right\}$), equation (4.22) gives

$$\tau_{ab,\infty} = \frac{4\left(K^2/\omega^2\right)}{R_{a,\infty}R_{b,\infty}} = \frac{K^2}{16\omega^2}\frac{1}{\sqrt{m^{(a)}B^{(a)}}\sqrt{m^{(b)}B^{(b)}}} \tag{4.26}$$

Substituting equation (4.26) into (4.21), results in the expression for the coupling loss factor given in equation (4.20).

In summary, conventional SEA estimates based on wave transmission are identical to the estimates of the coupled oscillator theory in the weak coupling limit.

4.3 Exact wave solution for one-dimensional subsystems: application to two rods

Ref. [6] contains a wave analysis for the response of two, one-dimensional subsystems coupled at their ends. Ensemble averages are taken by assuming that the total phase change a wave experiences as it propagates through the system, when taken $\mathrm{mod}(2\pi)$, is random and uniformly probable. The ensemble average subsystem energies, input power (for rain-on-the-roof) and coupling power were found. In this section these results for two, end-coupled rods are quoted and compared with the results of the coupled oscillator theory.

The coupling loss factor by this wave approach is given by

$$\omega n_a \eta_{ab}^{(w)} = \omega n_b \eta_{ba}^{(w)} = \frac{1}{2\pi}\frac{\tau_{ab,\infty}}{\sqrt{\left(1+\gamma_{ab}^2\right)\left(1+\delta_{ab}^2\right)} - \frac{\tau_{ab,\infty}}{2\pi}\left(\mu_a^{-1} + \mu_b^{-1}\right)} \tag{4.27}$$

(note that in [6] the modal overlaps were defined in terms of the noise bandwidth rather than the half-power bandwidth) where

$$\gamma_{ab}^2 = \frac{\cosh^2\left(\pi\left(\mu_a - \mu_b\right)/2\right)}{\sinh\left(\pi\mu_a\right)\sinh\left(\pi\mu_b\right)}\tau_{ab,\infty} \tag{4.28}$$

$$\delta_{ab}^2 = \frac{\sinh^2\left(\pi\left(\mu_a - \mu_b\right)/2\right)}{\sinh\left(\pi\mu_a\right)\sinh\left(\pi\mu_b\right)}\tau_{ab,\infty} \tag{4.29}$$

For two rods spring-coupled at an end, the transmission coefficient $\tau_{ab,\infty}$ is given by

$$\tau_{ab,\infty} = \frac{4K^2}{\omega^2}\frac{1}{(\rho_a A_a c_a)(\rho_b A_b c_b)} = 2\pi\omega n_a \eta_{ab}^{(c)} \tag{4.30}$$

In terms of the conventional SEA estimate the coupling loss factor predicted by the wave approach is

$$\eta_{ab}^{(w)} = \left[\frac{1}{\sqrt{(1+\gamma_{ab}^2)(1+\delta_{ab}^2)} - (\mu_a^{-1}+\mu_b^{-1})\omega n_a \eta_{ab}^{(c)}}\right]\eta_{ab}^{(c)} \tag{4.31}$$

Both the exact wave analysis and the coupled oscillator theory give the same expression for the coupling loss factor for the case of weak coupling (i.e. the conventional SEA estimate $\eta_{ab}^{(c)}$). Otherwise they differ in some respects.

First, there are two coupling parameters rather than one (γ) in the coupled oscillator results. Secondly, the functional dependence is different for large values of $\mu_{a,b}$ (due to the hyperbolic functions in equations(4.28) and (4.29)). The form of the exact wave result is more similar to the "blocked energy difference" expression of equation (4.7) than to the "actual energy difference" expression of equation (4.6).

If the difference in the modal overlaps is small then $\delta_{ab}^2 \approx 0$. If, furthermore, the modal overlaps are small then

$$\gamma_{ab}^2 \approx \frac{1}{\pi^2 \mu_a \mu_b}\tau_{ab,\infty} \tag{4.32}$$

Noting that for this case

$$\mu_a = n_a \Delta_a = \frac{L_a \Delta_a}{\pi c_a} \tag{4.33}$$

and that $\tau_{ab,\infty}$ and κ^2 are given by equations (4.30) and (4.15), it follows that

$$\gamma_{ab}^2 \approx \frac{\kappa^2}{\Delta_a \Delta_b} = \gamma^2 \tag{4.34}$$

where γ is the strength of coupling arising from the coupled oscillator theory, as given in equation (4.4).

In summary, for weak coupling both coupled oscillator expressions are identical to the result from the exact wave analysis. For strong coupling, if the modal overlaps are small enough then the coupling strength parameter becomes equal to that of the exact wave analysis. The "blocked energy difference" expression of equation (4.7) then asymptotes to that of the wave analysis. Otherwise there are differences in the coupling parameters, although there is the same general dependency of coupling loss factor on modal overlap.

4.4 Numerical estimates from energy distribution models: application to two plates

Apparent coupling loss factors (i.e. numerical estimates of the ensemble average CLFs) can be calculated from energy distribution (ED) models based on modal analysis of a specific realisation of the system. The ED models can be built up either in terms of the global modes of the whole system as described in [8], or in terms of the frequency and space averaged Green functions of the whole system as suggested in [12]. However, either approach can be computationally very expensive, especially for complex built-up systems. Finding analytical expressions for the ensemble statistics is intractable. Thus, while an "exact" numerical solution for a specific system can be found, the resulting ensemble averages are often inferred in an approximate manner, typically by Monte Carlo Simulation (MCS).

Consider two subsystems connected at a point and subjected to rain-on-the-roof excitation with constant power spectral densities S_a and S_b. The time-averaged input powers and energies, after frequency-, space- and ensemble-averaging, (i.e. averages over a broad frequency band, over all possible excitation positions and interface positions, and over an ensemble of subsystem properties), are related by

$$\begin{Bmatrix} \overline{P}_a \\ \overline{P}_b \end{Bmatrix} = \begin{bmatrix} \overline{\langle \mathrm{Re}\{Y_{aa}\}\rangle} & 0 \\ 0 & \overline{\langle \mathrm{Re}\{Y_{bb}\}\rangle} \end{bmatrix} \begin{Bmatrix} S_a \\ S_b \end{Bmatrix} \tag{4.35}$$

$$\begin{Bmatrix} \overline{E}_a \\ \overline{E}_b \end{Bmatrix} = \begin{Bmatrix} 2\overline{T}_a \\ 2\overline{T}_b \end{Bmatrix} = \begin{bmatrix} M_a\overline{\langle |Y_{aa}|^2\rangle} & M_a\overline{\langle |Y_{ab}|^2\rangle} \\ M_b\overline{\langle |Y_{ba}|^2\rangle} & M_b\overline{\langle |Y_{bb}|^2\rangle} \end{bmatrix} \begin{Bmatrix} S_a \\ S_b \end{Bmatrix} \tag{4.36}$$

In the above equations, $\langle \bullet \rangle$ denotes space- and frequency-averages, while $\overline{\bullet}$ represents ensemble averages. Note that the latter are found numerically and are hence approximations. $M_{a,b}$ are the masses of subsystems a and b, and Y mobility. From equations (4.35) and (4.36), an ED model of the coupled system can be formed, i.e.

$$\begin{Bmatrix} \overline{E}_a \\ \overline{E}_b \end{Bmatrix} = \begin{bmatrix} \dfrac{M_a\overline{\langle |Y_{aa}|^2\rangle}}{\overline{\langle \mathrm{Re}\{Y_{aa}\}\rangle}} & \dfrac{M_a\overline{\langle |Y_{ab}|^2\rangle}}{\overline{\langle \mathrm{Re}\{Y_{bb}\}\rangle}} \\ \dfrac{M_b\overline{\langle |Y_{ba}|^2\rangle}}{\overline{\langle \mathrm{Re}\{Y_{aa}\}\rangle}} & \dfrac{M_b\overline{\langle |Y_{bb}|^2\rangle}}{\overline{\langle \mathrm{Re}\{Y_{bb}\}\rangle}} \end{bmatrix} \begin{Bmatrix} \overline{P}_a \\ \overline{P}_b \end{Bmatrix} \tag{4.37}$$

By inverting this matrix, the apparent CLF can be estimated using this mobility approach as

$$\omega n_a \eta_{ab}^{(m)} = \frac{n_a}{M_a} \frac{\overline{\langle |Y_{ba}|^2\rangle}\,\overline{\langle \mathrm{Re}\{Y_{bb}\}\rangle}}{\overline{\langle |Y_{aa}|^2\rangle}\,\overline{\langle |Y_{bb}|^2\rangle} - \overline{\langle |Y_{ab}|^2\rangle}\,\overline{\langle |Y_{ba}|^2\rangle}} \tag{4.38}$$

Alternatively,

$$\omega n_b \eta_{ba}^{(m)} = \frac{n_b}{M_b} \frac{\overline{\langle |Y_{ab}|^2\rangle}\,\overline{\langle \mathrm{Re}\{Y_{aa}\}\rangle}}{\overline{\langle |Y_{aa}|^2\rangle}\,\overline{\langle |Y_{bb}|^2\rangle} - \overline{\langle |Y_{ab}|^2\rangle}\,\overline{\langle |Y_{ba}|^2\rangle}} \tag{4.39}$$

If only subsystem a is excited, the coupling loss factor can be simply estimated as

$$\omega n_a \eta_{ab}^{(m)} = \frac{\omega \eta_b \overline{E}_b}{\overline{E}_a/n_a - \overline{E}_b/n_b} = \frac{\omega \eta_b M_b \overline{\langle |Y_{ba}|^2\rangle}}{\dfrac{M_a}{n_a}\overline{\langle |Y_{aa}|^2\rangle} - \dfrac{M_b}{n_b}\overline{\langle |Y_{ba}|^2\rangle}} \tag{4.40}$$

The above approach might involve approximations and assumptions in determining the mobilities, but otherwise the analysis for a single specific system requires only the assumption of rain-on-the-roof excitation. Such one-off analyses are then averaged, and there are underlying assumptions that the frequency averages for single, specific cases, and numerical averages for a range of systems with specific properties, do indeed converge to the ensemble averages at the centre frequency. Thus, while estimates using this approach can in principle provide accurate numerical estimates, it must be borne in mind that these are, indeed, only numerical estimates. Finally, taking averages over a large sample of specific systems can be very expensive computationally.

4.4.1 A special case: two spring-coupled subsystems

For two specific, spring-coupled subsystems, the mobilities of the system can be expressed in terms of those of the uncoupled subsystems as

$$Y_{aa} = Y_{sr}^{a} - \frac{Y_{rI}^{a} Y_{sI}^{a}}{Y_{II}^{a} + Y_{II}^{b} + i\omega/K} \tag{4.41}$$

$$Y_{bb} = Y_{sr}^{b} - \frac{Y_{rI}^{b} Y_{sI}^{b}}{Y_{II}^{a} + Y_{II}^{b} + i\omega/K} \tag{4.42}$$

$$Y_{ab} = \frac{Y_{rI}^{a} Y_{sI}^{b}}{Y_{II}^{a} + Y_{II}^{b} + i\omega/K} \tag{4.43}$$

$$Y_{ba} = \frac{Y_{rI}^{b} Y_{sI}^{a}}{Y_{II}^{a} + Y_{II}^{b} + i\omega/K} \tag{4.44}$$

where s and r represent the source and response locations and I the interface location. In equations (4.41)-(4.44), the terms on the left-had side (i.e. without the superscripts a or b) correspond to the mobilities when the two subsystems are connected, while those with a superscripts a or b represent mobilities of each subsystem when uncoupled. For weak coupling and if there are many modes of each subsystem, these expressions can be frequency-, space- and ensemble-averaged asymptotically to give

$$\overline{\langle \mathrm{Re}\{Y_{aa}\}\rangle} \approx \frac{\pi n_a}{2M_a}, \quad \overline{\langle \mathrm{Re}\{Y_{bb}\}\rangle} \approx \frac{\pi n_b}{2M_b} \tag{4.45}$$

$$\overline{\langle |Y_{aa}|^2 \rangle} \approx \frac{\pi n_a}{2\eta_a \omega M_a^2}, \quad \overline{\langle |Y_{bb}|^2 \rangle} \approx \frac{\pi n_b}{2\eta_b \omega M_b^2} \tag{4.46}$$

$$\overline{\langle |Y_{ab}|^2 \rangle} = \overline{\langle |Y_{ba}|^2 \rangle} \approx \frac{K^2/\omega^2}{M_a^2 M_b^2}\left(\frac{\pi}{2\eta_a \omega}\right)\left(\frac{\pi}{2\eta_b \omega}\right) n_a n_b \tag{4.47}$$

Substituting equations (4.45)-(4.47) into equations (4.38)-(4.40), and assuming that $\{|Y_{aa}|,|Y_{bb}|\} >> \{|Y_{ab}|,|Y_{ba}|\}$, gives

$$\omega n_a \eta_{ab}^{(m)} = \omega n_b \eta_{ba}^{(m)} \approx \frac{\pi}{2}\kappa^2 n_a n_b \tag{4.48}$$

The asymptotic result given in equations (4.48) reduces to equation (3.31), which then leads to equations (4.14) and (4.20), respectively, for two spring-coupled rods and two spring-coupled plates.

4.5 Numerical examples

4.5.1 Two end-coupled rods

The parameters of the two rods in Figure 4.1 are assumed to be $\rho_a A_a = \rho_b A_b = 1$, $E_a A_a = E_b A_b = 1$, $L_a = \pi\left(1-\sqrt{26}/8\right)$, $L_b = \pi\sqrt{26}/8$. Consequently, the characteristic impedances and modal densities of the two rods are $Z_{a,\infty} = Z_{b,\infty} = 1Nm/s$, $n_a = 0.3626 rad^{-1}$ and $n_b = 1 - n_a = 0.6374 rad^{-1}$. The centre frequency remains constant at $\omega = 100 rad/s$ and the various estimates of coupling loss factor are shown as functions of damping loss factor η. Various values of the coupling spring stiffness are chosen, giving a range of different connection strengths.

Figure 4.3 shows the strength of coupling parameter γ^2 for different spring stiffness as a function of damping. It is seen that the coupling strength decreases as damping increases. The strong and weak coupling regions correspond to result of $\gamma^2 > 1$ and $\gamma^2 < 1$ respectively.

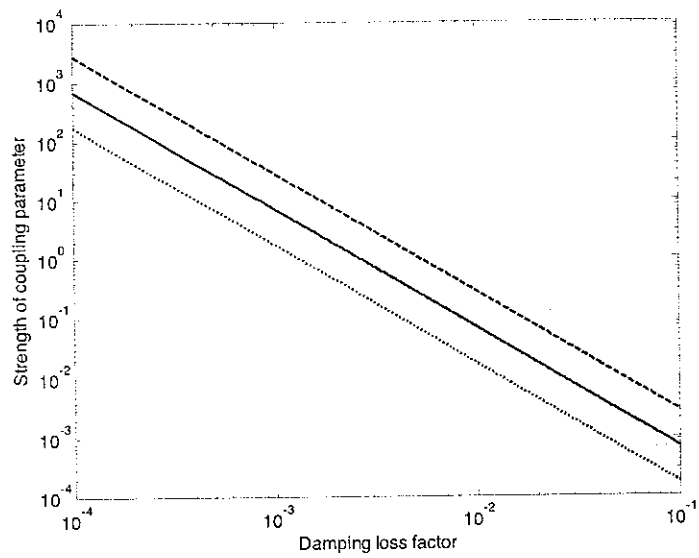

Figure 4.3 Coupling strength parameter of two spring-coupled rods for various K: .... K=10; ——— K=20; – – – K=40.

Figure 4.4 shows the various estimates of coupling los factor, namely the conventional estimate $\eta_{ab}^{(c)}$, the exact wave solution $\eta_{ab}^{(w)}$ and the two coupled oscillator results $\eta_{ab}^{(a)}$ and $\eta_{ab}^{(b)}$. The conventional estimate is of course independent of damping, the other estimates increasing with increasing damping when the coupling is strong (i.e. for low η), and asymptoting to the conventional estimate for weak coupling (i.e. as $\eta \to \infty$). For this case $\eta_{ab}^{(b)}$ is very close to (but is greater than) the exact wave estimate, while $\eta_{ab}^{(a)}$ is significantly smaller than $\eta_{ab}^{(w)}$ when the coupling is not sufficiently weak.

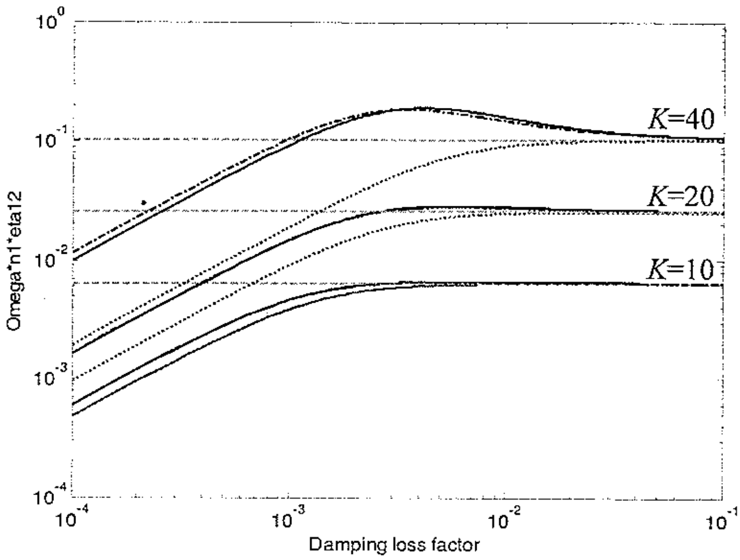

Figure 4.4 Coupling loss factors of two spring-coupled rods, K = 10, 20, 40 N/m: - - - $\eta_{ab}^{(c)}$; ……… $\eta_{ab}^{(a)}$; – · – $\eta_{ab}^{(b)}$; —— $\eta_{ab}^{(w)}$.

4.5.2 Two spring-coupled plates

Numerical results for the system comprising two coupled plates are shown in Figures 4.5 and 4.6. The material properties of both plates are $\rho_{a,b} = 10^3\text{kg/m}^3$, $E_{a,b} = 10^8\text{N/m}^2$ and Poisson's ratio $\upsilon = 0.38$, while the plate dimensions are $0.560 \times 0.405\text{m}$ and $0.665 \times 0.475\text{m}$, respectively, both with a thickness of 0.001 m. These parameters correspond to plate modal densities $n_a = 0.18 rad^{-1}$ and $n_b = 0.25 rad^{-1}$ and characteristic impedances $Z_{a,\infty} = Z_{b,\infty} = 0.79 N/m/s$. The edges of the plates are simply supported and the modes are found from well-known analytical expressions.

In the numerical results the centre frequency remains constant and the various estimates of coupling loss factor are shown as functions of damping loss factor η, both plates having the same loss factor. The centre frequency is taken as $\omega = 1000 rad/s$. Various values for the coupling spring stiffness are chosen, giving a range of different connection strengths $\left(\kappa^2 \approx 0.035,\ 0.14,\ 0.56\right)$, weak coupling holding if $\kappa^2/\omega^2 << \eta_a \eta_b$.

Figure 4.5 shows 240 different numerical estimates of ACLFs, each with different interface locations. (12 points on plate 1 and 20 points on plate 2 were randomly chosen, avoiding the regions close to the plate edges.) The frequency averages are taken over a band of width $100 rad/s$, which contains approximately 18 modes of plate a and 25 modes of plate b. The computational cost for getting such a solution, of course, is very high. Clearly, from Figure 4.5, there is significant variability in ACLF estimates which reduces as the damping increases.

Figure 4.6 shows the CLF estimated from the coupled oscillator theory (both blocked and actual energy difference expressions being shown) together with the conventional SEA estimate and the estimate found by averaging the powers and energies found from the Monte Carlo simulations (MCS). While the conventional SEA estimate $\eta_{ab}^{(c)}$ is of course independent of frequency, both the numerical and analytical expressions increase with damping for low damping, then asymptote to the conventional estimate (or close to it) as the damping increases sufficiently so that the coupling becomes weak. The CLF increases as K increases (and hence, of course, as $\tau_{ab,\infty}$ increases). The agreement between the coupled oscillator estimates and the MCS results is better for weaker connection. The computational cost of the analytical estimates is of course trivial. For this case $\eta_{ab}^{(a)}$ agrees better with the results of the MCS than does $\eta_{ab}^{(b)}$

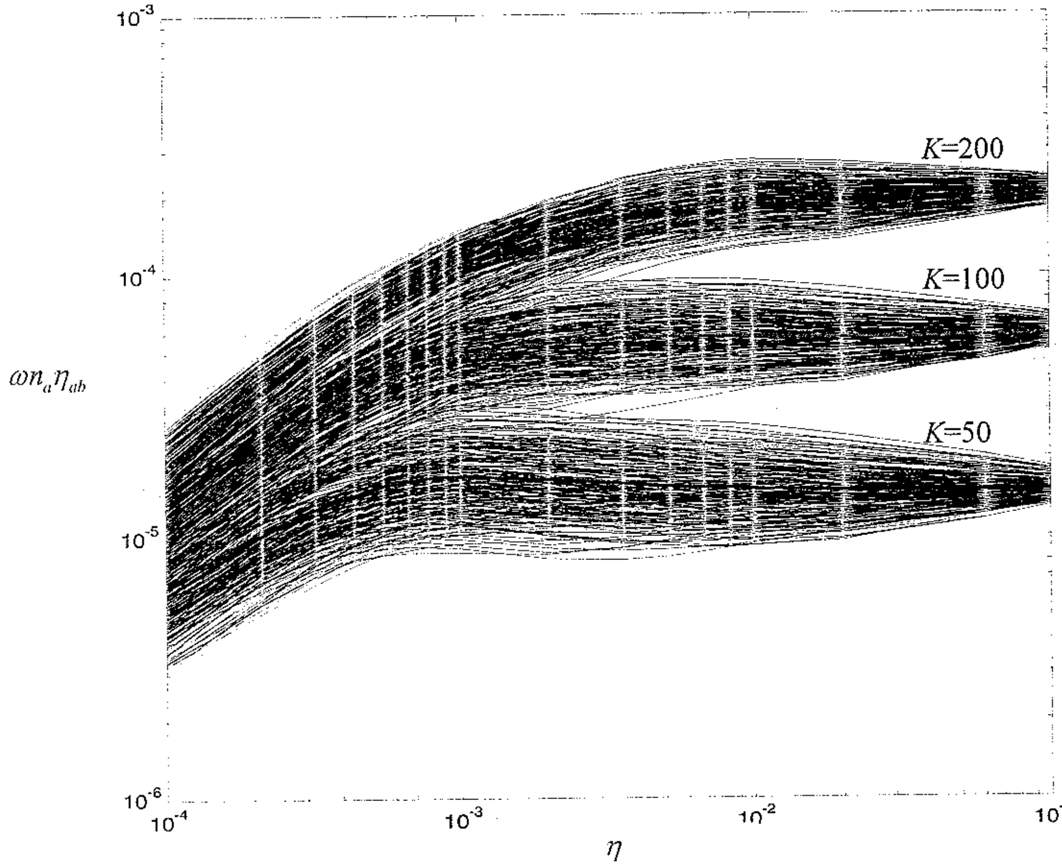

Figure 4.5 Apparent coupling loss factors from numerical estimates and average of 240 samples: —— K=50 N/m; - - - K=100 N/m, ········ K=200N/m.

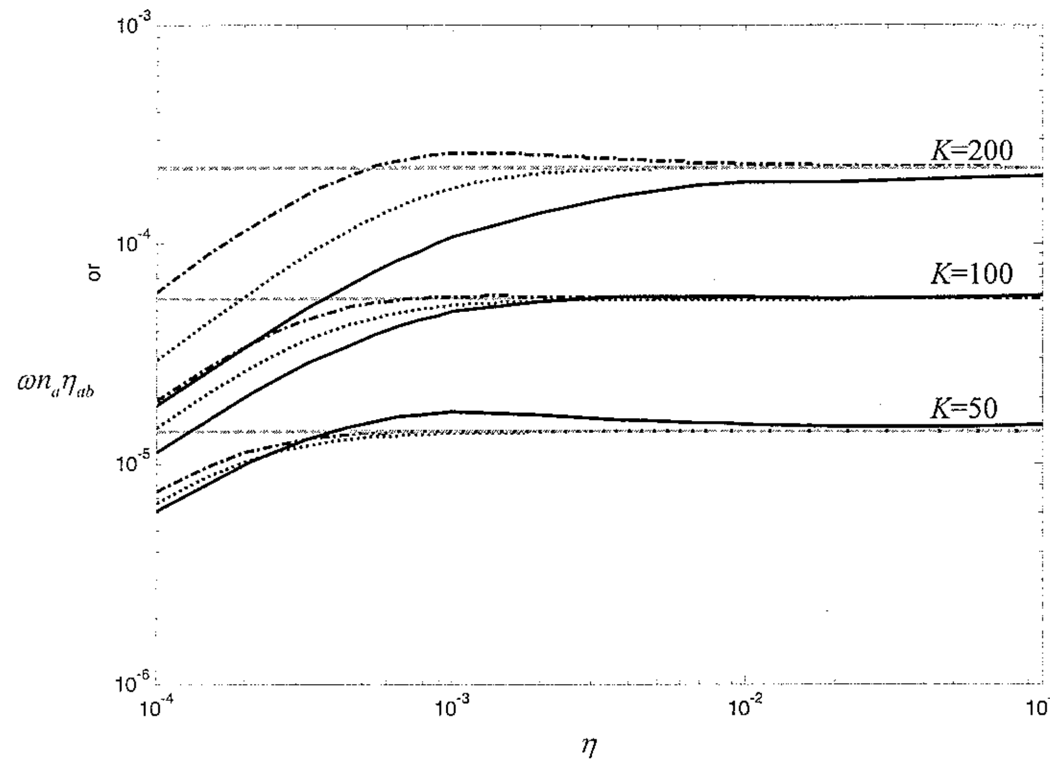

Figure 4.6 Coupling loss factors of two spring-coupled plates, $K = 50$, 100, 200 N/m: — — — $\eta_{ab}^{(c)}$; ········ $\eta_{ab}^{(a)}$; — · — $\eta_{ab}^{(b)}$; —— $\eta_{ab}^{(mcs)}$.

4.6 Concluding remarks

In this section coupled oscillator result were compared with conventional SEA estimates, an exact wave result and numerical (Monte Carlo) results, with both analytical and numerical comparisons being made. Conventional SEA estimates based on wave transmission are identical to the estimates of the coupled oscillator theory in the weak coupling limit. For weak coupling both coupled oscillator expressions are identical to the result from the exact wave analysis. For strong coupling, if the modal overlaps and/or system damping loss factors are small enough then the coupling strength parameter becomes equal to that of the exact wave analysis. The "blocked energy difference" expression of equation (4.7) then asymptotes to that of the wave analysis. Otherwise there are differences in the coupling parameters, although there is the same general dependency of coupling loss factor on modal overlap/damping loss factor. These conclusions were illustrated with numerical examples for the cases of two coupled rods and two coupled plates. The coupled oscillator expressions are seen to give good accuracy if the connection strength is weak enough, or if the coupling strength is weak.

5 CONCLUDING REMARKS

This memorandum concerned the SEA of coupled oscillators and coupled sets of oscillators. The properties of the oscillators are uncertain, and a statistical description is required, the system being drawn from an ensemble of similar systems whose properties are random. Consequently estimates of the ensemble average behaviour are required.

While the earliest studies in SEA concerned coupled oscillators, this work departs somewhat from these studies in that various assumptions are removed or relaxed. Consequently some of the results, conclusions and observation differ from those of conventional SEA.

First, the SEA of two coupled oscillators were considered. The oscillator properties were assumed to be random and ensemble averages found. These results were then extended to two coupled sets of oscillators (i.e. coupled multi-modal subsystems). While attention was focussed on spring-coupled sets, other forms of conservative coupling were considered: the conclusions are the same, the interaction of two oscillators depending on the separation of their natural frequencies and a term that determines the strength by which they are connected. The other cases considered were general, conservative coupling and spring-like coupling which is characteristic of the form of coupling that arises from fixed interface component mode synthesis. The application of the coupled oscillator theory to coupled continuous subsystems was described.

In the analysis one major assumption was removed: full account was taken of the correlation between the coupling parameters for an oscillator pair and their specific energies. For sets of oscillators, it was also assumed that the interaction of each oscillator pair is not affected by the presence of a third oscillator. Another assumption concerned the coupling power between each oscillator pair. Two approaches were suggested: the first is that the coupling power between each oscillator pair is proportional to their actual energies; the second is that the coupling power between each oscillator pair is proportional to their "blocked" energies. Very similar, but different, expressions for the ensemble averages of coupling power, energy response and coupling loss factor (CLF) are found. These become identical in the weak coupling limit.

Various conclusions were drawn about the qualitative and quantitative features of the behaviour under broadband excitation, some of which differ from those which are commonly assumed within SEA. It was seen that, for two oscillators, the coupling power or coupling loss factor can be written in terms of the "strength of connection" between the oscillators and a term involving their bandwidths and the separation of their natural frequencies. Equipartition of energy does not occur as damping tends to zero, except in the case where the uncoupled oscillators have identical natural frequencies. For coupled sets of oscillators the coupling loss factor depends on damping: it is proportional to damping at low damping and is independent of damping in the high damping, weak coupling limit. A parameter which describes the strength of coupling is identified: this depends on both the damping and a further parameter which describes the strength of connection.

Finally, applications and numerical examples were discussed. The examples of two spring-coupled rods and two spring-coupled plates were considered. Comparisons were made with the results of conventional SEA, an exact wave analysis for one-dimensional subsystems and numerical Monte Carlo simulations using an energy distribution approach. It was seen that conventional SEA estimates based on wave transmission are identical to the estimates of the coupled oscillator theory in the weak coupling limit. It was also seen that both coupled oscillator expressions are identical to the result from the exact wave analysis for weak coupling. For strong coupling, if the modal overlaps are small enough then the coupling strength parameter becomes equal to that of the exact wave analysis. The "blocked energy difference" expression then asymptotes to that of the wave analysis. Otherwise there are differences in the coupling parameters, although there is the same general dependency of coupling loss factor on modal overlap. Numerical examples showed good agreement between the coupled oscillator theory and Monte Carlo simulations for weak coupling or for weak connection: the computational cost of the former is of course negligible.

In summary, the coupled oscillator theory would seem to give accurate estimates of the coupling loss factors for either weak coupling or weak connection.

There were two motivations that prompted this work. The first was to re-examine the underlying physics of the interaction of oscillator sets, since the conventional SEA approaches lead to conclusions which are contradictory to results found from wave and finite element analysis (FEA), especially for stronger coupling. There was thus the desire to reconcile these different observations of the same system. The resulting conclusions relate to the strength of coupling, and a second factor, the strength of connection.

The second motivation was to lay the basis of a technique for estimating SEA parameters from discrete, FEA subsystem models, without the need to solve the global eigenvalue problem. Each subsystem can be modelled using FEA and its uncoupled modes determined. When two subsystems are joined these modal sets interact. Robust estimates of the coupling loss factors (CLFs) (i.e. estimates which are insensitive to the exact subsystem properties chosen) can be determined by analytically ensemble-averaging along the lines described here. practical applications of this approach are likely to be aimed at weak coupling cases (for parameter estimation), as well as strong coupling cases. This work is on-going, as is work to extend the results to estimate variability in the response of individual systems.

Does strong coupling matter? The opinion of the first author, at least, is that, yes, it is important in developing an understanding of the underlying physical phenomena that govern the interaction of uncertain dynamic systems. But does it matter in practical engineering applications? Probably no – if the coupling is strong, the difference between the actual energy, which is the quantity of interest in practice, and the energy predicted under an (erroneous) assumption of weak coupling is likely to be small. Indeed, assuming (erroneously) that the coupling is weak may well serve to make the model of the system better conditioned. Thus the second motivation, that of providing a systematic, FE-based approach to developing SEA-like models of complex systems without the need for solving the global eigenvalue problem – especially in the weak coupling limit – is perhaps the more valuable in a practical sense.

6 ACKNOWLEDGEMENTS

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APPENDIX A TWO SPRING-COUPLED OSCILLATORS UNDER HARMONIC EXCITATION

Consider the two-oscillator system shown in Figure 2.1. The equations of motion of the two oscillators are

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 = f_1 + k(x_2 - x_1) \quad \text{(A1)}$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2x_2 = f_2 + k(x_1 - x_2) \quad \text{(A2)}$$

Suppose that the oscillators are subjected to harmonic forces $f_1(t) = F_1e^{j\omega t}$ and $f_2(t) = F_2e^{j\omega t}$. Assuming time-harmonic response of the form $X_{1,2}e^{j\omega t}$ and suppressing the time dependence $e^{j\omega t}$, we find

$$\left(-\omega^2 m_1 + j\omega c_1 + k_1\right)X_1 = F_1 + k(X_2 - X_1) \quad \text{(A3)}$$

$$\left(-\omega^2 m_2 + j\omega c_2 + k_2\right)X_2 = F_2 + k(X_1 - X_2) \quad \text{(A4)}$$

Solving equations (A3) and (A4) gives

$$X_1 = \frac{1}{m_1m_2D(\omega)}\left(-m_2\left(\omega^2 - \frac{j\omega c_2}{m_2} - \frac{k_2+k}{m_2}\right)F_1 + kF_2\right) \quad \text{(A5)}$$

$$X_2 = \frac{1}{m_1m_2D(\omega)}\left(-m_1\left(\omega^2 - \frac{j\omega c_1}{m_1} - \frac{k_1+k}{m_1}\right)F_2 + kF_1\right) \quad \text{(A6)}$$

where

$$\begin{aligned} D(\omega) = (j\omega)^4 + (j\omega)^3\left(\frac{c_1}{m_1} + \frac{c_2}{m_2}\right) + (j\omega)^2\left(\frac{k_1+k}{m_1} + \frac{k_2+k}{m_2} + \frac{c_1c_2}{m_1m_2}\right) \\ + (j\omega)\frac{(k_1+k)c_2 + (k_2+k)c_1}{m_1m_2} + \frac{k_1k_2 + k(k_1+k_2)}{m_1m_2} \end{aligned} \quad \text{(A7)}$$

Let

$$\omega_1^2 = \frac{k_1+k}{m_1},\ \omega_2^2 = \frac{k_2+k}{m_2} \quad \text{(A8), (A9)}$$

$$\Delta_1 = \frac{c_1}{m_1},\ \Delta_2 = \frac{c_2}{m_2} \quad \text{(A10), (A11)}$$

Here $\omega_{1,2}$ are the "blocked" natural frequencies, i.e. the natural frequencies when one oscillator is fixed, while $\Delta_{1,2}$ are the half-power bandwidths. Equations (A5)-(A7) can be re-written as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \mathbf{H}(\omega)\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \text{(A12)}$$

where the frequency response matrix

$$\mathbf{H}(\omega)=\frac{1}{m_1 m_2 D(\omega)}\begin{bmatrix} m_2\left(\omega_2^2-\omega^2+j\omega\Delta_2\right) & k \\ k & m_1\left(\omega_1^2-\omega^2+j\omega\Delta_1\right)\end{bmatrix} \tag{A13}$$

and

$$\begin{aligned} D(\omega)&=(j\omega)^4+(j\omega)^3(\Delta_1+\Delta_2)+(j\omega)^2\left(\omega_1^2+\omega_2^2+\Delta_1\Delta_2\right) \\ &+(j\omega)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\left(\omega_1^2\omega_2^2-\frac{k^2}{m_1 m_2}\right)\end{aligned} \tag{A14}$$

The time-averaged kinetic energies of the oscillators are

$$K_1'=\frac{1}{4}m_1\omega^2|X_1|^2 \tag{A15}$$

$$K_2'=\frac{1}{4}m_2\omega^2|X_2|^2 \tag{A16}$$

Substituting equations (A12) and (A14) into (A15) and (A16) gives

$$\begin{aligned} K_1'=\frac{1}{4}\frac{m_1\omega^2}{m_1^2 m_2^2|D(\omega)|^2}&\Big\{m_2^2\left[\left(\omega^2-\omega_2^2\right)^2+\Delta_2^2\omega^2\right]|F_1|^2+k^2|F_2|^2 \\ &-2km_2\left[\left(\omega^2-\omega_2^2\right)\mathrm{Re}\left\{F_1F_2^*\right\}-\Delta_2\omega\,\mathrm{Re}\left\{jF_1F_2^*\right\}\right]\Big\}\end{aligned} \tag{A17}$$

$$\begin{aligned} K_2'=\frac{1}{4}\frac{m_2\omega^2}{m_1^2 m_2^2|D(\omega)|^2}&\Big\{m_1^2\left[\left(\omega^2-\omega_1^2\right)^2+\Delta_1^2\omega^2\right]|F_2|^2+k^2|F_1|^2 \\ &-2km_1\left[\left(\omega^2-\omega_1^2\right)\mathrm{Re}\left\{F_2F_1^*\right\}-\Delta_1\omega\,\mathrm{Re}\left\{jF_2F_1^*\right\}\right]\Big\}\end{aligned} \tag{A18}$$

The last terms in these expressions depend on the relative phase between the two forces.

Note that some of the potential energy is stored in the coupling spring: consequently this energy cannot be unambiguously attributed to one or other oscillator, so that the oscillator potential and total energies cannot be uniquely defined.

The time-averaged input powers can be written as

$$P_1'=\frac{1}{2}\mathrm{Re}\left\{F_1^*\left(j\omega X_1\right)\right\} \tag{A19}$$

$$P_2'=\frac{1}{2}\mathrm{Re}\left\{F_2^*\left(j\omega X_2\right)\right\} \tag{A20}$$

and, using (A12) and (A14), these become

$$\begin{aligned} P_1'=&\frac{|F_1|^2\Delta_1}{2m_1|D(\omega)|^2}\omega^2\left[\left(\omega^2-\omega_2^2\right)^2+\omega^2\Delta_2^2+\frac{\Delta_2k^2}{\Delta_1m_1m_2}\right] \\ &+\frac{\omega}{2m_1m_2|D(\omega)|^2}\left[\gamma_1\,\mathrm{Re}\left\{F_1^*F_2\right\}+\gamma_2\,\mathrm{Im}\left\{F_1^*F_2\right\}\right]\end{aligned} \tag{A21}$$

$$P_2' = \frac{|F_2|^2 \Delta_2}{2m_2 |D(\omega)|^2} \omega^2 \left[\left(\omega^2 - \omega_1^2\right)^2 + \omega^2 \Delta_1^2 + \frac{\Delta_1 k^2}{\Delta_2 m_1 m_2} \right] + \frac{\omega}{2m_1 m_2 |D(\omega)|^2} \left[\gamma_1 \operatorname{Re}\left\{F_2^* F_1\right\} + \gamma_2 \operatorname{Im}\left\{F_2^* F_1\right\} \right] \quad \text{(A22)}$$

where

$$\gamma_1 = k\omega \left[\left(\omega_2^2 \Delta_1 + \omega_1^2 \Delta_2\right) - \omega^2 \left(\Delta_1 + \Delta_2\right) \right] \quad \text{(A23)}$$

$$\gamma_2 = -k \left[\omega^4 - \omega^2 \left(\omega_1^2 + \omega_2^2 + \Delta_1 \Delta_2\right) + \left(\omega_1^2 \omega_2^2 - \frac{k^2}{m_1 m_2} \right) \right] \quad \text{(A24)}$$

The force in the coupling spring is

$$F = k\left(X_1 - X_2\right) \quad \text{(A25)}$$

Therefore the net energy flow through the coupling element is

$$P_{12}' = \frac{1}{2} \operatorname{Re}\left\{F\left(j\omega X_2\right)^*\right\} = \frac{k\omega}{2} \operatorname{Re}\left\{-jX_1 X_2^*\right\} \quad \text{(A26)}$$

$$P_{21}' = \frac{1}{2} \operatorname{Re}\left\{-F\left(j\omega X_1\right)^*\right\} = \frac{k\omega}{2} \operatorname{Re}\left\{-jX_2 X_1^*\right\} \quad \text{(A27)}$$

It can be seen that

$$P_{12}' = -P_{21}' \quad \text{(A28)}$$

This of course is expected because no power is lost in the coupling element. In terms of the applied forces the transmitted power is

$$P_{12}' = \frac{k^2 \omega^2}{m_1^2 m_2^2 |D(\omega)|^2} \left[\frac{1}{2} m_2 \Delta_2 |F_1|^2 - \frac{1}{2} m_1 \Delta_1 |F_2|^2 \right] + \frac{k\omega}{m_1^2 m_2^2 |D(\omega)|^2} \operatorname{Re}\left\{\alpha_1 F_1 F_2^* + \alpha_2 F_2 F_1^*\right\} \quad \text{(A29)}$$

where α_1 and α_2 are given by

$$\alpha_1 = \frac{m_1 m_2}{2} \omega \left[\left(\omega^2 - \omega_2^2\right) \Delta_1 + \left(\omega^2 - \omega_1^2\right) \Delta_2 \right] - j \frac{m_1 m_2}{2} \left[\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) + \Delta_1 \Delta_2 \omega^2 \right] \quad \text{(A30)}$$

$$\alpha_2 = -jk^2 \quad \text{(A31)}$$

In the limiting case $k \to 0$,

$$X_1 = \frac{F_1}{m_1 \left(\omega_1^2 - \omega^2 + j\omega\Delta_1\right)} \quad \text{(A32)}$$

$$X_2 = \frac{F_2}{m_2\left(\omega_2^2 - \omega^2 + j\omega\Delta_2\right)} \quad \text{(A33)}$$

$$P_1' = \Delta_1 E_1' = \Delta_1 \frac{\omega^2 |F_1|^2}{2m_1\left[\left(\omega^2 - \omega_1^2\right)^2 + \omega^2\Delta_2^2\right]} \quad \text{(A34)}$$

$$P_2' = \Delta_2 E_2' = \Delta_2 \frac{\omega^2 |F_2|^2}{2m_2\left[\left(\omega^2 - \omega_2^2\right)^2 + \omega^2\Delta_2^2\right]} \quad \text{(A35)}$$

$$P_{12}' = P_{21}' = 0 \quad \text{(A36)}$$

where

$$\omega_1^2 = \frac{k_1}{m_1},\ \omega_2^2 = \frac{k_2}{m_2} \quad \text{(A37), (A38)}$$

Equations (A32)-(A38) correspond to the cases where the two oscillators are physically uncoupled from each.

In much of the analysis, interest concerns the case where the excitations are random, broadband and uncorrelated. In such a situation the excitations are specified by their power spectral densities $S_{f_1}(\omega)$ and $S_{f_2}(\omega)$, while the cross-spectral densities $S_{f_1f_2}(\omega) = S_{f_2f_1}(\omega) = 0$. The spectral densities of the kinetic energies, input powers and coupling power are then from the above equations

$$S_{K_1}(\omega) = \frac{1}{4}\frac{m_1\omega^2}{m_1^2 m_2^2 |D(\omega)|^2}\left\{m_2^2\left[\left(\omega^2 - \omega_2^2\right)^2 + \Delta_2^2\omega^2\right]S_{f_1}(\omega) + k^2 S_{f_2}(\omega)\right\} \quad \text{(A39)}$$

$$S_{K_2}(\omega) = \frac{1}{4}\frac{m_2\omega^2}{m_1^2 m_2^2 |D(\omega)|^2}\left\{k^2 S_{f_1}(\omega) + m_1^2\left[\left(\omega^2 - \omega_1^2\right)^2 + \Delta_1^2\omega^2\right]S_{f_2}(\omega)\right\} \quad \text{(A40)}$$

$$S_{P_1}(\omega) = \frac{1}{2}\frac{\Delta_1}{m_1 |D(\omega)|^2}\omega^2\left[\left(\omega^2 - \omega_2^2\right)^2 + \omega^2\Delta_2^2 + \frac{\Delta_2 k^2}{\Delta_1 m_1 m_2}\right]S_{f_1}(\omega) \quad \text{(A41)}$$

$$S_{P_2}(\omega) = \frac{1}{2}\frac{\Delta_2}{m_2 |D(\omega)|^2}\omega^2\left[\left(\omega^2 - \omega_1^2\right)^2 + \omega^2\Delta_1^2 + \frac{\Delta_1 k^2}{\Delta_2 m_1 m_2}\right]S_{f_2}(\omega) \quad \text{(A42)}$$

$$S_{P_{12}}(\omega) = \frac{1}{2}\frac{k^2\omega^2}{m_1^2 m_2^2 |D(\omega)|^2}\left[m_2\Delta_2 S_{f_1}(\omega) - m_1\Delta_1 S_{f_2}(\omega)\right] \quad \text{(A43)}$$

APPENDIX B ENSEMBLE AVERAGE OF THE COUPLING TERM β

In equation (2.13), the coupling term β is given by

$$\beta=\left(\frac{k^2}{m_1m_2}\right)\frac{(\Delta_1+\Delta_2)}{(\omega_1^2-\omega_2^2)^2+(\Delta_1+\Delta_2)(\omega_2^2\Delta_1+\omega_1^2\Delta_2)} \tag{B1}$$

In the original approaches to SEA [1-5] the coupling loss factor is found by ensemble averaging β rather than, strictly correctly, ensemble averaging the powers and energies and *then* finding the ratio of coupling power and energy difference. If the ensemble is defined as in section 2.2 then the ensemble average of β is

$$\mathrm{E}[\beta]=\frac{\pi\kappa^2}{2\Omega} \tag{B2}$$

where the strength of connection (sometimes erroneously referred to as the strength of coupling)

$$\kappa^2=\frac{k^2}{m_1m_2\omega^2} \tag{B3}$$

and where ω is the centre frequency. This ensemble average is independent of damping. If the coupling loss factor is found in terms of this average it is equal to the true coupling loss factor in the weak coupling, high damping limit.

APPENDIX C GENERAL CONSERVATIVE COUPLING

In Ref. [5], the power balance and energy relations for two general, conservatively coupled oscillators as shown in Figure C.1 were investigated. As before, $m_{1,2}$, $c_{1,2}$ and $k_{1,2}$ are the mass, damping and stiffness of the corresponding oscillator, and m, k and g the mass, stiffness and gyroscopic constant of the coupling element. In the following, expressions for the power and energy responses are found along the lines followed in Appendix A for spring coupling. The results are identical to those found in Ref. [5]. Further expressions are then developed for the SEA power and energy relations.

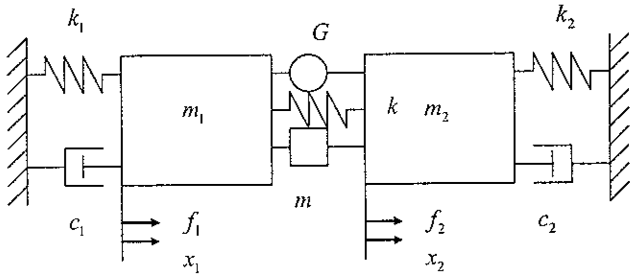

Figure C.1. Two general, conservatively coupled oscillators

The equations of motion of the two-oscillator system, when subjected to forces $f_{1,2}(t)$, can be written as

$$\left(m_1+\frac{m}{4}\right)\ddot{x}_1+c_1\dot{x}_1+\left(k_1+k\right)x_1+\frac{m}{4}\ddot{x}_2-g\dot{x}_2-kx_2=f_1(t) \qquad \text{(C1)}$$

$$\left(m_2+\frac{m}{4}\right)\ddot{x}_2+c_2\dot{x}_2+\left(k_2+k\right)x_2+\frac{m}{4}\ddot{x}_1+g\dot{x}_1-kx_1=f_2(t) \qquad \text{(C2)}$$

The instantaneous input and transmitted powers are $f_1\dot{x}_1$, $f_2\dot{x}_2$ and $k(x_1-x_2)\dot{x}_1-g\dot{x}_2\dot{x}_1+\frac{m}{4}(\ddot{x}_1+\ddot{x}_2)\dot{x}_1$, respectively. The total energy of each oscillator cannot be defined unambiguously because of the energy associated with the coupling elements. The energy may be defined in several reasonable ways, depending on how the energy in the coupling element is taken into account, and there are no substantial consequences of the definitions chosen. Here, the kinetic and potential energies of oscillator 1 are respectively defined as

$$T_1=\frac{1}{2}\left(m_1+\frac{m}{4}\right)\dot{x}_1^2 \qquad \text{(C3)}$$

and

$$V_1=\frac{1}{2}\left(k_1+k\right)x_1^2 \qquad \text{(C4)}$$

Now the following parameters are defined:

$$\omega_{1,2}^2 = \frac{k_{1,2} + k}{m_{1,2} + m/4} \tag{C5}$$

$$\Delta_{1,2} = \frac{c_{1,2}}{m_{1,2} + m/4} = \eta_{1,2}\omega_{1,2} \tag{C6}$$

$$\hat{\gamma} = \frac{g}{\sqrt{(m_1 + m/4)(m_2 + m/4)}} \tag{C7}$$

$$\lambda^2 = \frac{m_1 + m/4}{m_2 + m/4} \tag{C8}$$

$$\mu = \frac{m/4}{\sqrt{(m_1 + m/4)(m_2 + m/4)}} \tag{C9}$$

$$\hat{\kappa} = \frac{k}{\sqrt{(m_1 + m/4)(m_2 + m/4)}} \tag{C10}$$

Equations (C1)-(C2) then become

$$\ddot{x}_1 + \Delta_1\dot{x}_1 + \omega_1^2 x_1 + \frac{1}{\lambda}\left(\mu\ddot{x}_2 - \hat{\gamma}\dot{x}_2 - \hat{\kappa}x_2\right) = \frac{1}{m_1 + m/4} f_1(t) \tag{C11}$$

$$\ddot{x}_2 + \Delta_2\dot{x}_2 + \omega_2^2 x_2 + \lambda\left(\mu\ddot{x}_1 + \hat{\gamma}\dot{x}_1 - \hat{\kappa}x_1\right) = \frac{1}{m_2 + m/4} f_2(t) \tag{C12}$$

For time-harmonic excitations $F_{1,2}e^{j\omega t}$ the responses $X_{1,2}e^{j\omega t}$ are

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \mathbf{H}(\omega)\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{C13}$$

where the frequency response functions are

$$H_{11}(\omega) = \left(\frac{1}{m_1 + m/4}\right)\left[\frac{-\omega^2 + j\omega\Delta_2 + \omega_2^2}{D(\omega)}\right] \tag{C14}$$

$$H_{22}(\omega) = \left(\frac{1}{m_2 + m/4}\right)\left[\frac{-\omega^2 + j\omega\Delta_1 + \omega_1^2}{D(\omega)}\right] \tag{C15}$$

$$H_{12}(\omega) = \frac{1}{\lambda}\left(\frac{1}{m_2 + m/4}\right)\left[\frac{\omega^2\mu + j\omega\hat{\gamma} + \hat{\kappa}}{D(\omega)}\right] \tag{C16}$$

$$H_{21}(\omega)=\left(\frac{1}{m_1+m/4}\right)\left[\frac{\omega^2\mu-j\omega\hat{\gamma}+\hat{\kappa}}{D(\omega)}\right] \tag{C17}$$

$$\begin{aligned}D(\omega)=&(j\omega)^4\left(1-\mu^2\right)+(j\omega)^3\left(\Delta_1+\Delta_2\right)+(j\omega)^2\left(\omega_1^2+\omega_2^2+\Delta_1\Delta_2+\hat{\gamma}^2+2\mu\hat{\kappa}\right)\\&+(j\omega)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\left(\omega_1^2\omega_2^2-\hat{\kappa}^2\right)\end{aligned} \tag{C18}$$

Now suppose that the forces $f_{1,2}(t)$ are statistically independent, stationary and random with constant spectral densities S_{f_1} and S_{f_2} over some excitation frequency band B. The spectral densities of the kinetic and potential energies and input and coupling powers then follow in the same manner as equations (A38) to (A42) in terms of the spectral densities of the forces and the frequency responses. The total powers and energies, found by integrating over the frequency band of excitation (c.f. section 2.1 for spring-coupled oscillators) are then finally

$$P_1=\frac{\pi S_{f_1}}{m_1+m/4},\ P_2=\frac{\pi S_{f_2}}{m_2+m/4} \tag{C19}$$

$$P_{12}=\frac{q}{\left(1-\mu^2\right)Q}\left[\frac{\pi S_{f_1}}{\left(m_1+m/4\right)\Delta_1}-\frac{\pi S_{f_2}}{\left(m_2+m/4\right)\Delta_2}\right] \tag{C20}$$

$$T_1=\frac{1}{2\left(1-\mu^2\right)\Delta_1\Delta_2 Q}\left[\frac{\pi S_{f_1}}{m_1+m/4}p-\frac{\pi S_{f_2}}{m_2+m/4}q\right] \tag{C21}$$

$$V_1=\frac{\omega_1^2}{2\left(\omega_1^2\omega_2^2-\hat{\kappa}^2\right)\Delta_1\Delta_2 Q}\left[\frac{\pi S_{f_1}}{m_1+m/4}s+\frac{\pi S_{f_2}}{m_2+m/4}t\right] \tag{C22}$$

where

$$\begin{aligned}Q=&\left[\left(\omega_1^2-\omega_2^2\right)^2+\left(\Delta_1+\Delta_2\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]+\frac{\mu^2}{\Delta_1\Delta_2}\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)^2\\&+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\left(\frac{1}{\Delta_1}+\frac{1}{\Delta_2}\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\frac{\left(\Delta_1+\Delta_2\right)^2}{\Delta_1\Delta_2}\hat{\kappa}^2\end{aligned} \tag{C23}$$

$$\begin{aligned}p=&\left\{\Delta_2\left[\left(\omega_1^2-\omega_2^2\right)^2+\left(\Delta_1+\Delta_2\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]-\mu^2\left[\omega_2^4\left(\Delta_1+\Delta_2\right)\right.\right.\\&\left.\left.+\left(\Delta_2^2-2\omega_2^2\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\hat{\kappa}^2\left(\Delta_1+\Delta_2\right)\right\}\end{aligned} \tag{C24}$$

$$\begin{aligned}q=&\left\{\mu^2\left[\Delta_1\omega_2^4+\Delta_2\omega_1^4+\Delta_1\Delta_2\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]\right.\\&\left.+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\hat{\kappa}^2\left(\Delta_1+\Delta_2\right)\right\}\end{aligned} \tag{C25}$$

$$s = \left\{\Delta_2\omega_2^2\left[\left(\omega_1^2-\omega_2^2\right)^2+(\Delta_1+\Delta_2)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]+\mu^2\omega_2^4\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right.$$
$$\left.+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\omega_2^4\left(\Delta_1+\Delta_2\right)-\hat{\kappa}^2\left[\left(\Delta_1+\Delta_2\right)\left(\Delta_2^2-2\omega_2^2\right)+\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]\right\} \tag{C26}$$

$$t = \left\{\mu^2\omega_1^2\omega_2^2\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\omega_1^2\omega_2^2\left(\Delta_1+\Delta_2\right)\right.$$
$$\left.+\hat{\kappa}^2\left[\Delta_1\Delta_2\left(\Delta_1+\Delta_2\right)+\left(\omega_1^2\Delta_1+\omega_2^2\Delta_2\right)\right]\right\} \tag{C27}$$

These equations are analogous to equations (2.4) to (2.9) for the spring-coupled case but are substantially more complicated. The time-averaged energy responses T_2 and V_2 of oscillator 2 can be determined by replacing the subscript 1 to 2 in equations (C21)-(C22).

These results were derived in Ref. [5]. However, they are not very convenient for subsequent analysis of power and energy relations for the two oscillators. Alternatively, it is more convenient perhaps to write the power transmission and energy responses in equations (C20)-(C22) into forms similar to those given in Appendix A, i.e., in terms of the input powers. These are described below.

Define the parameter $\nu^2 = \hat{\kappa}^2/\omega_1^2\omega_2^2$. The coupling power and energies can be simply re-written as

$$P_{12} = \frac{q}{\left(1-\mu^2\right)Q}\left[\frac{P_1}{\Delta_1}-\frac{P_2}{\Delta_2}\right] \tag{C28}$$

$$T_1 = \frac{P_1}{2\left(1-\mu^2\right)\Delta_1} - \frac{q}{2\left(1-\mu^2\right)\Delta_1}\frac{1}{Q}\left[\frac{P_1}{\Delta_1}-\frac{P_2}{\Delta_2}\right] \tag{C29}$$

$$V_1 = \frac{P_1}{2\left(1-\nu^2\right)\Delta_1} - \frac{t}{2\left(1-\nu^2\right)\Delta_1}\frac{1}{Q}\left[\frac{P_1}{\Delta_1}-\frac{P_2}{\Delta_2}\right] \tag{C30}$$

The differences in the time-average kinetic and potential energies of the two oscillators are found to be equal, with

$$T_1 - T_2 = V_1 - V_2 = \frac{\left(\omega_1^2-\omega_2^2\right)^2+\left(\Delta_1+\Delta_2\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)}{2Q}\left[\frac{P_1}{\Delta_1}-\frac{P_2}{\Delta_2}\right] \tag{C31}$$

As a result, the coupling power is seen to be proportional to the oscillator energy difference (or the kinetic or potential energy difference) such that

$$P_{12} = \beta\left[E_1-E_2\right] = \beta\left[2\left(T_1-T_2\right)\right] \tag{C32}$$

where the coupling parameter β is given by

$$\beta=\frac{\left\{\mu^2\left[\Delta_1\omega_2^4+\Delta_2\omega_1^4+(\Delta_1+\Delta_2)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]+\left(\hat{\gamma}^2+2\mu\hat{\kappa}\right)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)+\hat{\kappa}^2(\Delta_1+\Delta_2)\right\}}{\left(1-\mu^2\right)\left[\left(\omega_1^2-\omega_2^2\right)^2+(\Delta_1+\Delta_2)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)\right]} \qquad \text{(C33)}$$

Note that when $m=0$ and $g=0$, equation (C33) becomes

$$\beta=\frac{k^2}{m_1 m_2}\frac{(\Delta_1+\Delta_2)}{\left(\omega_1^2-\omega_2^2\right)^2+(\Delta_1+\Delta_2)\left(\omega_2^2\Delta_1+\omega_1^2\Delta_2\right)} \qquad \text{(C34)}$$

Equation (C34) is the power-flow energy-difference proportionality derived in Appendix A when the two oscillators are connected via a spring. Finally note that the coupling parameter β depends on a group of parameters related to the coupling mechanism, while the denominator depends on the oscillator properties and, in particular, the difference in their natural frequencies, in a similar (but far more complicated) form for spring-coupled oscillators given in equation (2.13).

APPENDIX D TWO OSCILLATORS CONNECTED BY A FIXED SPRING

Consider the two-oscillator system shown in Figure D.1. This form of model can arise when fixed interface (Craig-Bampton) component mode synthesis (CMS) methods are used to model coupled subsystems. Oscillator 1 (m_1, k_1 and c_1) and oscillator 2 (m_2, k_2 and c_2) are subjected to two harmonic forces $f_1(t) = F_1 e^{j\omega t}$ and $f_2(t) = F_2 e^{j\omega t}$, respectively.

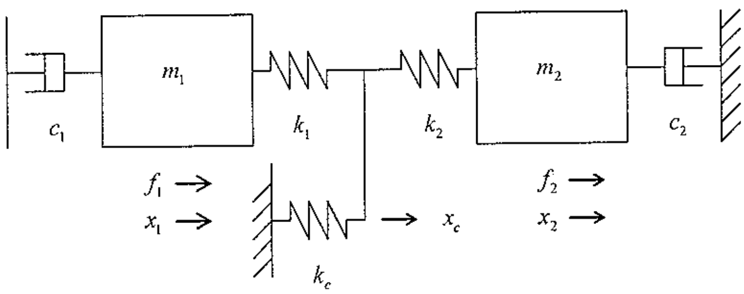

Figure D.1 Two oscillators connected via a fixed spring.

The equations governing the motion of the masses are

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1(x_1 - x_c) = f_1 \tag{D1}$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2(x_2 - x_c) = f_2 \tag{D2}$$

The interface coordinate is such that

$$k_1x_1 + k_2x_2 = (k_1 + k_2 + k_c)x_c \tag{D3}$$

Once again there is some ambiguity in the system energy, since there the potential energy stored the interface spring k_c cannot be unambiguously ascribed to one or other oscillator.

Define

$$\omega_1^2 = (1-\mu_1)\frac{k_1}{m_1};\ \omega_2^2 = (1-\mu_1)\frac{k_2}{m_2};\ \Delta_{1,2} = \frac{c_{1,2}}{m_{1,2}} \tag{D4}$$

where

$$\mu_1 = \frac{k_1}{k_1 + k_2 + k_c};\ \mu_2 = \frac{k_2}{k_1 + k_2 + k_c} \tag{D5}$$

Here ω_1 (ω_2) given in equation (D4) is the natural frequency of oscillator 1 (2) when the motion of mass 2 (1) is fixed. Specially, when $k_c \rightarrow \infty$, which corresponds to a fixed interface $x_c \rightarrow 0$, equations (D4) gives

$$\omega_1^2 = \frac{k_1}{m_1};\ \omega_2^2 = \frac{k_2}{m_2} \tag{D6}$$

For time harmonic excitations $F_{1,2}e^{j\omega t}$, the displacements of the two oscillators $X_{1,2}e^{j\omega t}$ can be determined from the frequency response functions of the system as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \mathbf{H}(\omega)\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{D7}$$

where

$$H_{11}(\omega) = \frac{1}{m_1}\frac{\omega_2^2 + j\omega\Delta_2 - \omega^2}{D(\omega)};\ H_{22}(\omega) = \frac{1}{m_2}\frac{\omega_1^2 + j\omega\Delta_1 - \omega^2}{D(\omega)} \tag{D8}$$

$$H_{12}(\omega) = H_{21}(\omega) = \frac{1}{m_1}\frac{\mu_1}{1-\mu_2}\frac{\omega_2^2}{D(\omega)} = \frac{1}{m_2}\frac{\mu_2}{1-\mu_1}\frac{\omega_1^2}{D(\omega)} \tag{D9}$$

and where

$$\begin{aligned} D(\omega) &= (j\omega)^4 + (j\omega)^3(\Delta_1+\Delta_2) + (j\omega)^2(\omega_1^2+\omega_2^2+\Delta_1\Delta_2) \\ &\quad + (j\omega)(\omega_1^2\Delta_2 + \omega_2^2\Delta_1) + \left(\omega_1^2\omega_2^2 - \mu_1\mu_2\frac{k_1k_2}{m_1m_2}\right) \end{aligned} \tag{D10}$$

The interface displacement $X_c e^{j\omega t}$ is

$$X_c = H_{1c}(\omega)F_1 + H_{1c}(\omega)F_2 \tag{D11}$$

where

$$\begin{aligned} H_{1c}(\omega) &= \frac{1}{m_1}\frac{\mu_1}{D(\omega)}\left[\frac{\omega_2^2}{1-\mu_2} + j\omega\Delta_2 - \omega^2\right]; \\ H_{2c}(\omega) &= \frac{1}{m_2}\frac{\mu_2}{D(\omega)}\left[\frac{\omega_1^2}{1-\mu_1} + j\omega\Delta_1 - \omega^2\right] \end{aligned} \tag{D12}$$

For random, stationary, statistically independent, broadband excitation with spectral densities S_{f_1} and S_{f_2} the powers and energy responses (twice the kinetic energy) are found to be

$$P_1 = \frac{\pi S_{f_1}}{m_1};\ P_2 = \frac{\pi S_{f_2}}{m_2} \tag{D13}$$

$$P_{12} = \frac{\chi}{Q}\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) \tag{D14}$$

$$E_1 = \left(1 - \frac{\chi}{\Delta_2 Q}\right)\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\chi}{\Delta_1 Q}\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) \tag{D15}$$

where

$$\chi = \mu_1\mu_2\frac{k_1k_2}{m_1m_2}(\Delta_1+\Delta_2) \tag{D16}$$

$$Q = (\omega_1^2 - \omega_2^2)^2 + (\Delta_1+\Delta_2)(\omega_1^2\Delta_2 + \omega_2^2\Delta_1) + \mu_1\mu_2\frac{k_1k_2}{m_1m_2}\frac{(\Delta_1+\Delta_2)^2}{\Delta_1\Delta_2} \tag{D17}$$

The difference in oscillator energies is then

$$E_1 - E_2 = \left(1 - \frac{\chi}{Q}\left(\frac{1}{\Delta_1} + \frac{1}{\Delta_2}\right)\right)\left(\frac{\pi S_{f_1}}{m_1\Delta_1} - \frac{\pi S_{f_2}}{m_2\Delta_2}\right) - \frac{\chi}{Q}\left(\frac{\pi S_{f_1}}{m_1\Delta_1}\frac{1}{\Delta_2} - \frac{\pi S_{f_2}}{m_2\Delta_2}\frac{1}{\Delta_1}\right) \quad \text{(D18)}$$

Note, from the last term on the right-hand side of (D18), that the coupling power (equation (D14)) is proportional to the energy difference only if $\Delta_1 = \Delta_2$. The differences are likely to be small, especially when the coupling or connection is weak. However, it should be noted that the fixed-interface modes may not provide a basis for an accurate SEA model for coupled sets of oscillators when the damping bandwidths of the two sets are very different.

APPENDIX E ENSEMBLE AVERAGE ENERGY BY A RANDOM POINT PROCESS TECHNIQUE

In deriving the ensemble mean energy in equation (3.10), the term $\sum\sum\left|k_{jk}^{ab}\right|^2/Q_{jk}^{ab}$ was averaged using number variance theory [10]. The same result can be found by using a so-called '*random point processing*' technique Ref. [11] as described below.

Assume $B << \omega_c$ and also

$$\omega_c = \left(\omega_j^a + \omega_k^b\right)/2,\ \Delta = \left(\Delta_j^a + \Delta_k^b\right)/2, \tag{E1}$$

Combining these with the simplifications made in equation (3.8), equation (3.5) reduces to

$$Q_{jk}^{ab} \approx 4\omega_c^2\left[\left(\omega_j^a - \omega_k^b\right)^2 + \left(\Delta^2 + \frac{\left|k_{jk}^{ab}\right|^2}{m_a m_b \omega_c^2}\right)\right] \tag{E2}$$

Equations (E1) and (E2) lead to

$$\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{\left|k_{jk}^{ab}\right|^2}{Q_{jk}^{ab}} \approx \frac{1}{4\omega_c^2}\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{\left|k_{jk}^{ab}\right|^2}{\left(\omega_j^a - \omega_k^b\right)^2 + \left(\Delta^2 + \kappa^2\right)} \tag{E3}$$

where

$$\kappa^2 = \frac{\mathrm{E}\left[\left|k_{jk}^{ab}\right|^2\right]}{m_a m_b \omega_c^2} \tag{E4}$$

is the coupling strength parameter. Note that in equation (E3), the denominator has been approximated by replacing $\left|k_{jk}^{ab}\right|^2$ with its mean value $\mathrm{E}\left[\left|k_{jk}^{ab}\right|^2\right]$. This approximation introduces errors because it ignores correlation between the terms $\left|k_{jk}^{ab}\right|^2$ and $1/Q_{jk}^{ab}$ and because averaging the inverse of a quantity is not equal to the inverse of its average. Errors introduced by this approximation are particularly small if κ^2 is small compared to Δ^2. Each term $\left|k_{jk}^{ab}\right|^2/Q_{jk}^{ab}$ in equation (E3) may be regarded as being like an impulse, which "sparks" whenever $\omega_j^a \approx \omega_k^b$. Consequently, $\left(\omega_j^a - \omega_k^b\right)$ forms a set of random point processes.

Let $\delta_{jk} = \left(\omega_j^a - \omega_k^b\right)$. Equation (E3) can then be re-written as

$$\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{\left|k_{jk}^{ab}\right|^2}{Q_{jk}^{ab}} \approx \frac{1}{4\omega_c^2}\sum_{t=1}^{N_a\times N_b}\frac{\left|k_{jk}^{ab}\right|^2}{\left(\delta_t\right)^2 + \left(\Delta^2 + \kappa^2\right)} \tag{E5}$$

where $\delta_{t(j,k)} = \delta_{jk}$. By using random point process theory ([11], equation (14)), the ensemble mean of the quantity given in equation (E5) becomes

$$\mathrm{E}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{\left|k_{jk}^{ab}\right|^2}{Q_{jk}^{ab}}\right]=\frac{1}{4\omega_c^2}\left\{\mathrm{E}\left[\left|k_{jk}^{ab}\right|^2\right]\int_{-\infty}^{+\infty}\frac{n_Q}{(\delta_t)^2+\left(\Delta^2+\kappa^2\right)}d\delta_t\right\} \quad \text{(E6)}$$

where $n_Q = n_a n_b \Omega$. This finally yields

$$\mathrm{E}\left[\sum_{j=1}^{N_a}\sum_{k=1}^{N_b}\frac{\left|k_{jk}^{ab}\right|^2}{Q_{jk}^{ab}}\right]=\frac{\kappa^2}{4\omega_c^2}\frac{\pi}{\sqrt{\Delta^2+\kappa^2}}n_a n_b \Omega \quad \text{(E7)}$$

Combining equation (E7) with equations (3.10) and (3.19), the ensemble mean energy of subsystem a is, finally

$$\bar{E}_a=\frac{\pi S_a}{m_a\Delta_a}n_a-\frac{\pi}{2}\frac{n_a n_b\kappa^2}{\Delta_a\sqrt{1+\kappa^2/(\Delta_1\Delta_2)}}\left(\frac{\pi S_a}{m_a\Delta_a}-\frac{\pi S_b}{m_b\Delta_b}\right) \quad \text{(E8)}$$

Equation (E8) is the same as equation (3.21). In the special case where $\Delta_1 \approx \Delta_2 \approx \Delta$, equation (E8) leads to

$$\bar{E}_a=\frac{\pi S_a}{m_a\Delta_a}n_a-\frac{\pi}{2}\frac{n_a n_b\kappa^2}{\sqrt{\Delta^2+\kappa^2}}\left(\frac{\pi S_a}{m_a\Delta_a}-\frac{\pi S_b}{m_b\Delta_b}\right) \quad \text{(E9)}$$

APPENDIX F INTERMODAL COUPLING STIFFNESS OF TWO SPRING-COUPLED CONTINUOUS SUBSYSTEMS

The theory derived in Chapters 2 and 3 concerned the coupling of pairs of oscillators. In this Appendix the theory is applied to two continuous, multi-modal subsystems joined by a spring. The spring couples the modes of the subsystems, each mode being regarded as an oscillator. The intermodal coupling stiffness and strength of connection are derived.

The motion in subsystem a can be written in terms of a modal sum as

$$w^{(a)}\left(x^{(a)}\right)=\sum_j q_j^{(a)}\phi_j^{(a)}\left(x^{(a)}\right) \tag{F1}$$

where $q_j^{(a)}$ and $\phi_j^{(a)}$ are the jth modal amplitude and mode shape ($x^{(a)}$ may be a vector if the subsystem is 2- or 3-dimensional). The mode shapes are such that modal mass is given by

$$m_j^{(a)}=\int_{(a)} m^{(a)}\left(x\right)\phi_j^{(a)2}\left(x\right)dx \tag{F2}$$

where $m_j^{(a)}$ is the modal mass of the jth mode and $m^{(a)}\left(x\right)$ is the mass density.

Suppose the subsystems are coupled by a spring of stiffness K attached between the points $x_I^{(a)}$ and $x_I^{(b)}$ and which exerts a force F_I on each plate. The jth modal amplitude thus satisfies

$$m_j^{(a)}\left[\omega_j^{(a)2}+i\Delta_j^{(a)}\omega-\omega^2\right]q_j^{(a)}=f_j^{(a)}+F_I\phi_j^{(a)}\left(x_I^{(a)}\right) \tag{F3}$$

where $f_j^{(a)}$ is the modal force applied by the external excitations. The interface force F_I is

$$F_I=K\left[w^{(b)}\left(x_I^{(b)}\right)-w^{(a)}\left(x_I^{(a)}\right)\right] \tag{F4}$$

Combining this with equation (F1) and the equivalent equation for subsystem b gives

$$m_j^{(a)}\left[\omega_j^{(a)2}+i\Delta_j^{(a)}\omega-\omega^2\right]q_j^{(a)}=f_j^{(a)}-K\left[\sum_n q_n^{(a)}\phi_n^{(a)}\left(x_I^{(a)}\right)-\sum_k q_k^{(b)}\phi_k^{(b)}\left(x_I^{(b)}\right)\right]\phi_j^{(a)}\left(x_I^{(a)}\right) \tag{F5}$$

Note that the spring couples mode j in subsystem a to the modes in subsystem b (and also to the other modes in subsystem a). The intermodal coupling stiffness between modes j and k in subsystems a and b is

$$k_{jk}^{(ab)}=K\phi_j^{(a)}\left(x_I^{(a)}\right)\phi_k^{(b)}\left(x_I^{(b)}\right) \tag{F6}$$

Hence the strength of connection κ between the mode pair is such that

$$\kappa_{jk}^2=\frac{K^2\phi_j^{(a)2}\left(x_I^{(a)}\right)\phi_k^{(b)2}\left(x_I^{(b)}\right)}{m_j^{(a)}m_k^{(b)}\omega^2} \tag{F7}$$

Note that the coupling stiffness depends on the mode shapes.

Now consider the case where the coupling points are randomly located over the subsystems. The average coupling term can be found by averaging over all possible locations $x_I^{(a)}$ and

$x_l^{(b)}$. In particular, suppose that the probability density function of $x_l^{(a)}$ is proportional to the mass density $m^{(a)}(x)$. The strength of connection, averaged over all interface points in each subsystem, is then

$$\overline{\kappa_{jk}^2} = \frac{K^2}{\omega^2} \frac{\int_{(a)} m^{(a)}(x)\phi_j^{(a)2}(x)\,dx}{m_j^{(a)} \int_{(a)} m^{(a)}(x)\,dx} \frac{\int_{(b)} m^{(b)}(x)\phi_k^{(b)2}(x)\,dx}{m_k^{(b)} \int_{(b)} m^{(b)}(x)\,dx} \tag{F8}$$

In view of the definition of modal mass (equation (F2)) this reduces to

$$\overline{\kappa_{jk}^2} = \frac{K^2}{M^{(a)} M^{(b)} \omega^2} \tag{F9}$$

where

$$M^{(a)} = \int_{(a)} m^{(a)}(x)\,dx \tag{F10}$$

is the total mass of subsystem a.

APPENDIX G: NUMERICAL CALCULATION OF APPARENT COUPLING LOSS FACTORS FOR TWO SPRING-COUPLED PLATES

In Ref. [8], a procedure for calculating the apparent coupling loss factors of a built-up system is described. The procedure is based on an energy distribution (ED) model of the system. The ED model is formed in terms of the global modes of the whole system, which can be obtained from a finite element model of the system, for example. The results are postprocessed to determine the energy influence coefficient matrix, which relates subsystem energies and input powers for rain-on-the-roof excitation. This matrix is inverted to yield numerical estimates of the coupling loss factors. Because these are estimates from a single numerical model (or possibly from an average of a number of such models) and are not found by an ensemble averaging procedure, the estimates are referred to as apparent coupling loss factors (ACLFs) and generally differ from the CLFs which relate ensemble average powers and energies.

In this Appendix this approach is used to produce estimates of ACLFs for two, spring-coupled plates, using numerical FE models developed by Thite (personal communication). Results are compared to those from the subsystem-FRF approach described in section 4.3.2. The system parameters are listed in Table G1. The apparent coupling loss factors are shown in Figure G1. There is good agreement between the two approaches. The subsystem-FRF based approach does not however require solution of the global eigenvalue problem. These numerical approaches are used elsewhere to estimate ACLFs for comparison with the coupled-oscillator results.

Numerical investigations also revealed that for very low damping, the ACLF estimates are very sensitive to the exact locations of the coupling spring. In this case, the system has a very low modal overlap and the variance of the ACLFs becomes very large.

Table G1. System parameters and physical properties (SI units)

Elastic modulus (N/m^2)	2×10^{11}	Length of coupled edge (m)	$L_{y1}=L_{y2}=0.9$
Density (kg/m^3)	8×10^3	Plate area (m^2)	$S_1=0.9$, $S_2=1.26$
Poisson's ratio	0.3	Modal density (/Hz)	$n_1=0.0297$, $n_2=0.0416$
Thickness (m)	0.01	System total modal density	$n_1+n_2=0.0714$
Spring stiffness (N/m)	$K=5\text{x}10^6$, $1\text{x}10^7$, $2\text{x}10^7$, $3\text{x}10^7$		
Connection points (m)	$(X_1,Y_1)=(0.7, 0.55)$, $(X_2,Y_2)=(0.3, 0.55)$		
Centre frequency (Hz)	1500	Frequency band (Hz)	400

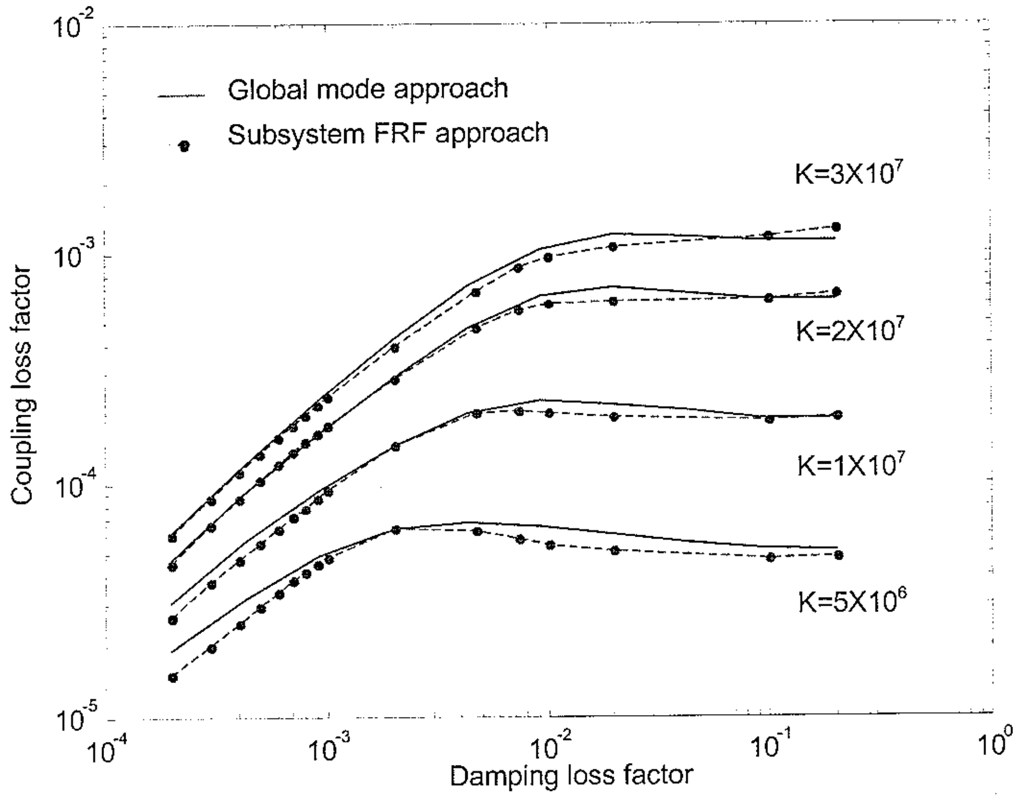

Figure G1. Apparent coupling loss factors for two, spring-coupled plates.

APPENDIX H LIST OF SYMBOLS

A	Area.
$\mathbf{A}$	Energy influence coefficient (EIC) matrix.
B	Excitation bandwidth.
c_1, c_2	Damping constant of oscillators 1 and 2.
c_a, c_b	Wave speed of subsystems a and b.
E	Total energy; elastic modulus.
E_j^a, E_k^b	The frequency averages of energies of oscillators j and k in sets a and b.
$f(t)$	Forces.
F	Force amplitudes.
$\mathbf{H}$	Frequency response function (FRF) matrix.
k	Stiffness, coupling stiffness between oscillators
k_{jk}^{ab}	Coupling stiffness between oscillators j and k in sets a and b.
K	Spring stiffness between subsystems a and b.
K'	Discrete frequency kinetic energy.
L	Length
$\mathbf{L}$	Coupling loss factor (CLF) matrix.
m_1, m_2	Masses of oscillators 1 and 2.
m_a, m_b	Modal mass of subsystems a and b.
m_j^a, m_k^b	Masses of oscillators j and k in sets a and b.
M_a, M_b	Total masses of subsystems a and b.
N_a, N_b	Numbers of modes of subsystems a and b within a specific frequency band.
n_a, n_b	Modal densities of subsystems a and b.
p	Joint probability density function.
P_1', P_2'	Discrete frequency input powers of oscillators 1 and 2.
P_1, P_2	Total input power
$\bar{P}_1$, $\bar{P}_2$	Ensemble average input power
P_a, P_b	The frequency averaged input powers of subsystems a and b.
P_j^a, P_k^b	The frequency averaged input powers of oscillators j and k in sets a and b.
P_{12}'	Discrete frequency coupling power between oscillators 1 and 2.
P_{12}	Total coupling power between oscillators 1 and 2 in a frequency band of force.
$\bar{P}_{12}$	Ensemble average of P_{12} in a frequency band of natural frequency distribution.
P_{ab}	Coupling power between oscillator sets a and b.
P_{jk}^{ab}	Frequency averaged coupling power between oscillators j and k in sets a and b.
Q	Coupling parameter between oscillators 1 and 2.

Q_{jk}^{ab}	Coupling parameter between oscillators j and k in sets a and b.
$R_{a,\infty}$, $R_{b,\infty}$	Real parts of the characteristic impedances of systems a and b.
S_a, S_b	Spectral densities of rain-on-the-roof forcing on subsystems a and b.
S_j^a, S_k^b	Spectral densities of forces on oscillators j and k in sets a and b.
S_{f_1}, S_{f_2}	Spectral densities of $f_1(t)$ and $f_2(t)$.
x_I^a, x_I^b	Interface locations on subsystems a and b.
Y	Mobility
Z	Impedance
β	Coupling term between oscillators 1 and 2.
β_{jk}^{ab}	Coupling term between oscillators j and k in sets a and b.
B	Bending stiffness.
γ	Strength of coupling parameter.
δ	Natural frequency spacing between oscillators 1 and 2.
Δ	Damping bandwidth.
Δ_1, Δ_2	The modal bandwidths of oscillators 1 and 2.
Δ_a, Δ_b	The modal bandwidths of subsystems a and b.
Δ_j^a, Δ_k^b	The modal bandwidths of oscillators j and k in sets a and b.
ξ_1, ξ_2	The amplitudes of time-harmonic response of oscillators 1 and 2.
η_1, η_2	Damping loss factors of oscillators 1 and 2.
η_{12}, η_{21}	Coupling loss factors between oscillators 1 and 2.
η_a, η_b	Damping loss factors of oscillator sets a and b.
η_{ab}, η_{ba}	Coupling loss factors between subsystems a and b.
κ	Strength of connection parameter.
μ	Modal overlap (based on the half-power bandwidth)
Ω	Natural-frequency-distribution band.
ρ_a, ρ_b	Mass densities of rod a and rod b.
$\tau_{ab,\infty}$	Transmission coefficient between two infinite system a and b.
ϕ	Mode shape.
ω	Frequency.
ω_1, ω_2	Natural frequencies of oscillators 1 and 2.
ω_c	Centre frequency of a frequency band.
ω_j^a, ω_k^b	The natural frequencies of oscillators j and k in sets a and b.

Superscripts

$\bar{}$	Ensemble average

Subscripts

a, b	Subsystem
1, 2	Oscillator number