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FLUID DYNAMICS AND ACOUSTICS GROUP

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by

T G Leighton

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Authorized for issue by
Professor R J Astley, Group Chairman

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ABSTRACT

The most common nonlinear equations of motion for the pulsation of a spherical gas bubble in an infinite body of liquid arise in the various forms of the Rayleigh-Plesset equation, expressed in terms of the dependency of the bubble radius on the conditions pertaining in the gas and liquid. However over the past few decades several important analyses have begun with a heuristically-derived form of the Rayleigh-Plesset equation which considers the bubble volume, instead of the radius, as the parameter of interest, and for which the dissipation term is not derived from first principles. The predictions of these two sets of equations can differ in important ways, largely through differences between the methods chosen to incorporate damping. As a result this report derives the Rayleigh-Plesset equation in terms of the bubble volume from first principles in such a way that it has the same physics for dissipation (viscous shear) as is used in the radius frame.

LIST OF SYMBOLS

c	sound speed in the liquid
D	the material derivative
p	the sum of all steady and unsteady pressures outside the bubble wall
p'	the pressure at a boundary within the fluid
p_{g}	the sum of all steady and unsteady pressures in the gas
$p_{\mathrm{g,e}}$	the value of p_{g} when $R = R_0$
p_{i}	the sum of all steady and unsteady pressures in the bubble interior
$p_{\mathrm{i,e}}$	the value of p_{i} when $R = R_0$
p_{v}	vapour pressure
p_{L}	pressure in the liquid at the bubble wall
p_0	the static pressure in the liquid just outside the bubble wall
$p_{\infty}(t)$	the value of p very far from the bubble
r	range from the bubble centre
$R(t)$	bubble radius
R_0	equilibrium bubble radius
t	time
$\vec{u}$	the liquid particle velocity

$V(t)$	bubble volume
V_0	equilibrium bubble volume
κ	Polytropic index of the gas
γ	ratio of specific heats for the gas
λ	the wavelength of the insonifying field.
ε_r'	Principle component of strain tensor in the radial direction
ϕ_{KE}	the kinetic energy in the liquid
ρ_0	liquid density (assumed to be constant)
ψ	amplitude function
σ	the surface tension
η	shear viscosity of the liquid
η_B	bulk viscosity of the liquid
Φ	velocity potential
$\sum \vec{F}_{ext}$	the vector summation of all body forces

1 Introduction

The most popular nonlinear equation for description of the nonlinear response of a gas bubble in liquid to a driving pressure field is the Rayleigh-Plesset equation. This can be derived from first principles using the bubble radius R as the dynamic parameter. However there exist heuristic formulations based on a form of the Rayleigh-Plesset equation in which the bubble volume V is used as the dynamic parameter, where the damping is not derived from first principles. The predictions of the two approaches do not always agree, and this study was undertaken to derive to derive a form of the Rayleigh-Plesset equation in which the bubble volume V is used as the dynamic parameter, and where the physics describing the dissipation is identical to that used when the Rayleigh-Plesset equation is cited in the radius frame.

This paper will proceed by using the following common assumptions: The bubble exists in an infinite medium. The bubble stays spherical at all times during the pulsation. Spatially uniform conditions exist within the bubble. The bubble radius is much smaller than the wavelength of the driving sound field. There are no body forces acting (e.g. gravity). Bulk viscous effects can be ignored. The density of the surrounding fluid is much greater than that of the internal gas. The gas content is constant.

2 Derivation of the Rayleigh-Plesset equation in terms of the bubble radius

The Rayleigh-Plesset equation is usually derived by considering the dynamics of the bubble radius, R. This can be done in two basic ways:

- Differentiation with respect to R of the balance between the kinetic energy in the liquid and the potential energy in the gas;
- Integration of the Navier Stokes equation.

These two methods will be shown in sections 2.1 and 2.2. The section 3 will discuss ways of deriving the Rayleigh-Plesset equation in terms of the dynamics of the bubble volume.

2.1 Derivation of the Rayleigh-Plesset equation in terms of the bubble radius using an energy balance

Consider a spherical gas bubble which pulsates in an incompressible liquid as a result of an insonifying field (the long wavelength limit being assumed throughout this book). The fluid velocity $u(r,t)$ falls off as an inverse square law with range r as a result of the assumption of liquid incompressibility, which implies that:

$$u(r,t) = \frac{R^2(t)}{r^2(t)}\dot{R}(t) \tag{1}$$

where the bubble has radius $R(t)$ and wall velocity $\dot{R}(t)$. As the bubble radius changes, for example from an equilibrium value R_0 to some other, work is done on the bubble by that pressure which would exist at the location of the centre of the bubble were the bubble not to be present. If the spatial scales over which this pressure changes are much greater than the bubble radius, this almost equals the liquid pressure far[1] from the bubble, $p_\infty = p_0 + P(t)$, which includes a dynamic component $P(t)$. The difference between this work, and that done by the pressure p_L at the bubble wall, equals the kinetic energy in the liquid (ϕ_{KE}):

$$\phi_{KE} = \frac{1}{2}\rho_0 \int_{r=R}^{r=\infty} 4\pi r^2 u^2 dr = 2\pi\rho_0 R^3 \dot{R}^2 \tag{2}$$

This balance can be expressed as follows:

$$\int_{R_0}^{R} (p_L - p_\infty) 4\pi R^2 dR = 2\pi\rho_0 R^3 \dot{R}^2 \tag{3}$$

[1] If t_b is the timescale over which the boundary moves, then the condition $r >\sim c_\infty t_b$ describes what is commonly understood to be ‘far from the bubble’.

Differentiation of this with respect to *R*, noting that

$$\frac{\partial \dot{R}^2}{\partial R} = \frac{1}{\dot{R}}\frac{\partial \dot{R}^2}{\partial t} = 2\ddot{R} \tag{4}$$

gives

$$p_L(t) - p_\infty = \rho_0\left(R\ddot{R} + \frac{3\dot{R}^2}{2} \right) + O(\dot{R}/c) \tag{5}$$

The term on the left arises from the difference in work done at the bubble wall and remote from the bubble; and the terms on the right arise from the kinetic energy imparted to the liquid. If the pressure far from the bubble p_∞ comprises both a static component p_0 and an applied driving pressure $P(t)$, then this can be expressed in (5) to give the so-called Rayleigh-Plesset equation of motion [1]:

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(p_L(t) - p_0 - P(t) \right) + O(\dot{R}/c) \tag{7}$$

2.2 Derivation of the Rayleigh-Plesset equation in terms of the bubble radius using the Navier Stokes equation

In the following derivation, the use of the dot notation in this, and the subsequent equations of motion, indicates the use of the material derivative [1§2.2.2], i.e.:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \tag{8}$$

where $\vec{u}$ is the liquid particle velocity. For the discussion of the pulsation of a single bubble whose centre remains fixed in space, as occurs in this report, the convective term (the second term on the right) is zero. Before applying the equations of this book, critical evaluation should be made of their applicability, given this restriction. Models of translating bubbles need careful evaluation. Even where bubbles are assumed to pulsate only, if they exist in a dense cloud then the convective term may be significant [2].

The following derivation relies assumes that the material outside the gas bubble wall is incompressible, and assumes that spatially uniform conditions are assumed to exist within the bubble.

When these assumptions are applied for the case of a gas bubble in a liquid, the equations for the conservation of energy within the liquid can be coupled to that of the diffusion of dissolved gas within it, and to the equation for conservation of mass in the liquid:

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\vec{\nabla}\cdot\vec{u}=0 \qquad \textit{(Continuity equation)} \qquad (9)$$
$$\Rightarrow \frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0$$

where $\vec{u}$ is the liquid particle velocity and ρ is the liquid density; and to the equation for conservation of momentum in the liquid:

$$\rho\frac{D\vec{u}}{Dt}=\rho\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\vec{\nabla})\vec{u}\right)=\rho\sum\vec{F}_{ext}-\vec{\nabla}p+\left(\frac{4\eta}{3}+\eta_B\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})-\eta\vec{\nabla}\times\vec{\nabla}\times\vec{u}$$

(*Navier Stokes equation)* (10)

where *p* represents the sum of all steady and unsteady pressures.

Equation (10) simplifies in a number of ways for limits which are often appropriate to gas bubbles in water [1§2.3.2]. The term $\eta\vec{\nabla}\times\vec{\nabla}\times\vec{u}$ encompasses the dissipation of acoustic energy associated with, amongst other things, vorticity, and hence is zero in conditions of irrotational flow (required for the definition of a velocity potential). The term $\left(4\eta/3+\eta_B\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$ represents the product of viscous effects (through the shear

η and bulk η_B viscosities of the liquid), with the gradient of $\vec{\nabla}\cdot\vec{u}$ (which, from (9), represents in turn the liquid compressibility). As an interaction term, it is generally small. Note that setting it to zero does not imply that all viscous effects are neglected, but simply that they appear only through the boundary condition. Lastly, the term $\sum\vec{F}_{ext}$ represents the vector summation of all body forces which are neglected in the formulations of this report. If it is then assumed that the bubble remains spherical at all times and pulsates in an infinite body of liquid, then because of spherical symmetry, the particle velocity in the liquid $\vec{u}$ is always radial and of magnitude *u(r,t)*, and equations (9) and (10) reduce, respectively, to:

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial(r^2\rho u)}{\partial r}=0 \tag{11}$$

and

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\,. \qquad \textit{(Euler's equation)} \tag{12}$$

The bubble radius $R(t)$ oscillates about some equilibrium radius R_0 with bubble wall velocity $\dot{R}(t)$.

Approximations of this sort are required because the solution of the conservation equations of continuum mechanics both within and outside the bubble, with suitable boundary conditions at the wall, is not simple [1, 3-15].

The derivation of the Rayleigh-Plesset equation given previously (section 2.1) was based on equating the kinetic and potential energies. To follow the approach discussed above, equation (12) is combined with the linearised wave equation for velocity potential Φ

$$\nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=0\,, \tag{13}$$

given[2]

$$\vec{u} = \vec{\nabla}\Phi \, , \tag{14}$$

where in the spherically symmetric conditions assumed above, equation (13) has the solution

$$\Phi = \frac{\psi(t - r/c)}{r} \, , \tag{15}$$

and the amplitude function ψ has yet to be determined. At first sight it may seem odd to apply a linearised wave equation, given that the intended equation of dynamics is nonlinear, but this is justified provided that the nonlinearity introduced into the system by the bubble we are describing is much greater than the nonlinearity of the system when our bubble is removed (which is usually the case for single bubbles in an otherwise bubble-free medium). In the long wavelength limit, equation (13) simplifies into the equation of mass conservation in an incompressible liquid,

$$\nabla^2\Phi = 0 \, , \tag{16}$$

because [16]

$$\left| \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} \right| \Big/ \nabla^2\Phi \sim \left(R/\lambda \right)^2 << 1 . \tag{17}$$

where λ is the wavelength of the insonifying field. In this limit, both (15) and (16) give

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[2] Note that some authors use the alternative convention $\vec{u} = -\vec{\nabla}\Phi$

$$\Phi = \frac{\psi(t)}{r}, \tag{18}$$

where ψ can be evaluated by application to (16) of the boundary condition $\left|\partial\Phi/\partial r\right|_{r=R} = \dot{R}$ (which follows from (14)). This gives $\psi = -R^2\dot{R}$ and

$$\Phi(r,t) = -R^2\dot{R}/r \tag{19}$$

which recovers the incompressible relation (1). Another consequence is that (12) becomes

$$\frac{\partial^2\Phi}{\partial t\partial r} + \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial r}\right)^2 + \frac{1}{\rho_0}\frac{\partial p}{\partial r} = 0 \Rightarrow \qquad (R \ll r_1 \ll \lambda) \tag{20}$$

$$\int_r^{r_1}\frac{\partial^2\Phi}{\partial t\partial r'}dr' + \frac{1}{2}\int_r^{r_1}\frac{\partial}{\partial r'}\left(\frac{\partial\Phi}{\partial r'}\right)^2 dr' = -\int_r^{r}\frac{1}{\rho_0}\frac{\partial p}{\partial r'}dr'.$$

where the use of the invariant density ρ_0 reflects the assumed liquid incompressibility. Using the independence of the variables *r* and *t*, and given that r_1 is sufficiently far from the bubble that the liquid pressure there is dominated, not by the presence of the bubble but rather by the static pressure p_0 and the applied one $P(t)$, then (20) gives the pressure at a location *r* in the liquid, in terms of the velocity potential:

$$\frac{\partial\Phi(r,t)}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi(r,t)}{\partial r}\right)^2 = -\frac{p(r,t) - p_0 - P(t)}{\rho_0}. \tag{21}$$

The Rayleigh-Plesset equation (7) is obtained simply by evaluating (21) at the bubble wall, where application of (19) implies:

$$\Phi(r,t)\big|_{r=R} = -R\dot{R}, \tag{22}$$

$$\frac{\partial\Phi(r,t)}{\partial t} = -\frac{2R\dot{R}^2 + R^2\ddot{R}}{r} \Rightarrow \left.\frac{\partial\Phi(r,t)}{\partial t}\right|_{r=R} = -R\ddot{R} - 2\dot{R}^2,$$

$$\frac{\partial\Phi(r,t)}{\partial r}=\frac{R^2\dot{R}}{r^2}\Rightarrow\left.\frac{\partial\Phi(r,t)}{\partial r}\right|_{r=R}=\left.\frac{R^2\dot{R}}{r^2}\right|_{r=R}=\dot{R},$$

the last of these being a logical consequence of (14). Applications of these relationships at the bubble wall (where $p(r=R,t)=p_{\mathrm{L}}(t)$) to (21) gives:

$$\left.\frac{\partial\Phi(r,t)}{\partial t}\right|_{r=R}+\frac{1}{2}\left(\frac{\partial\Phi(r,t)}{\partial r}\right)^2\bigg|_{r=R}=-\frac{p(r=R,t)-p_0-P(t)}{\rho_0}\Rightarrow \tag{23}$$

$$R\ddot{R}+\frac{3\dot{R}^2}{2}=\frac{p_{\mathrm{L}}(t)-p_0-P(t)}{\rho_0}$$

the Rayleigh-Plesset equation in terms of the bubble radius.

In similar vein, application of (22) to the body of the liquid gives the pressure field radiated by the bubble, where the retarded time $t_r=t-r/c$ is recovered in place of t to impose a finite sound speed on this 'incompressible' medium:

$$P_{b1}(r)=\rho_0\left(\frac{R(t_r)}{r}\left(R(t_r)\ddot{R}(t_r)+2\dot{R}^2(t_r)\right)-\frac{\dot{R}^2(t_r)}{2}\left(\frac{R(t_r)}{r}\right)^4\right) \tag{24}$$

The radiated pressure comprises a component which decays to as r^{-4} [1§3.3.1(b)], such that far from the bubble wall the radiated field is dominated by the r^{-2} term, and

$$P_{b1}(r)\approx\frac{\rho_0 R(t_r)}{r}\left(R(t_r)\ddot{R}(t_r)+2\dot{R}^2(t_r)\right)=\frac{\rho_0\ddot{V}(t_r)}{4\pi r} \tag{25}$$

since $V=4\pi R^3/3\Rightarrow\dot{V}=4\pi R^2\dot{R}\Rightarrow\ddot{V}=d(4\pi R^2\dot{R})/dt$.

3. Derivation of the Rayleigh-Plesset equations in terms of bubble volume from the energy balance

Consider a spherical gas bubble which pulsates in an incompressible liquid as a result of an insonifying field (the long wavelength limit being assumed throughout this book). The fluid velocity $u(r,t)$ falls off as an inverse square law with range r as a result of the assumption of liquid incompressibility. The mass of liquid which moves at the bubble wall, in time Δt, is $\rho_0 \dot{V}(t)\Delta t$, whilst that at range r is $\rho_0 4\pi r^2 u(t)\Delta t$ t, , which implies that:

$$u(r,t) = \frac{\dot{V}(t)}{4\pi r^2} \tag{26}$$

where the bubble has volume *V(t)* and wall volume velocity $\dot{V}(t)$. As the bubble volume changes, for example from an equilibrium value V_0 to some other, work is done on the bubble by that pressure which would exist at the location of the centre of the bubble were the bubble not to be present. If the spatial scales over which this pressure changes are much greater than the bubble radius, this almost equals the liquid pressure far[3] from the bubble, $p_\infty = p_0 + P(t)$, which includes a dynamic component *P(t)*. The difference between this work, and that done by the pressure p_L at the bubble wall, equals the kinetic energy in the liquid:

$$\phi_{KE} = \frac{1}{2}\rho_0 \int_{r=R}^{r=\infty} 4\pi r^2 u^2 dr = \frac{\rho_0 \dot{V}^2(t)}{8\pi R} \tag{27}$$

This balance can be expressed as follows:

$$\int_{V_0}^{V} (p_L - p_\infty) dV = \frac{\rho_0 \dot{V}^2(t)}{8\pi R} = \frac{\rho_0 \dot{V}^2(t)}{8\pi}\left(\frac{4\pi}{3V}\right)^{1/3} \tag{28}$$

Differentiation of this with respect to *V*, noting that

[3] See footnote number 1.

$$\frac{\partial \dot{V}^2}{\partial V} = \frac{1}{\dot{V}}\frac{\partial \dot{V}^2}{\partial t} = 2\ddot{V} \tag{29}$$

gives

$$\frac{1}{8\pi}\left(\frac{4\pi}{3V}\right)^{1/3}\left(2\ddot{V} - \frac{\dot{V}^2}{3V}\right) = \frac{1}{\rho_0}\left(p_\mathrm{g} + p_\mathrm{v} - p_\sigma - p_0 - P(t)\right) + O(\dot{R}/c) \tag{30}$$

4. The liquid pressure

Having obtain the Rayleigh-Plesset equation in terms of the pressure in the liquid at the bubble wall (p_L) in radius (equations (7) and (23)) or volume (equation (30)) form, the next stage in generating a formulation suitable for predicting bubble dynamics is to find appropriate expressions for p_L. To do this in a way which accounts for all loss mechanisms in a nonlinear and time-dependent manner is very complicated, and most simulations rely on assumptions to simplify the problem. The simplest solution assumes no dissipation occurs, and that solution is derived in the section 4.1.

4.1 The liquid pressure assuming no dissipation

Initially this was done by neglecting all forms of dissipation. Evaluation of p_L for use in the equations of motion then becomes a question of calculating p_i, the pressure inside the bubble. After the initial step of expressing the internal pressure in the bubble in terms of the sum of the gas (p_g) and vapour (p_v) pressures, correcting for the Laplace pressure introduced through the effect of surface tension (p_σ):

$$\begin{aligned} &p_\mathrm{i} = p_\mathrm{g} + p_\mathrm{v} = p_\mathrm{L} + p_\sigma \Rightarrow \\ &p_\mathrm{L} = p_\mathrm{g} + p_\mathrm{v} - p_\sigma \end{aligned}, \tag{31}$$

where

$$p_\sigma = 2\sigma / R = 2\sigma\left(\frac{3V}{4\pi}\right)^{1/3}, \tag{32}$$

Evaluation of (31) when the bubble is at equilibrium size and the pressures take values appropriate for that size in the absence of any driving pressure ($R = R_0; V = V_0; p_\mathrm{i} = p_\mathrm{i,e}; p_\mathrm{g} = p_\mathrm{g,e}$; see [1§2.1]) gives:

$$p_\mathrm{i,e} = p_\mathrm{g,e} + p_\mathrm{v} = p_0 + \frac{2\sigma}{R_0} = p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3}, \tag{33}$$

This requires an understanding of thermal losses from the gas.

By far the most common way of calculating p_g is to appeal to a polytropic law. It involves calculating the pressure in the gas at a given bubble size by comparing it with the pressure at equilibrium. From (33), that the gas pressure at equilibrium ($p_\mathrm{g,e}$) is equal to the sum of the static pressure in the liquid just outside the bubble wall (p_0), plus the Laplace pressure at equilibrium $2\sigma/R_0$ (where σ is the surface tension [1§2.1]), minus that component due to vapour (p_v). Hence when the bubble has radius R the pressure in the gas will be:

$$p_\mathrm{g} = p_\mathrm{g,e}\left(\frac{R_0}{R}\right)^{3\kappa} = \left(p_0 + \frac{2\sigma}{R_0} - p_\mathrm{v}\right)\left(\frac{R_0}{R}\right)^{3\kappa} \tag{34}$$

in terms of the bubble radius, and

$$p_\mathrm{g} = p_\mathrm{g,e}\left(\frac{V_0}{V}\right)^{3\kappa} = \left(p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} - p_\mathrm{v}\right)\left(\frac{V_0}{V}\right)^{\kappa} \tag{35}$$

in volume terms. The use of the polytropic index adjusts the relationship between bubble volume and gas pressure (effectively, the ‘spring constant’ of the bubble) to account for heat flow across the bubble wall, but crucially it ignores *net* thermal losses from the bubble (see below). Therefore if the Rayleigh-Plesset equation is evaluated using a polytropic law, the result would, without correction, ignore two of the major sources of dissipation: net thermal losses and, through the incompressible assumption, radiation losses. Approximate corrections, which artificially enhance the viscosity to account for thermal and radiation damping, are available through use of enhancements to the viscosity, although these are only partially effective. These enhancements are based on the same physics as the 'linear' damping coefficients.

The pressure in the liquid in the radius frame is therefore found by substituting Substitution of (34) into (31) gives

$$p_{\mathrm{L}} = p_{\mathrm{g}} + p_{\mathrm{v}} - p_{\sigma} = \left(p_0 + \frac{2\sigma}{R_0} - p_{\mathrm{v}} \right) \left(\frac{R_0}{R} \right)^{3\kappa} + p_{\mathrm{v}} - p_{\sigma} \tag{36}$$

and substitution of this into (7) or (23) gives

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(p_{\mathrm{g}} + p_{\mathrm{v}} - p_{\sigma} - p_0 - P(t) \right) + O(\dot{R}/c) \Rightarrow \tag{37}$$

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(\left(p_0 + \frac{2\sigma}{R_0} - p_{\mathrm{v}} \right) \left(\frac{R_0}{R} \right)^{3\kappa} + p_{\mathrm{v}} - \frac{2\sigma}{R} - p_0 - P(t) \right) + O(\dot{R}/c)$$

which is the Rayleigh-Plesset equation expressed in terms of the bubble radius with no dissipation included.

Similarly, substitution of (35) into (31) gives

$$p_{\mathrm{L}} = p_{\mathrm{g}} + p_{\mathrm{v}} - p_{\sigma} = \left(p_0 + 2\sigma \left(\frac{4\pi}{3V_0} \right)^{1/3} - p_{\mathrm{v}} \right) \left(\frac{V_0}{V} \right)^{\kappa} + p_{\mathrm{v}} - 2\sigma \left(\frac{4\pi}{3V} \right)^{1/3} \tag{38}$$

and substitution of this into (30) gives the polytropic version of the Rayleigh-Plesset equation in terms of bubble volume, with no dissipation:

$$\frac{1}{8\pi}\left(\frac{4\pi}{3V} \right)^{1/3} \left(2\ddot{V} - \frac{\dot{V}^2}{3V} \right) = \frac{1}{\rho_0}\left(p_{\mathrm{g}} + p_{\mathrm{v}} - p_{\sigma} - p_0 - P(t) \right) + O(\dot{R}/c) \Rightarrow \tag{40}$$

$$\frac{1}{8\pi}\left(\frac{4\pi}{3V} \right)^{1/3} \left(2\ddot{V} - \frac{\dot{V}^2}{3V} \right) = \frac{1}{\rho_0}\left(\left(p_0 + 2\sigma \left(\frac{4\pi}{3V_0} \right)^{1/3} - p_{\mathrm{v}} \right) \left(\frac{V_0}{V} \right)^{\kappa} + p_{\mathrm{v}} - 2\sigma \left(\frac{4\pi}{3V} \right)^{1/3} - p_0 - P(t) \right) + O(\dot{R}/c)$$

which is the Rayleigh-Plesset equation expressed in terms of the bubble volume with no dissipation included.

4.2 The inclusion of viscous losses

The most common approach in the radius form of the equation is to neglect all forms of loss except viscous losses (an assumption which is often not valid), which enables an expression for the liquid pressure to be obtained by dynamically matching normal stresses across at the bubble wall. This liquid pressure can be obtained by dynamically matching normal stresses across at the bubble wall:

$$p_L = p_i - \frac{2\sigma}{R} - \frac{4\eta \dot{R}}{R} \tag{41}$$

which, for η=0, is what could be obtained by substituting (34) into (31). This approach explicitly introduces viscous damping. It has been derived for the radius frame [1§4.2.1b, 17] and can also be derived for the volume frame.

The relationships of section 4.1 were derived for an inviscid liquid, but when the shear viscosity of the liquid is finite. Finite shear modifies the liquid pressure at the bubble wall (p_{L}) such that it differs from the pressure at a boundary with in the fluid (p') by an amount proportional to the principle rate of strain in the radial direction ($\varepsilon_r' = \partial u / \partial r$) as follows [1§4.2.1b equation 4.73]:

$$p_{\mathrm{L}} = p' - 2\eta\varepsilon_r' = p' - 2\eta \frac{\partial u}{\partial r} \, . \tag{42}$$

This can readily be converted into forms relevant to the radius and volume frames, using their respective incompressibility relations, i.e.

$$u(r,t) = \frac{R^2(t)}{r^2(t)} \dot{R}(t) \Rightarrow \frac{\partial u(r,t)}{\partial r} = -\frac{2R^2\dot{R}}{r^3} \tag{43}$$

from (1) (for the radius frame) and (from (26) for the volume frame):

$$\frac{\partial u(r,t)}{\partial r} = -\frac{\dot{V}(t)}{2\pi r^3} \tag{44}$$

Substitution of (43) into (42) gives, for the radius frame:

$$p_{\mathrm{L}} = p' + \frac{4\eta R^2 \dot{R}}{r^3}. \tag{45}$$

whilst substitution of (44) into (42) gives the equivalent expression for the volume frame:

$$p_{\mathrm{L}} = p' + \frac{\eta \dot{V}(t)}{\pi r^3}. \tag{46}$$

These expressions ((45) and (46)) can now be used to derive (41) and an equivalent expression for the volume frame.

4.2(a) Calculation of the losses in the radius frame

Within the body of the liquid, Bernoulli's equation follows from the integration of a suitably reduced form of the Navier Stokes equation (equation (10)) [1§4.2.1b]:

$$\frac{p'}{\rho_0} = \frac{p_\infty}{\rho_0} - \frac{\partial \Phi}{\partial t} - \frac{u^2}{2} \qquad \textit{(Bernoulli's equation)} \tag{47}$$

Substitution for the velocity potential from (19) and for the velocity from the incompressibility relation (1) into (12), and evaluation of the result at the bubble wall ($r=R$), gives:

$$\frac{p'}{\rho_0} = \frac{p_\infty}{\rho_0} - \left.\frac{\partial \Phi(r,t)}{\partial t}\right|_{r=R} - \frac{1}{2}\left(\left.\frac{R^2(t)}{r^2(t)} \dot{R}(t)\right|_{r=R} \right)^2 \tag{48}$$

which is readily evaluated using (22):

$$\frac{p'}{\rho_0} = \frac{p_\infty}{\rho_0} + R\ddot{R} + \frac{3\dot{R}^2}{2}. \tag{49}$$

Eliminating p' from (49) and (45) (evaluated at the bubble wall) gives:

$$R\ddot{R} + \frac{3\dot{R}^2}{2} = \frac{1}{\rho_0}\left(p_{\mathrm{L}} - \frac{4\eta \dot{R}}{R} - p_\infty \right). \tag{50}$$

which, when (36) is used to substitute for p_{L}, gives

$$R\ddot{R}+\frac{3\dot{R}^2}{2}=\frac{1}{\rho_0}\left(\left(p_0+\frac{2\sigma}{R_0}-p_{\mathrm{v}}\right)\left(\frac{R_0}{R}\right)^{3\kappa}+p_{\mathrm{v}}-\frac{2\sigma}{R}-\frac{4\eta\dot{R}}{R}-p_0-P(t)\right). \tag{51}$$

where the assumption is made that the pressure in the liquid far from the bubble (p_∞) is a summation of steady (p_0) and time-varying ($P(t)$) components (i.e. $p_\infty = p_0 + P(t)$).

4.2(b) Calculation of the losses in the volume frame

To incorporate shear into the Rayleigh-Plesset equation in the volume frame, substitutions into (47) can be made for the liquid velocity using (26) and, an appropriate velocity potential (14) can be calculated from (26) using the relevent boundary conditions:

$$\Phi(r,t)=-\frac{\dot{V}(t)}{4\pi r}. \tag{52}$$

In this way equation (47) can be evaluated at the bubble wall ($r=R$):

$$\begin{aligned}\frac{p'}{\rho_0}&=\frac{p_\infty}{\rho_0}-\left.\frac{\partial\Phi(r,t)}{\partial t}\right|_{r=R}-\frac{1}{2}\left(\left.\frac{\dot{V}(t)}{4\pi r^2}\right|_{r=R}\right)^2\Rightarrow\\ \frac{p'}{\rho_0}&=\frac{p_\infty}{\rho_0}+\frac{\ddot{V}}{4\pi R}-\frac{\dot{V}^2(t)}{2\left(4\pi R^2\right)^2}\end{aligned} \qquad \textit{(Bernoulli's equation)} \tag{53}$$

Elimination of p' from (46) and (53) gives:

$$\begin{aligned}&\frac{\ddot{V}}{4\pi R}-\frac{\dot{V}^2(t)}{2\left(4\pi R^2\right)^2}=\frac{1}{\rho_0}\left(p_{\mathrm{L}}-p_\infty-\frac{4\eta\dot{V}(t)}{3V}\right)\Rightarrow\\ &\frac{1}{4\pi}\left(\frac{4\pi}{3V}\right)^{1/3}\left(\ddot{V}-\frac{\dot{V}^2(t)}{6V}\right)=\frac{1}{\rho_0}\left(p_{\mathrm{L}}-p_\infty-\frac{4\eta\dot{V}(t)}{3V}\right)\end{aligned}. \tag{54}$$

Substituting for p_{L} from (38) gives:

$$\frac{\ddot{V}}{4\pi R} - \frac{\dot{V}^2(t)}{2\left(4\pi R^2\right)^2} = \frac{1}{\rho_0}\left(p_{\mathrm{L}} - p_\infty - \frac{4\eta \dot{V}(t)}{3V} \right) \Rightarrow \tag{55}$$

$$\frac{1}{4\pi}\left(\frac{4\pi}{3V}\right)^{1/3}\left(\ddot{V} - \frac{\dot{V}^2(t)}{6V}\right) = \frac{1}{\rho_0}\left(\left(p_0 + 2\sigma\left(\frac{4\pi}{3V_0}\right)^{1/3} - p_{\mathrm{v}}\right)\left(\frac{V_0}{V}\right)^{\kappa} + p_{\mathrm{v}} - 2\sigma\left(\frac{4\pi}{3V}\right)^{1/3} - p_0 - P(t)\text{-}$$

which is the Rayleigh-Plesset equation in terms of the bubble volume, including viscous damping.

5. Conclusions

It has been possible to derive a form of the Rayleigh-Plesset equation from first principles which incorporates viscous damping.

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