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Theory for Acoustic Propagation in Materials which can Support Stress and which contain Gas Bubbles, with Applications to Acoustic Effects in Marine Sediment and Tissue

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UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

FLUID DYNAMICS AND ACOUSTICS GROUP

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Authorized for issue by
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ABSTRACT

Whilst there is a considerable body of work in the literature on the theory of acoustic propagation in marine sediment, the incorporation of gas bubbles into such theories is done with the inclusion of assumptions which severely limit the applicability of those models to practical gas-laden marine sediments.

Section 2 develops a theory appropriate for predicting the acoustically-driven dynamics of a single spherical gas bubble embedded in an incompressible lossy elastic solid. Use of this theory to calculate propagation parameters requires calculation of the gas pressure component of section 2, and the options are outlined in section 3. The incorporation of radiation losses is discussed in section 4. Section 5 discusses how the entire scheme can be incorporated into a nonlinear, time-dependent propagation model.

LIST OF SYMBOLS

c	the sound speed in the solid for compressional waves of infinitesimal amplitude.
C_p	the specific heat of the gas at constant pressure
K_g	the thermal conductivity of the gas within the bubble
p	the sum of all steady and unsteady pressures outside the bubble wall
p_i	the sum of all steady and unsteady pressures in the gas
p_{v}	vapour pressure
p_0	the static pressure in the liquid just outside the bubble wall
$p_\infty(t)$	the value of p very far from the bubble
$R(t)$	bubble radius
R_0	equilibrium bubble radius
R_ε	the radial displacement of the bubble wall
T	Gas temperature
T_{rr}, $T_{\theta\theta}$ and $T_{\phi\phi}$	the components of the stress tensor in the solid
$\vec{u}$	the liquid particle velocity.
$\bar{u}_g$	radial velocity in the gas
γ	ratio of specific heats for the gas
ε_{rr}	the component of the strain tensor in the radial direction

λ_s and G_s	Lamé constants
ρ	liquid density
ρ_g	density in the gas
σ	the surface tension
η	shear viscosity of the liquid
η_B	bulk viscosity of the liquid
η_s	shear viscosity of the solid
$\sum \vec{F}_{ext}$	the vector summation of all body forces

1 Introduction

Whilst there is a considerable body of work in the literature on the theory of acoustic propagation in marine sediment, the incorporation of gas bubbles into such theories is done with the inclusion of assumptions which severely limit the applicability of those models to practical gas-laden marine sediments.

The assumption of quasi-static gas dynamics limits the applicability of the resulting theory to cases where the frequency of insonification is very much less than the resonances of any bubbles present. It also eliminates from the model all bubble resonance effects, which often of are overwhelming practical importance when marine bubble populations are insonified. This limitation becomes more severe as gas-laden marine sediments are probed with ever-increasing frequencies.

The assumption of monochromatic steady-state bubble dynamics, where the bubbles pulsate in steady state, is inconsistent with the use of short acoustic pulses to obtain range resolution.

The assumption of monodisperse bubble populations is inconsistent with the wide range of bubble sizes that are found in marine sediments.

The ubiquitous assumption of linear bubble pulsations becomes increasingly questionable as acoustic fields of increasing amplitudes are used to overcome the high attenuations, and the resulting poor-signal-to-noise ratios (SNRs), often encountered in marine sediments.

This report outlines a theory which does not require the above assumptions. Some assumptions are still maintained, notably that the bubbles in question interact with the sound field through volumetric pulsation. Whilst this does not necessarily mean that the bubbles should be spherical at all times, it is through this assumption that the theory encompasses the volumetric pulsations. It is well-known that there are classes of bubbles in sediment which do not behave in this way (e.g. those which bear a closer resemblance of 'slabs of gas' and 'gas-filled cracks', than they do to gas-filled spheres).

In this first analysis the assumption is also maintained that the sediment outside of each individual bubble may be treated as incompressible. Whilst this greatly eases the analysis, the extent to which it is correct will depend on the characteristics of the sediment. The result of this assumption is that acoustic radiation damping is neglected. Furthermore the sediment outside of the bubble is assumed to be a lossy elastic solid, and no bubble-bubble interactions are assumed to occur.

It should be noted that this analysis is also relevant to acoustic propagation through tissue, provided that the latter can be treated as an incompressible lossy elastic solid.

Section 2 will develop formulation appropriate for predicting the acoustically-driven dynamics of a single spherical gas bubble embedded in an incompressible lossy elastic solid. Section 3 will outline the options for evaluating the gas pressure component of section 2, and section 4 discusses the incorporation ofpropagation losses. Section 5 discusses how the entire scheme can be incorporated into a propagation model.

2 Theory for the dynamics of a single gas bubble in an incompressible lossy elastic solid

In the following derivation, the use of the dot notation in this, and the subsequent equations of motion, indicates the use of the material derivative [1§2.2.2], i.e.:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \qquad (1)$$

where $\vec{u}$ is the liquid particle velocity. For the discussion of the pulsation of a single bubble whose centre remains fixed in space, as occurs in this report, the convective term (the second term on the right) is zero. Before applying the equations of this theory, critical evaluation should be made of their applicability, given this restriction. Models of translating bubbles need careful evaluation. Even where bubbles are assumed to pulsate only, if they exist in a dense cloud then the convective term may be significant [2].

The following derivation relies assumes that the material outside the gas bubble wall is incompressible, and assumes that spatially uniform conditions are assumed to exist within the bubble.

When these assumptions are applied for the case of a gas bubble in a liquid, the equations for the conservation of energy within the liquid can be coupled to that of the diffusion of dissolved gas within it, and to the equation for conservation of mass in the liquid:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}\cdot\vec{u} = 0$$
$$\Rightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0 \qquad . \tag{2}$$

This is the well-known equation of continuity, where $\vec{u}$ is the liquid particle velocity and ρ is the liquid density; and to an equation for conservation of momentum in the liquid, specifically the Navier Stokes equation:

$$\rho\frac{D\vec{u}}{Dt} = \rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) = \rho\sum\vec{F}_{ext} - \vec{\nabla}p + \left(\frac{4\eta}{3} + \eta_B\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \eta\vec{\nabla}\times\vec{\nabla}\times\vec{u} \tag{3}$$

where p represents the sum of all steady and unsteady pressures.

Equation (3) simplifies in a number of ways for limits which are often appropriate to gas bubbles in water [1§2.3.2]. The term $\eta\vec{\nabla}\times\vec{\nabla}\times\vec{u}$ encompasses the dissipation of acoustic energy associated with, amongst other things, vorticity, and hence is zero in conditions of irrotational flow (required for the definition of a velocity potential). The term $\left(4\eta/3+\eta_B\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$ represents the product of viscous effects (through the shear η and bulk η_B viscosities of the liquid), with the gradient of $\vec{\nabla}\cdot\vec{u}$ (which, from (2), represents in turn the liquid compressibility). As an interaction term, it is generally small. Note that setting it to zero does not imply that all viscous effects are neglected, but simply that they appear only through the boundary condition. Lastly, the term $\sum \vec{F}_{ext}$ represents the vector summation of all body forces which are neglected in the formulations of this report. If it is then assumed that the bubble remains spherical at all times and pulsates in an infinite body of liquid, then because of spherical symmetry, the particle velocity in the liquid $\vec{u}$ is always radial and of magnitude *u(r,t)*, and equations (2) and (3) reduce, respectively, to:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho u)}{\partial r} = 0 \tag{4}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \,. \qquad \textit{(Euler's equation)} \tag{5}$$

The situation is somewhat different for a single gas bubble in an incompressible lossy elastic solid. The bubble radius $R(t)$ oscillates about some equilibrium radius R_0 with bubble wall velocity $\dot{R}(t)$. Euler's equation for liquids must be modified for solids as follows

$$\rho_s\left(\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{rr}\right) - \frac{T_{\theta\theta} + T_{\phi\phi}}{r} \tag{6}$$

where ρ_s is the bulk density of the solid material outside of the bubble wall, u_s is the particle velocity in the elastic solid and T_{rr}, $T_{\theta\theta}$ and $T_{\phi\phi}$ are the components of the stress tensor. Note that because the trace of the stress tensor is zero in elastic solids (as it also is in Newtonian liquids), the following relationship will be assumed valid [3]:

$$T_{rr} = -(T_{\theta\theta} + T_{\phi\phi}) \,. \tag{7}$$

Equation (6) will now be integrated through the solid (from R to $r=\infty$), using the assumption of liquid incompressibility, which implies that:

$$u_{\mathrm{s}}(r,t)=\frac{R^2(t)}{r^2(t)}\dot{R}(t) \tag{8}$$

where the bubble has radius *R(t)* and wall velocity $\dot{R}(t)$. The integration process can be divided into a series of smaller integrals:

$$\int_R^\infty \rho_{\mathrm{s}}\frac{\partial u_{\mathrm{s}}}{\partial t}\mathrm{d}r=\int_R^\infty \frac{\rho_{\mathrm{s}}}{r^2}\frac{\partial(R^2(t)\dot{R}(t))}{\partial t}\mathrm{d}r=\int_R^\infty \rho_{\mathrm{s}}\left(\frac{R^2\ddot{R}+2R\ \dot{R}^2}{r^2}\right)\mathrm{d}r \tag{9}$$

$$=\rho_{\mathrm{s}}\left[\frac{-R^2\ddot{R}-2R\ \dot{R}^2}{r}\right]_R^\infty=\rho_{\mathrm{s}}(R\ \ddot{R}+2\dot{R}^2).$$

$$\int_R^\infty \rho_{\mathrm{s}}u_{\mathrm{s}}\frac{\partial u_{\mathrm{s}}}{\partial r}\mathrm{d}r=\int_R^\infty \frac{\rho_{\mathrm{s}}}{2}\frac{\partial u_{\mathrm{s}}^2}{\partial r}\mathrm{d}r=\frac{\rho_{\mathrm{s}}}{2}\left(u_{\mathrm{s}}^2(r=\infty,t)-u_{\mathrm{s}}^2(R\ ,t)\right)=-\frac{\rho_{\mathrm{s}}\dot{R}^2}{2}. \tag{10}$$

$$\int_R^\infty \frac{1}{r^2}\frac{\partial(r^2T_{rr})}{\partial r}\mathrm{d}r=\int_R^\infty \frac{r^2}{r^2}\frac{\partial T_{rr}}{\partial r}\mathrm{d}r+\int_R^\infty \frac{T_{rr}}{r^2}\frac{\partial r^2}{\partial r}\mathrm{d}r \tag{11}$$

$$=\int_R^\infty \frac{\partial T_{rr}}{\partial r}\mathrm{d}r+\int_R^\infty \frac{2T_{rr}}{r}\mathrm{d}r=T_{rr}(r=\infty,t)-T_{rr}(R\ ,t)+\int_R^\infty \frac{2T_{rr}}{r}\mathrm{d}r$$

$$\int_R^\infty -\frac{(T_{\theta\theta}+T_{\phi\phi})}{r}\mathrm{d}r=\int_R^\infty \frac{T_{rr}}{r}\mathrm{d}r \tag{12}$$

Combining these subsidiary integrals allows the integration of (6) to be undertaken from across the solid and liquid phases (i.e. from R to $r=\infty$):

$$\rho_{\mathrm{s}} R\ddot{R} + \frac{3}{2}\rho_{\mathrm{s}}\dot{R}^2 = p_{\mathrm{s}}(R,t) - p_{\infty}(t) + T_{rr}(r=\infty,t) - T_{rr}(R,t) + \int_R^{\infty} 3\frac{T_{rr}}{r}\,\mathrm{d}r \tag{13}$$

noting that for this case, $T_{rr}(r=\infty,t)$ can be taken to equal zero, giving

$$\rho_{\mathrm{s}} R\ddot{R} + \frac{3}{2}\rho_{\mathrm{s}}\dot{R}^2 = \left(p_{\mathrm{s}}(R,t) - T_{rr}(R,t)\right) - p_{\infty}(t) + \int_R^{\infty} 3\frac{T_{rr}}{r}\,\mathrm{d}r \tag{14}$$

The bracketed term on the right of equation (14) can readily be found using the boundary condition at the bubble wall ($r=R$):

$$p_{\mathrm{s}}(R,t) - T_{rr}(R,t) = p_{\mathrm{g}} - \frac{2\sigma}{R} - \frac{\partial\sigma}{\partial R} \tag{15}$$

where σ is the surface tension, and $\partial\sigma/\partial R$ represents a radial force which results from the variation in the concentration of surface active molecules on the bubble wall as the bubble pulsates, although this is normally assumed to be zero [3].

Substitution of (15) into (13) gives:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_{\mathrm{s}}}\left(p_{\mathrm{g}} - \frac{2\sigma}{R} - \frac{\partial\sigma}{\partial R} - p_{\infty}(t) + \int_R^{\infty} 3\frac{T_{rr}}{r}\,\mathrm{d}r \right) \tag{16}$$

which can be readily evaluated to form time histories of the bubble response using the techniques familiar for gas bubbles in liquids, provided that it is possible to determine T_{rr}, the radial component of the stress tensor in the sediment.

The radial component of the stress tensor in the dissipative elastic solid consists of two parts, encompassing respectively the elastic and dissipative characteristics of the solid. The elastic constituent [4] can be expressed in terms of the Lamé constants λ_{s} and G_{s} (the latter also being known as the modulus of rigidity):

$$T_{rr} = (\lambda_s + 2G_s)\frac{\partial \varepsilon_{rr}}{\partial r} + 2\lambda_s \frac{\varepsilon_{rr}}{r} \tag{17}$$

where ε_{rr} is the component of the strain tensor in the radial direction which, for small displacements, is given by:

$$\varepsilon_{rr} = \left(\frac{R}{r}\right)^2 R_\varepsilon \,. \tag{18}$$

where R_ε is the radial displacement of the bubble wall [3]. Note that this solid has been assumed to be incompressible (equation (8)), and for such solids the Lamé coefficient λ_s becomes so large as to be undefined. However, as will be shown later, this does not cause problems in the current calculation.

The second constituent of the radial component of the stress tensor in the dissipative elastic solid $T_{s,rr}$ reflects the losses associated with the internal friction within it. If the velocity gradient is small, the higher order terms can be neglected, and the damping becomes proportional to the first derivative of the velocity [5], $2\eta_s(\partial u/\partial r)$, where η_s is the shear viscosity of the solid. Church [3] notes that this is equivalent to assuming that the dilational viscosity is negligible [6]. The extent to which this is valid in gas-laden sediment will depend on the specific case.

Taking both the elastic and lossy characteristics of the solid together, the radial component of the stress tensor is:

$$T_{rr} = -\frac{4R^2}{r^3}(G_s R_\varepsilon + \eta_s \dot{R}) \tag{19}$$

The assumption of solid incompressibility has caused terms involving the Lamé coefficient λ_s to cancel out, avoiding the problems which could have been caused by its undefined valued for an incompressible solid. The integral for the solid in equation (16) can now be evaluated:

$$\int_R^\infty \frac{3T_{rr}}{r}\mathrm{d}r = -\frac{4}{R}(G_\mathrm{s}R_\varepsilon + \eta_\mathrm{s}\dot{R}) \tag{20}$$

Equation (16) can now be expressed with the integrals evaluated using (20):

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho_\mathrm{s}}\left(p_\mathrm{g} - \frac{2\sigma}{R} - \frac{\partial\sigma}{\partial R} - p_\infty(t) - \frac{4}{R}(G_\mathrm{s}R_\varepsilon + \eta_\mathrm{s}\dot{R})\right). \tag{21}$$

Equation (21) forms the basis of predicting the dynamics of a single bubble in a lossy elastic solid. Section 3 will outline the options for evaluating the gas pressure component of this, and Section 4 discusses how the entire scheme can be incorporated into a propagation model.

3 Methods for calculating the gas pressure and the effect on thermal damping

By far the most common way of calculating p_g (required for evaluation of (20)) is to appeal to a polytropic law). This involves calculating the pressure in the gas at a given bubble size by comparing it with the pressure at equilibrium. The latter is equal to the sum of the static pressure in the liquid just outside the bubble wall (p_0), plus the Laplace pressure at equilibrium $2\sigma/R_0$ (where σ is the surface tension [1§2.1]), minus that component due to vapour (p_v). Hence when the bubble has radius R the pressure in the gas will be:

$$p_\mathrm{g} = \left(p_0 + \frac{2\sigma}{R_0} - p_\mathrm{v}\right)\left(\frac{R_0}{R}\right)^{3\kappa} \tag{22}$$

This adjusts the relationship between bubble volume and gas pressure (effectively, the ‘spring constant’ of the bubble) to account for heat flow across the bubble wall, but crucially it ignores *net* thermal losses from the bubble (see below). Therefore if (21) is evaluated using a polytropic law, the result would, without correction, ignore two of the major sources of dissipation: net thermal losses and, through the incompressible assumption, radiation losses. Approximate corrections, which artificially enhance the viscosity to account for thermal and radiation damping, are available through use of enhancements to the viscosity [3], although these are only partially effective. These enhancements are based on the same physics as the 'linear' damping coefficients

A more accurate option, which would keep the nonlinear character of (21) uncompromised, would be obtained by combining the continuity and energy relations for a perfect gas with spatially uniform pressure (p_i) to provide an exact expression for the velocity field in terms of the temperature gradient. This reduces the problem to an ordinary differential equation for the internal pressure, with a nonlinear partial differential equation for the temperature field, for a bubble which is spherical at all times. Furthermore, if it is assumed that vapour effects are negligible, and that the bubble wall temperature does not change (an assumption which can be justified by estimating temperature changes when the heat flux from the thermal boundary layer in the gas is equated to that entering the boundary layer just beyond the bubble wall), then these two assumptions effectively make consideration of the effect of thermal dissipation on p_g primarily an issue of the gas dynamics. For most common cases, it is acceptable to assume a constant meniscus temperature equal to the undisturbed liquid temperature, with *T(r,t)* representing the time-varying temperature field within the bubble [9]. If the density and radial velocity in the gas are ρ_g and $\vec{u}_g$ respectively (there are no tangential velocity components), then, the continuity equation for the gas is:

$$\frac{D\rho_g}{Dt} + \rho_g \vec{\nabla} \cdot \vec{u}_g = 0 \qquad (23)$$

and the equation for the conservation of energy is

$$\rho_g C_p \frac{DT}{Dt} + \left.\frac{\partial \rho_g}{\partial t}\right|_p \frac{T}{\rho_g}\frac{Dp_i}{Dt} = \vec{\nabla}\cdot(K_g \vec{\nabla} T) \tag{24}$$

where viscous heating in the gas in neglected; where C_p is the specific heat of the gas at constant pressure, which in this derivation is assumed to be constant[1]; and where the thermal conductivity of the gas within the bubble, K_g, is a function of the gas temperature [7, 8]:

$$\frac{K_g}{[WK/m]} = 2.6526\times 10^{-4}\frac{T^{0.74}}{[K]} \tag{25}$$

Recall that only a single value $p_i(t)$ is required to describe completely the spatially uniform pressure in the bubble, and that the notation indicates use of the convective derivative. Applying a perfect gas law having constant specific heat at constant pressure

$$\rho_g C_p T = \frac{\gamma p_i}{\gamma - 1} \tag{26}$$

$$\left.\frac{\partial \rho_g}{\partial T}\right|_p = -\frac{\rho_g}{T} \tag{27}$$

to the combination of the two conservation laws ((23),(24)), integration of the spherically symmetric system gives the radial velocity field in the gas:

$$u_g = \frac{1}{\gamma p_i}\left((\gamma - 1)K_g\frac{\partial T}{\partial r} - \frac{r\dot{p}_i}{3}\right) \tag{28}$$

in terms of the temperature gradient and the convective derivative of the pressure. By applying the boundary condition that u_g must equal the velocity of the bubble wall at

[1] In most studies of non-inertial cavitation it has been enough to assume that the specific heat of the gas is constant. If the gas temperature changes become great, the temperature dependence needs to be included.

the location of the wall, (28) can be recast to give a differential equation for the spatially uniform pressure within the bubble

$$\dot{p}_{\mathrm{i}} = \frac{3}{R}\left((\gamma - 1) K_{\mathrm{g}} \left.\frac{\partial T}{\partial r}\right|_R - \gamma p_{\mathrm{i}} \dot{R} \right) \tag{29}$$

Clearly the temperature gradient needs to be evaluated if (29) is to be of used in a bubble equation of motion. There is flexibility in the route now taken, using for example the equation of continuity coupled with the equation of state of a perfect gas. Alternatively one can use the enthalpy equation in nonconservation form, and by doing so Prosperetti *et al.* [9] obtained (30) from (24):

$$\frac{\gamma}{\gamma - 1}\left(\frac{\partial T}{\partial t} + u_{\mathrm{g}} \frac{\partial T}{\partial r} \right) \frac{p_{\mathrm{i}}}{T} - \dot{p}_{\mathrm{i}} = \vec{\nabla} \cdot (K_{\mathrm{g}} \vec{\nabla} T) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(K_{\mathrm{g}} r^2 \frac{\partial T}{\partial r} \right) \tag{30}$$

Evaluation of (30) requires the radial velocity field from (28), and allowance for the dependence on gas thermal conductivity K_{g} on temperature during the oscillation (25). With these, the pressure within the bubble is calculated, which can be used to resolve the dependency on p_{g} of the various equations of motion. Of the options for numerical integration of this scheme, Prosperetti *et al.* [9] chose a finite-difference, second order predictor-corrector method. Unless an extremely small time step was used, the accumulated error prevented integration over too many cycles. Kamath and Prosperetti [10] describe a collocation method, the Galerkin method with a fixed number of terms, and an adaptive Galerkin method with a variable number of terms (an adaptive Galerkin-Chebyshev spectral method), the latter proving to be the most precise and efficient. The accuracy of the pseudospectral method can be assessed by using the computed temperature field and pressure to calculate the total mass of gas within the bubble [10, 11].

However despite the severe problems associated with the alternative route (*i.e.* the polytropic one, see above) few workers calculate of p_{g} using these formulations. This is perhaps because, unlike the polytropic model, the alternative described above does not provide a simple equation for gas pressure. Instead they give a set of equations to determine average temperature, and then using the perfect gas law to obtain the spatially-averaged pressure. By far the more common route has been to appeal to a polytropic law. This approach will give an answer, but this will contain a degree of inaccuracy (see above) that is rarely quantified.

4 The options for incorporating radiation losses

The analysis of section 2 included the assumption that the solid outside of the bubble wall was incompressible. This assumption eliminates acoustic radiation damping from the formulation, unless it is added back in using an *ad hoc* approach such as through the augmentation of the viscosity to compensate [12, 13].

A more proper approach would be to include into the equation of motion for a bubble a term $R\dot{p}(R,t)/c$, where $p_{\mathrm{s}}(R,t)$ is the sum of all steady and unsteady pressure just outside the bubble wall, and c is sound speed in the solid for compressional waves of infinitesimal amplitude [14, 15]:

$$R\ddot{R}+\frac{3}{2}\dot{R}^{2}=\frac{1}{\rho_{\mathrm{s}}}\left(p_{\mathrm{g}}-\frac{2\sigma}{R}-\frac{\partial\sigma}{\partial R}+\frac{R\dot{p}_{\mathrm{s}}(R,t)}{c}-p_{\infty}(t)-\frac{4}{R}(G_{\mathrm{s}}R_{\varepsilon}+\eta_{\mathrm{s}}\dot{R})\right). \tag{31}$$

where, from (15),

$$p_{\mathrm{s}}(R,t)=p_{\mathrm{g}}-T_{rr}(R,t)+\frac{2\sigma}{R}+\frac{\partial\sigma}{\partial R}. \tag{32}$$

This is one of several possible approaches [16- 20].

5 Incorporating this formulation into an acoustic propagation simulation

Once (21) or (31) has been used to obtain radius time history data for bubbles, an acoustic propagation simulation can be constructed which incorporates nonlinear time-dependent bubble oscillations. Key to evaluation of (21) and (31) is the choice of the method for calculating the gas pressure (section 3) and selection of G_s and η_s for the gassy sediment in question. Whilst estimates of these might be obtained from the literature, it is vitally important to appreciate the assumptions inherent in their calculation, so as to avoid compromising (21) and (31) (for example by inserting values of G_s and η_s which have been calculated for a sediment under assumption of quasi-static bubble dynamics, which compromises the efforts to avoid having to make such an assumption through section 2).

Having through (21) or (31) evaluated radius/time histories, the bubble population can be divided into appropriate size bins, and a representative bubble size allocated for each bin. For each representative bubble, volume/pressure plots can be derived in the manner outlined by Leighton *et al*. [21]. Summation of the volumes of these provides the attenuation, which can be calculated for the steady-state or for short pulses, and the sound speed through use of the spines of these loop.

In calculating the attenuation, it is important to appreciate that if the polytropic law of section 3 is used, thermal losses will not be included (unless a ‘thermal viscosity’ is calculated). Furthermore, the assumption of incompressibility in the solid precludes the inclusion of acoustic radiation losses from (21) (unless an ‘acoustic viscosity’ is calculated).

Therefore if (21) on its own is used, the only losses associated with the bubble motion are viscous losses at the bubble wall. If the gas pressure is calculated through use of (28) to (30), then thermal losses are also included. If, instead of (21), equation (31) is used as the equation of motion for the bubble, then acoustic radiation losses will also be included.

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