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SIGNAL PROCESSING & CONTROL GROUP

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by

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Abbreviated title: Spatial-temporal correlation

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Abstract

The spatial correlation function of the sound in a diffuse field is a quantity widely used in many reverbrant room acoustic applications. Although results for the spatial and temporal correlation for pure tone and narrow-band diffuse fields have already been developed, these have not been generalised for other signal types. This work presents a generalised derivation of the diffuse field spatial-temporal correlation which can be used for sound fields generated by stationary broadband signals with given power spectral density. It is shown that the spatial-temporal correlation depends entirely on the temporal-correlation of the plane waves composing the diffuse field, and that the temporal correlation of the diffuse field is the same as the temporal correlation of these plane waves. A simulation using the plane wave model is presented to verify the theoretical results for tonal and broadband diffuse sound fields.

PACS numbers:

I. Introduction

The model of diffuse sound fields is widely used in the analysis of the sound in enclosures. In particular, the spatial and temporal correlation of the sound in a diffuse field is a useful measure that finds applications in areas from reverberation room measurements [1],[2] to active control of sound [3]. The spatial and temporal correlation function for a diffuse pure-tone sound field was first derived and tested experimentally by Cook *et al* [4]. A plane wave model was assumed for the sound field, with waves arriving from all directions, and a spatial-averaging calculation was used for the correlation function. It was shown that the spatial correlation behaves as $sin(kr)/kr$ function for waves arriving from three dimensions, with k and r denoting the wave number and distance, respectively. In the same work the spatial correlation for an acoustic signal made from a band of frequencies was also computed as the average over frequency of the single-frequency spatial correlation function, and was shown to contain high-order terms of kr. A more detailed calculation of the correlation functions for tonal and narrow-band fields can be found in [2] and [5]. Jacobsen [5] presented a comprehensive analysis of the diffuse field and a computation of the correlation function for both spatial and temporal variations.

Although previous work provided useful results for the pure tone case and the narrow-band noise case, no result for the spatial correlation of a more general sound signal appears to exist. This could be useful in cases where the sound is generated by a broadband random noise source, for example, with a given power spectral density. This paper presents a derivation of the spatial-temporal correlation function in a diffuse field for such a sound signal, and shows that the spatial-temporal correlation is an explicit function of the temporal auto-correlation of the signal generating the diffuse field. The correlation functions for pure-tone and narrow-band sound fields are then shown to be special cases of the more general formulation. The paper is concluded with simulations that verify the theoretical results.

II. Generalised spatial-temporal correlation

A. Derivation based on the plane waves model

The plane wave model of a diffuse field assumes an infinite number of plane waves, arriving uniformly from all directions, with random amplitudes and phases. In this analysis it is assumed more generally that plane waves arriving from different directions are uncorrelated to each other, due to the spatial behaviour of the diffuse field. It is also assumed that each plane wave has the same temporal auto-correlation, and hence power spectral density, which can be justified by assuming that all waves originate from the same acoustic source(s), and that the sound is stationary with respect to both time and space.

The pressure at position r_1 and time t_1 is therefore an infinite summation of all plane waves written as [1]:

$$p(r_1, t_1) = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} p_n(r_1, t_1) \tag{1}$$

Where p_n is a single plane wave arriving from a given direction.

The spatial-temporal correlation function can now be computed by taking the expectation over many samples of diffuse field, where it is assumed that the field is stationary over both time and space, so that:

$$R(r_1,r_2;t_1,t_2)=R(r,\tau)=E\{p(r_1,t_1)p(r_2,t_2)\} \tag{2}$$

where $r=|r_2-r_1|$ and $\tau=|t_2-t_1|$. It should be noted that to compute the correlation coefficient, $\rho(r,\tau)\in[-1,1]$, the correlation function is divided by the power of the signal, which is given by $R(0,0)$. The correlation function can now be written by substituting equation (1) in equation (2), as:

$$R(r,\tau)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N}E\{p_n(r_1,t_1)p_m(r_2,t_2)\} \tag{3}$$

Since plane waves from different directions are assumed uncorrelated, only terms in which $n=m$ contribute, and the correlation function can be written as:

$$R(r,\tau)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}E\{p_n(r_1,t_1)p_n(r_2,t_2)\}=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}R_n(r,\tau) \tag{4}$$

A spherical co-ordinate system is next assumed, as in Figure 1, with $\boldsymbol{r}_1$ placed at the origin, such that $\boldsymbol{r}_1=0$, and $\boldsymbol{r}_2$ placed along the z-axis a distance r from $\boldsymbol{r}_1$, as shown in Figure 2. Since it is assumed that the diffuse field is spatially stationary, the spatial correlation depends only on the relative distance between the two locations, and not the actual locations. The above locations were chosen to simplify the mathematical formulation. The infinite summation, in the limit, converges to a double integral over θ and ϕ, covering all plane-waves arrival directions. The area element for a sphere with a unit radius in this case is $\sin\theta d\theta d\phi$ [8], which replaces the area element of $4\pi/N$ in the summation. The correlation function can therefore be written as:

$$R(r,\tau)=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}R_{\phi,\theta}(r,\tau)\sin\theta\cdot d\phi\cdot d\theta \tag{5}$$

where $R_{\phi,\theta}$ is the correlation function for the plane wave arriving from direction (ϕ,θ). The distance r as in Figure 2 is now transformed to a propagation delay for each individual plane wave, which depends on the angle θ, as follows:

$$\tau_\theta = \frac{r}{c}\cos(\theta) \tag{6}$$

Since all waves are assumed to carry a signal with the same temporal auto-correlation function, the correlation $R_{\phi,\theta}$, can be written using equation (6) as:

$$R_{\phi,\theta}(r,\tau) = R_{\phi,\theta}(0,\tau-\tau_\theta) = R_0(\tau-\tau_\theta) \tag{7}$$

Substituting equation (7) in equation (5), the correlation function can be written as:

$$R(r,\tau) = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} R_0\left(\tau - \frac{r}{c}\cos\theta\right)\sin\theta \cdot d\phi \cdot d\theta \tag{8}$$

Equation (8) suggests that the correlation of the diffuse field can be calculated as the spatial average over ϕ and θ of the of the plane-wave correlation function. This results was used previously for the calculation of the correlation of a pure-tone diffuse field [2],[4], where as the result above is more general to any given plane-wave correlation, and includes temporal variations.

The integral in equation (8) with respect to ϕ reduces to a constant of 2π, and the equation is written as:

$$R(r,\tau) = \frac{1}{2}\int_0^{\pi} R_0\left(\tau - \frac{r}{c}\cos\theta\right)\sin\theta d\theta \tag{9}$$

The parameter t is introduced below as:

$$t = \tau - \frac{r}{c}\cos\theta \tag{10}$$

and is substituted in the integral, with $dt = \frac{r}{c}\sin\theta d\theta$, to produce the following equation:

$$R(r,\tau) = \frac{c}{2r}\int_{\tau-\frac{r}{c}}^{\tau+\frac{r}{c}} R_0(t)dt \tag{11}$$

This equation shows that the spatial-temporal correlation of the pressure in a diffuse field is entirely dependent on the temporal auto-correlation of the signal carried by the plane waves composing the diffuse field.

It should be noted that this expression is not well defined for r=0, so in practice the integral must first be calculated before the value of the correlation at r=0 can be evaluated. A special case of this equation is the correlation dependent on space only, with τ=0, which is written as:

$$R(r,0) = \frac{c}{2r}\int_{-\frac{r}{c}}^{\frac{r}{c}} R_0(t)dt \tag{12}$$

B. Frequency domain formulation

The temporal auto-correlation of the signal carried by the plane waves can be written in terms of the inverse Fourier transform of the power spectral density of the signal [6] as:

$$R_0(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0(\omega)e^{j\omega\tau}d\omega \tag{13}$$

Substituting this equation for the auto-correlation in equation (11), the spatial-temporal correlation in a diffuse field can be written as:

$$R(r,\tau)=\frac{c}{2r}\int_{\tau-\frac{r}{c}}^{\tau+\frac{r}{c}}R_0(t)dt=\frac{c}{4\pi r}\int_{\tau-\frac{r}{c}}^{\tau+\frac{r}{c}}\int_{-\infty}^{\infty}S_0(\omega)e^{j\omega t}d\omega dt$$
$$=\frac{c}{4\pi r}\int_{-\infty}^{\infty}S_0(\omega)\int_{\tau-\frac{r}{c}}^{\tau+\frac{r}{c}}e^{j\omega t}dtd\omega=\frac{c}{2\pi r}\int_{-\infty}^{\infty}S_0(\omega)\frac{1}{\omega}e^{j\omega\tau}\sin\left(\omega\frac{r}{c}\right)d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\omega)sinc\left(\frac{\omega r}{c}\right)e^{j\omega\tau}d\omega \quad (14)$$

This shows that the spatial-temporal correlation is the inverse Fourier transform of the power spectral density of the signal exiting the plane waves when weighted by the spatial *sinc* function.

An interesting result is derived for the special case of r=0 in this equation, so that the temporal auto-correlation of the acoustic signal in the diffuse sound field becomes:

$$R(0,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\omega)e^{j\omega\tau}d\omega=R_0(\tau) \quad (15)$$

This result clearly shows that the temporal correlation of the signal in the diffuse sound field is identical to the temporal correlation of the signal carried by the individual plane waves composing the diffuse field. This result suggests that $R_0(\tau)$ in equation (11) can be replaced by $R(0,\tau)$ which is the temporal auto-correlation of the acoustic signal in the diffuse field.

III. Examples

A. Pure tone diffuse field

Some special cases can be calculated from the integral equation for the spatial correlation as derived above. For example, if the pressure signal of the plane waves composing the diffuse field is a pure tone at frequency ω, with a normalised temporal auto-correlation function given by $R_0(\tau)=\cos(\omega\tau)$, the spatial-temporal correlation of the sound in the diffuse field is calculated using equation (11) as:

$$
\begin{aligned}
R(r,\tau) &= \frac{c}{2r}\int_{\tau-\frac{r}{c}}^{\tau+\frac{r}{c}}\cos(\omega t)dt = \frac{c}{2r}\frac{1}{\omega}\left[\sin\left(\omega\tau+\omega\frac{r}{c}\right)-\sin\left(\omega\tau-\omega\frac{r}{c}\right)\right] \\
&= \frac{c}{\omega r}\sin\left(\omega\frac{r}{c}\right)\cos(\omega\tau) = \frac{\sin(kr)}{kr}\cos(\omega\tau)
\end{aligned}
\tag{16}
$$

where $k=\omega/c$. This result is widely known and has been previously derived for a pure tone diffuse field [1],[4],[5]. A plot of this correlation function for τ=0 as a function of *kr* is presented in Figure 3 (dashed curve).

B. Narrow-band diffuse field

A narrow-band random noise is considered next, with a power spectral density of $2\pi/(\omega_2-\omega_1)$ at a frequency range between ω_1 and ω_2. For simplicity, τ is set to zero, and the spatial correlation can be computed using equation (14) as:

$$
R(r,0)=\frac{1}{4\pi}\int_{-\infty}^{\infty}S_0(\omega)sinc(kr)d\omega = \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}sinc(kr)d\omega
\tag{17}
$$

A similar result and some suggestions for solutions were reported in [2] and [4]. Extending this result to include temporal variations, the corresponding spatial-temporal correlation function can be expressed using equation (14) as:

$$
R(r,\tau)=\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}sinc(kr)e^{j\omega\tau}d\omega
\tag{18}
$$

C. Broad-band diffuse field

The formulation derived in this paper provides the means to calculate the spatial-temporal correlation for other pressure signals than pure tone or narrow band noise. Consider a diffuse field composed of plane waves carrying a signal with a power spectral density of a white noise, which is filtered by a low-pass filter with a single pole at ω_0 and hence has a frequency response of $\omega_0/(\omega_0 + j\omega)$. The power spectral density of the signal when normalised to a unit power is then given by $2\omega_0/(\omega_0^2 + \omega^2)$ [6]. The auto-correlation, which is the inverse Fourier transform of the power spectral density is then given by $R_0(\tau) = e^{-\omega_0|\tau|}$ [6]. For simplicity, only the spatial dependence of the correlation function is considered, which is:

$$R(r,0) = \frac{c}{2r}\int_{-\frac{r}{c}}^{\frac{r}{c}} e^{-\omega_0|t|}dt = \frac{c}{r}\int_{0}^{\frac{r}{c}} e^{-\omega_0 t}dt = \frac{\left(1-e^{-\omega_0 r/c}\right)}{\omega_0 r/c} = \frac{\left(1-e^{-k_0 r}\right)}{k_0 r} \qquad (19)$$

where $k_0=\omega_0/c$. Since a normalised signal was used, the value of $R(0)$, found using L'Hopital's rule [8], is unity in this case. Figure 4 shows the power spectral density of the low-pass filtered plane wave signal as a function of the normalised frequency ω/ω_0, and Figure 3 shows the spatial correlation of the diffuse field in this case (solid curve). This is also compared to the correlation of a pure tone diffuse field at a frequency ω_0.

The correlation of the diffuse field for other types of signals can be examined using the analysis of the low-pass filtered signal. For example, increasing the cut-off frequency ω_0 to infinity can approximate a white-noise signal. In this case $k_0 r$ will be infinitely large for any value of $r>0$, with a corresponding infinitesimally small value for the correlation. The correlation is therefore unity for $r=0$ and zero elsewhere, making the field spatially uncorrelated for any $r>0$.

IV. Simulations

Diffuse field simulations were performed to verify the theoretical results obtained for the spatial correlation. A diffuse field model was used as in equation (1) where a number of spatially uniformly distributed plane wave were superimposed to simulate the diffuse field. Overall 1145 plane waves were used to compose the diffuse field, where plane waves arriving from different directions were designed to be uncorrelated, as detailed below. Spherical co-ordinates were used to generate the diffuse field, as detailed in the Appendix. Each plane wave carried a signal that was constructed in the time domain as a sampled sequence, using a sampling frequency of 100kHz to allow for accurate time resolution. The propagation delay of the plane waves was then simulated by shifting the signal an appropriate number of samples. As in the theoretical analysis, the spatial correlation along direction $\boldsymbol{r}$ for $\theta=0$ was chosen for simplicity. The spatial correlation was calculated using equation (2), for $\tau=0$, normalised by the power of the signals, where the expectation operation was approximated by averaging 10,000 samples of different diffuse fields.

In the first simulation a pure tone diffuse field was constructed, with the individual plane waves having pure tone waveforms at 500Hz, each with a complex amplitude generated by two independent normally distributed random variables corresponding to the real and imaginary parts of the complex magnitude [5],[7]. Figure 5 shows the theoretical and simulated spatial correlation as a function of the distance in wavelengths. The simulation results follows well the theoretical *sinc*(kr) function.

In the second simulation each plane wave carried a time domain signal generated by filtering normally distributed white noise with a low-pass filter having a single pole at

ω_0=2π500 rad/sec, and a frequency response function $\omega_0/(\omega_0 + j\omega)$. Since each plane wave was generated using a different random variable, the plane waves were all uncorellated. The auto-correlation of the plane waves, however, is decaying exponentially with time, with $R_0(\tau) = e^{-\omega_0|\tau|}$, as suggested above. Figure 6 shows the spatial correlation calculated theoretically, as in equation (19) (dashed curve) and the correlation obtained using the simulation. The simulation results follows the theoretical $\frac{1-e^{-k_0 r}}{k_0 r}$ function accurately, which supports the theoretical derivation presented above.

Conclusions

A derivation for the spatial-temporal correlation of a sound signal with arbitrary power spectral density in a diffuse field was developed in this paper. This is calculated as the integral over the temporal correlation or the power spectral density of the signal exiting the diffuse field. The formulation is therefore general and can be used for sound signals with any given spectrum. Experimental verification of the theoretical results for broadband diffuse fields, and the use of this formulation in applications involving diffuse sound fields such as reverbrant room measurements and active sound control, are suggested for future study.

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Appendix

The diffuse field in this study was simulated by N planes waves arriving from uniformly distributed directions in space. Spherical co-ordinates were used, as in [5] (see Figure 1), to represent the diffuse field as a double summation over the angles θ_i and ϕ_j, as follows:

$$p(r_1,t_1) = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{i=1}^{I} \sum_{j=1}^{J} p_{i,j}(r_1,t_1) \tag{20}$$

where I is the integer value of $\sqrt{\frac{\pi N}{4}}$, and J is the integer value of $2I \sin\theta_i$, such that the plane wave $p_{i,j}$ arrives from the direction:

$$(\theta_i, \phi_j) = \left(\frac{i\pi}{I}, \frac{2j\pi}{J} \right) \tag{21}$$

Using this formulation, the plane waves are spatially uniformly distributed. The semi-circle over the angle θ will have I waves, while the full circle over ϕ for θ=0 will have $2I$ waves, and a number decreasing by $\sin\theta$ to account for the smaller diameter of the circles around ϕ for smaller values of θ. In this case a total number of N waves compose the diffuse field.

It should be noted that the formulation with I being the integer value of $\sqrt{N}$, and J the integer value of $\frac{\pi^2 I}{8}\sin\theta_i$, as suggested in [5], will result in a slightly higher density of waves in the θ direction compared to the ϕ direction. The resulting waves will therefore not be completely uniformly distributed in space. This, however, did not affect the final spatial correlation result as presented in [5].

Equations (14) and (15) were used to generate the diffuse fields in the simulations above, with I=30, so that a total of N=1145 plane waves composed the diffuse field.

Figure captions

Figure 1 The spherical co-ordinates system

Figure 2 Incident plane wave arriving at positions r_1 and r_2 distance r apart.

Figure 3 The spatial correlation of the diffuse field composed of plane waves carrying a low-pass filtered random noise (solid curve), and plane waves carrying tonal signal at the same frequency as the low-pass filter cut-off frequency (dashed curve).

Figure 4 The power spectral density of the low-pass filtered signal carried by the plane waves as a function of the normalised frequency ω/ω_0.

Figure 5 The spatial correlation of the diffuse field for pure tone excitation, theoretical result (dashed curve) and simulated result (solid curve).

Figure 6 The spatial correlation of the diffuse field for low-pass filtered random excitation, theoretical result (dashed curve) and simulated result (solid curve).

Figures

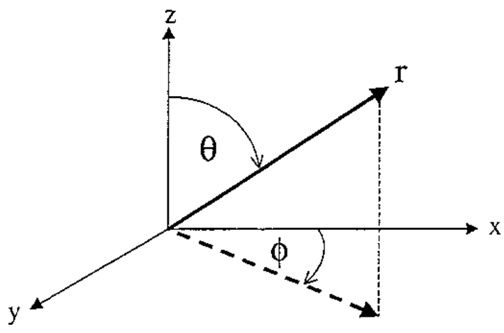

Figure 1

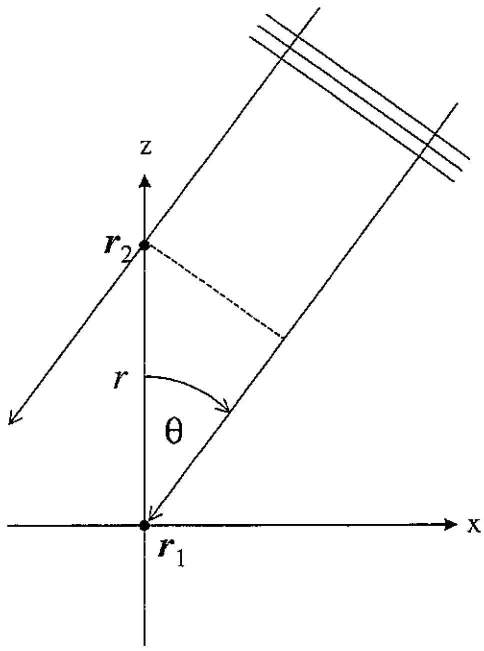

Figure 2

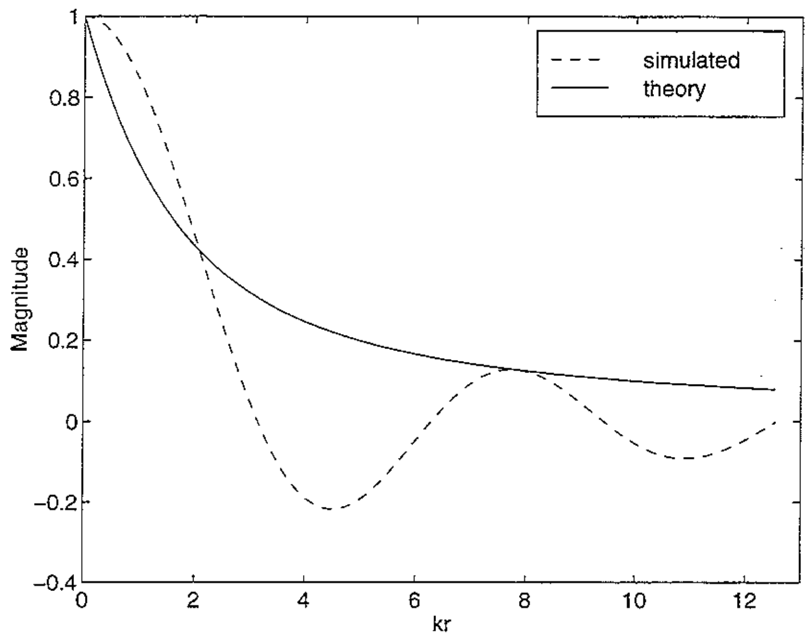

Figure 3

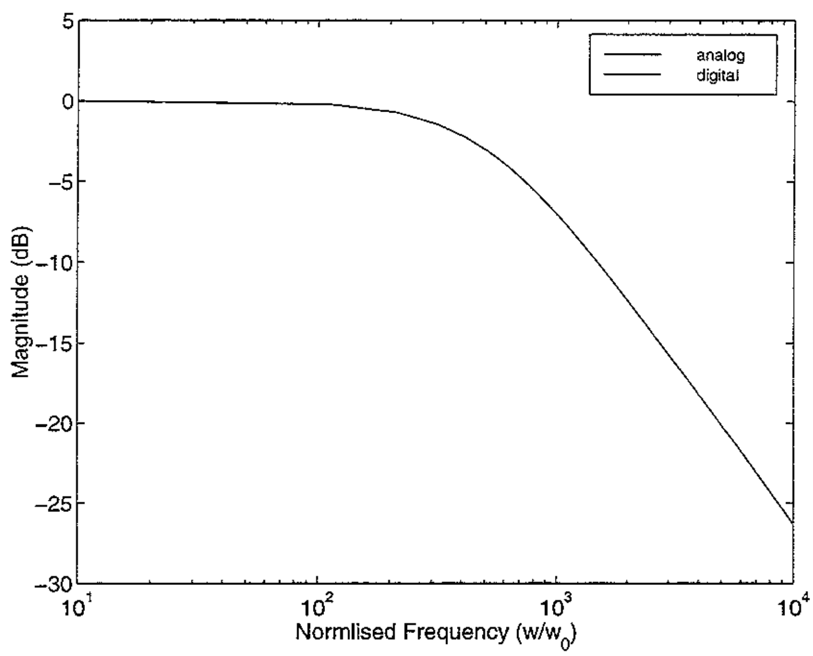

Figure 4

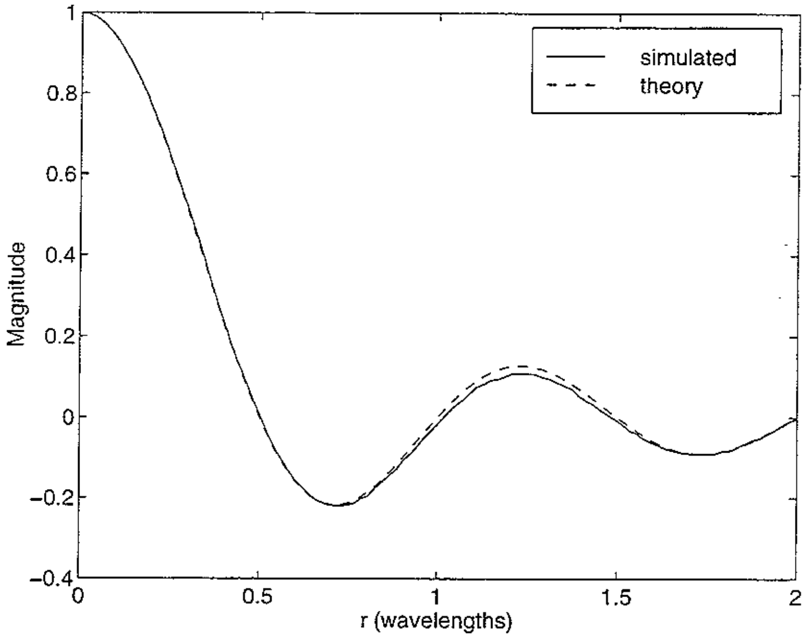

Figure 5

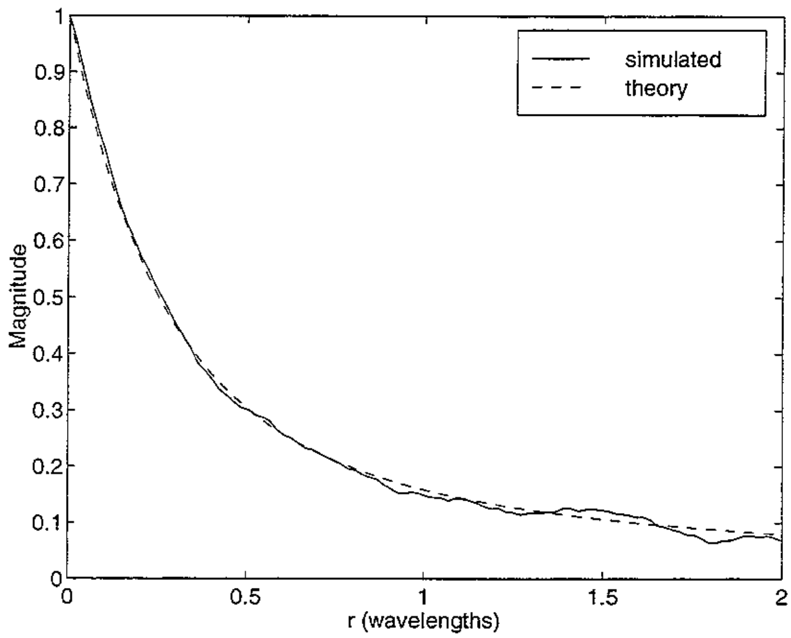

Figure 6