isvr Institute of Sound and Vibration Research

Coupling Strength as an Indicator of the Applicability of Statistical Energy Analysis

S. Finnveden

ISVR Technical Report No 268

May 1997

University of Southampton

SCIENTIFIC PUBLICATIONS BY THE ISVR

Technical Reports are published to promote timely dissemination of research results by ISVR personnel. This medium permits more detailed presentation than is usually acceptable for scientific journals. Responsibility for both the content and any opinions expressed rests entirely with the author(s).

Technical Memoranda are produced to enable the early or preliminary release of information by ISVR personnel where such release is deemed to the appropriate. Information contained in these memoranda may be incomplete, or form part of a continuing programme; this should be borne in mind when using or quoting from these documents.

Contract Reports are produced to record the results of scientific work carried out for sponsors, under contract. The ISVR treats these reports as confidential to sponsors and does not make them available for general circulation. Individual sponsors may, however, authorize subsequent release of the material.

COPYRIGHT NOTICE

(c) ISVR University of Southampton All rights reserved.

ISVR authorises you to view and download the Materials at this Web site ("Site") only for your personal, non-commercial use. This authorization is not a transfer of title in the Materials and copies of the Materials and is subject to the following restrictions: 1) you must retain, on all copies of the Materials downloaded, all copyright and other proprietary notices contained in the Materials; 2) you may not modify the Materials in any way or reproduce or publicly display, perform, or distribute or otherwise use them for any public or commercial purpose; and 3) you must not transfer the Materials to any other person unless you give them notice of, and they agree to accept, the obligations arising under these terms and conditions of use. You agree to abide by all additional restrictions displayed on the Site as it may be updated from time to time. This Site, including all Materials, is protected by worldwide copyright laws and treaty provisions. You agree to comply with all copyright laws worldwide in your use of this Site and to prevent any unauthorised copying of the Materials.

UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

STRUCTURAL DYNAMICS GROUP

Coupling Strength as an Indicator of the Applicability of Statistical Energy Analysis

by

S. Finnveden

ISVR Technical Report No. 268

May 1997

© Institute of Sound & Vibration Research

ABSTRACT

The response of two spring-coupled elements is investigated in an attempt to develop a unifying approach to the weak coupling criterion for applying Statistical Energy Analysis (SEA). Recently, Bessac, Gagliardini and Guyader presented a series expansion technique for analysis of high frequency vibrations in spring coupled structures while Fahy and James proposed a measurement method for indicating SEA coupling strength. In each article a new measure of coupling strength is presented. Here, the similarity between these two measures is demonstrated. They are both related to the concept of 'local' and 'global' modes and, in effect, they are equivalent to the measure previously proposed by Finnveden and Mace. As a consequence it is possible to further our understanding of SEA. The study is mainly a compilation of previous results while some new results for the transient response of two coupled oscillators are derived. Also, in an intermediate step of the analysis, the "standard" SEA equations for the energy flow between two weakly coupled, statistically defined, single d.o.f. elements are formulated, for the first time.

1. INTRODUCTION

Statistical Energy Analysis (SEA) is a convenient approximate method for predicting high frequency vibrations in built-up structures. It is quick to apply and requires only general structural information. It is believed that SEA applies if there are many resonances in each element and if these are weakly coupled. However, there is no universally agreed criterion for coupling strength. Therefore, much effort has been put into investigation of the SEA hypothesis that energy flow between two directly coupled elements is proportional to the difference in their modal energies. Recently, Bessac, Gagliardini and Guyader [1] and Fahy and James [2] published investigations of the high frequency energy flow between two multi-modal elements. In each article a new and very interesting measure of coupling strength is presented. The object of this work is to show the similarity between these two measures. It will be shown that, in effect, they are equivalent to the measure previously proposed by Finnveden and Mace [3, 4, 5]. As a consequence, it is possible to further our understanding of Statistical Energy Analysis.

When applying SEA, the investigated structure is divided into 'elements' and equations for the conservation of energy are formulated with the vibration energies as independent variables. This formulation is facilitated by two assumptions: 1) In each element the response is determined by its 'resonances', that is, vibrations are described by those of the elements' eigenmodes having resonance frequencies within the considered frequency band; 2) The energy flow between coupled elements is proportional to the difference of their modal energy, (i.e., Coupling Power Proportionality).

The first assumption enables us to define an 'element'. In many cases it enables response calculations (e.g., velocity, sound pressure or stress levels) from SEA results. The first assumption is related to the (perhaps not rigorously defined) concept of 'local' and 'global' modes. Local modes are those proper to an isolated element. In coupled structures only global modes exist; that is, they are solutions to an eigenvalue problem for the entire structure. However, modes may be localised, meaning that all or most vibration energy in a mode may be found within an element. Since it is localised, not much energy is exchanged at the boundary. Hence, in principle, it is possible to find conservative boundary conditions for an element such that the local modes' resonance frequencies and their mode shapes are not much different from those of the global modes. In such a situation assumption 1) is approximately fulfilled and one is entitled to say that the structure's response is given by its local modes.

The second assumption, of coupling power proportionality, is based on the result derived by Lyon and Scharton showing it to be exactly true for the frequency-averaged energy flow between two conservatively coupled oscillators [6]. This result is then, in an *ad hoc* manor, extended to apply to the coupling between sets of oscillators. The assumption is fundamental to the formulation of the SEA equations. It is also the basis for the calculation of the parameters in these equations. Moreover, because of this assumption, these calculations are simplified since it is then implicit that the energy flow between two directly coupled elements could be considered while neglecting the rest of the structure.

First, in Section 2, some previous work on SEA and aspects of the recent articles by Bessac et. al. [1] and Fahy and James [2] are discussed. Subsequently, in Section 3, by combining the analysis by Bessac et al. [1] with those by Langley [7] and Skudrzyk [8], for two point-coupled elements, the second SEA assumption is proved for weak coupling. A criterion for the validity of this assumption is determined. It is found to be the same as that which applies for ensemble averages in one-dimensional systems [3-5]. It is then also demonstrated that the coupling measure defined by Fahy and James is, in effect, equivalent to this measure. Because of the equality of the measures of coupling strength discussed, the significance of the measurement method developed by Fahy and James is enhanced. With this method it is possible to perform simple in situ measurements to determine whether or not SEA applies. Finally, while the second SEA assumption appears to be the more suspect (historically most effort has been made to prove or find limits for the coupling power proportionality), it is argued that the coupling strength measure indicated in references [1-5, 7] is essentially related to the first SEA assumption. That is, it is argued that if the responses in two coupled elements are with some effectiveness described by their uncoupled resonant modes, coupling is weak and SEA applies.

2. PREVIOUS WORK

In this section some results from the literature and some minor reflections are assembled. All are useful in the analysis in Section 3.

2.1 Response in a single element to broad band excitation

It is well known that the frequency averaged input power to a weakly, viscously, damped single-degree-of-freedom oscillator from a white force excitation of bandwidth Ω is given by

$$P_{in} \approx \frac{1}{\Omega}\int_{\Omega} |f|^2\, Y\, d\omega = |f|^2 \frac{\pi}{2m\Omega}, \quad \omega_1 \in \Omega,$$
$$P_{in} \approx 0, \quad \omega_1 \notin \Omega, \tag{1}$$

where

$$\omega_1 = \sqrt{k/m}, \quad Y = -i\omega\big/\left(k - i\omega c - \omega^2 m\right), \tag{2}$$

where a time dependence $e^{-i\omega t}$ is implicitly assumed and where f is the rms complex amplitude of the force, Y is the input mobility, k is the spring stiffness, c is the viscous damping coefficient and m is the mass of the oscillator. Equation (1) is asymptotically exact in the limit of zero damping. It is very accurate for small damping in which case it is also valid for mass- or stiffness- proportional damping.

In SEA, the one-oscillator result is generalised to apply for an element, described by its local modes

$$P_{in} = \langle f^2 \rangle Y_c ; \quad Y_c = \frac{\pi}{2} \frac{n}{m} ; \quad n = \frac{N}{\Omega} , \tag{3}$$

where $\langle f^2 \rangle$ is the mean square modal force on the element, m is the mass, N is the number of resonance frequencies within the frequency band Ω and n is the modal density. Y_c is for large elements equal to the characteristic mobility, the mobility for an infinite element. Equation (3) is derived for rain-on-the-roof excitation or as the expected value of the input power for a point force at a random location within the element. (Derivations are found, e.g., in [9, Section IV, 4] and in [10].)

Equation (3) applies for frequency band averages. Skudrzyk [8] showed that at higher frequencies (above the first few resonances) the space-averaged point mobility for a particular frequency is bounded by

$$Y_c \, \pi \, M/2 \leq Y \leq Y_c / (\pi \, M/2) , \tag{4}$$

where M is the modal overlap, that is, the ratio of the bandwidth of a resonance to the average frequency spacing between resonances. $\pi \, M/2$ is the modal overlap based on the noise bandwidth whereas M is based on the 3 dB bandwidth

$$M = \eta \omega \, n , \tag{5}$$

where η is the loss factor. The upper limit in equation (4) is reached at resonance and the lower at anti resonance. It should be noted, these limits are not exact but depend on the position in the element. However, for uniform elements (beams, plates, air-volumes, etc.) the upper limit does not much differ from the one given.

2.2 SEA OF THE ENERGY FLOW BETWEEN TWO ELEMENTS

In SEA, the independent variables are the modal energies

$$Em = E/n , \tag{6}$$

where E is the vibration energy of an element within a frequency band. The governing equation is that of conservation of energy which is written

$$\begin{bmatrix} M_1 + C & -C \\ -C & M_2 + C \end{bmatrix} \begin{bmatrix} Em_1 \\ Em_2 \end{bmatrix} = \begin{bmatrix} P_{in1} \\ P_{in2} \end{bmatrix} , \tag{7}$$

where the coupling coefficient C is defined by this equation. In the new edition of Lyon and DeJong's book [11] it is proposed that the modal energies, defined by equation (6) should be termed 'modal power potential'. To further this suggestive definition, it is here proposed that the non-dimensional parameter C should be termed the 'modal energy conductivity', or, when the context is given, simply the 'conductivity'.

For two elements, coupled at a point via a spring, the conductivity is [10, equation (23a)]

$$C = \frac{2}{\pi}\frac{\langle \mathrm{Re}(Y_1)\rangle \langle \mathrm{Re}(Y_2)\rangle}{\left|\langle Y_1\rangle + \langle -i\omega/k_c + Y_2\rangle\right|^2} = \frac{2}{\pi}\frac{Y_{c1}\,Y_{c2}}{\left|\langle Y_1\rangle + \langle -i\omega/k_c + Y_2\rangle\right|^2}\ , \tag{8}$$

where k_c is the coupling spring stiffness, and where $\langle\ \rangle$ denotes that appropriate averaging (frequency, space, ensemble, etc.) should be performed.

For two end-coupled rods, C is, in the limit of large elements, given by [5]

$$C = \frac{2}{\pi}\frac{Y_{c1}\,Y_{c2}}{Y_{c1}^2 + Y_{c2}^2 + (\omega/k_c)^2}\ . \tag{9}$$

If coupling is weak in the sense that the mobility of the spring is much higher than the characteristic mobilities of the elements, both equations (8) and (9) simplify to

$$C = \frac{2k_c^2}{\pi\omega^2}\,Y_{c1}\,Y_{c2}\ . \tag{10}$$

2.3 DERIVATION OF SEA EQUATIONS

Langley derived equations similar to the SEA equations for linearly vibrating systems with conservatively coupled elements which are rain-on-the-roof excited [7]. The coupling coefficients are found from the solution to the equations of motion for the entire system which, for practical applications, is a drawback. However, the theoretical implications are immense.

The independent variables in Langley's formulation of the equation of conservation of energy are

$$\hat{E}_i = \pi\,\hat{T}_i \Big/ \int_\Omega \int_{V_i} \mathrm{Re}(\rho\,Y_i)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\omega\ , \tag{11}$$

where $\hat{T}_i$ is the kinetic energy in element i within the frequency band Ω, V_i is the element's 'volume', ρ is the mass density and Y_i is the point mobility.

As shown by Langley, if the mass density is uniform, and if the first SEA assumption is fulfilled, that is, if the point mobility of each element is given with sufficient accuracy by those of its uncoupled modes that have resonance frequencies within the considered frequency band, then equation (3) applies and the independent variables are

$$\hat{E}_i = 2\hat{T}_i / N_i\ , \tag{12}$$

where N_i is the number of modes having resonance in the frequency band Ω. If the response is given by resonant modes then to a good approximation $2\hat{T}_i = E_i$, where E_i is the total vibration energy within the frequency band.

Thus, using the results by Langley, it is possible to define weak coupling by the requirement that the frequency and space averaged point mobility in each element in a structure is governed by its resonant modes and that it is, within the accuracy required, unaffected by the presence of coupled elements. If these criteria are fulfilled for a two-element structure, the coupling power proportionality is proved; the second SEA assumption is demonstrated to be a consequence of the first SEA assumption. For a many-element structure, Langley's analysis shows that if the first assumption is fulfilled the conservation of energy is expressed with coupling powers proportional to the difference of modal energies. However, the importance of the indirect couplings is still an unsettled question [5, 12, 13].

2.4 ENSEMBLE-AVERAGED ENERGIES IN ONE-DIMENSIONAL STRUCTURES

Mace [4] and Finnveden [5] calculated ensemble averaged energies and energy flows in structures with one-dimensional end-coupled elements. The ensembles are defined so that at each frequency, and for each uncoupled element, there is an equal probability that the nearest resonance is above or below this frequency. This definition implies that the ensembles are, for each element, defined via stochastic Helmholtz numbers

$$kL = \theta + \phi, \;\; \theta = \langle kL \rangle, \;\; \phi = \mathrm{R}[-\pi/2, \pi/2], \tag{13}$$

where k is the wavenumber, L is the element length, θ is the expected value of the Helmholtz number and ϕ is a rectangular distributed stochastic variable. The energies and energy flows are stochastic functions and their expected values are calculated on this basis. Mace considered two-element structures with arbitrary wave-guide elements in which only one wave-type may propagate [4]. Finnveden restricted the analysis to elements described by the Helmholtz equation while considering three elements [5]. The four major findings, relevant for the present analysis, are: (*i*) The relations between ensemble averaged energies and energy flows depend only on non-dimensional numbers: C (one for each coupling) and M_i (one for each element). (*ii*) If the factor γ is small,

$$\gamma = C/(\pi \, M_1 \, M_2), \tag{14a}$$

that is if

$$\gamma < 1, \tag{14b}$$

the so-called 'travelling wave' estimates, e.g., equation (10), apply for the two-element structure. If γ is larger than unity, the coupling factors decrease and, if the modal overlaps M_1 and M_2 are of the same order, this decrease is directly proportional to the square root of their product. (Recently, this conclusion has been confirmed by Yap and Woodhouse for plates [14].) (*iii*) For the three-element structure, when γ is small for both couplings, the travelling wave estimate of the modal energy conductivity applies, and the indirect couplings vanish. That is, only when the inequality (14b) is fulfilled, the coupling power proportionality applies for the ensemble averages in a three-element structure [5]. (*iv*) As shown by Mace [15], the criterion (14b) marks a qualitative change in the characteristics of the energy flow between two coupled reverberent elements. When coupling is weak according to the criterion (14b), the response in each element is dominated by its local modes; a directly excited element may, significantly, only accept energy from the source via its local resonances and, similarly, a coupled element may only accept energy from another element via its own resonances. Thus, if γ is small the ensemble averaged energy flow between two elements is governed by those members of the ensemble that have at least one pair of equal uncoupled resonance frequencies. On the contrary, when γ is large, energy is both accepted and transferred at the resonance frequencies of the coupled systems, these differing from the uncoupled frequencies. The conclusion is that the coupling measure γ signifies, from an ensemble-averaged point of view, whether response is local or global.

The conclusions presented above are further enhanced by the discussion in the recent article by Mace [16]. (Please note, Mace uses a different expression for γ; for reverberant elements (and if C is not close to its limit value one) this expression is to a fair approximation $\gamma_{mace} = \sqrt{\gamma}$. For the qualitative discussion here this difference is of no importance.)

2.5 COUPLING EIGENVALUES

Bessac, Gagliardini and Guyader investigated two coupled elements, each having a motion described in terms of its uncoupled modes [1]. These modes form a complete basis and may therefore be used to express the solutions to the coupled equations of motion. It is shown that the response of the coupled structure is given by

$$[\mathbf{I}-\mathbf{C}]\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix} = \begin{bmatrix}\mathbf{w}_{b1}\\ \mathbf{w}_{b2}\end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix}\mathbf{0} & \mathbf{C}_{1.2}\\ \mathbf{C}_{2.1} & \mathbf{0}\end{bmatrix}, \tag{15}$$

where the vector $\mathbf{w}_i$ contains the modal velocity amplitudes in element i, $\mathbf{w}_{bi}$ contains the uncoupled modal amplitudes, $\mathbf{I}$ is an identity matrix and $\mathbf{C}$ is a non-dimensional coupling matrix. If the magnitude of $\mathbf{C}$ is small, that is, if

$$\lambda = \max\left|\text{eigenvalues}(\mathbf{C})\right| < 1, \tag{16}$$

it is possible to use a series expansion to solve equation (15),

$$\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix} = \left[\mathbf{I}+\mathbf{C}+\mathbf{C}^2+\cdots\right]\begin{bmatrix}\mathbf{w}_{b1}\\ \mathbf{w}_{b2}\end{bmatrix}. \tag{17}$$

It should be pointed out that this series expansion method has previously been used to solve the same problem by, e.g., Newland [17]. However, in reference [1] a criterion for the convergence of (17) is derived. The special case of spring coupling, is considered. It is shown that in this case the magnitude of λ is given by

$$\lambda^2 = \left|(k_c/\omega)^2 \; Y_1 Y_2\right|. \tag{18}$$

The authors then investigate coupling involving many degrees of freedom and seek solutions of the form (17) using only the coupling path, the part of **C**, that is given by its largest eigenvalue. In the present work, only point coupling via one spring is considered; however, the analysis in [1] implies that the results possibly have wider applications.

2.6 TRANSIENT EXCITATION OF COUPLED OSCILLATORS

Pinnington and Lednik investigated the application of the transient version of SEA for prediction of failure due to shock excitation of equipment mounted on larger structures [18]. The equipment, and the structure on which it is mounted, are modelled as single d.o.f. system, see Figure 1. Of special interest for the secondary system are the total vibration energy, maximum amplitude, the rate of initial rise in vibration energy and the time to peak energy (of which only the latter, the 'rise time', is of interest here). Analysis was made for the case of 'equal' resonators, having equal uncoupled resonance frequencies and loss factors, but not necessarily the same mass. The initial conditions are

$$x_1 = x_2 = \dot{x}_2 = 0, \quad \dot{x}_1 = 1/m_1,$$

where the 'dot' signifies the time derivative. Upon considering the envelope of the secondary oscillator's kinetic energy, it was found that its maximum energy occurred at a time t_m given by (please note the print error in [18])

$$\tan(\chi\, t_m/2\omega_1) = \chi/\eta_1\,\omega_1^2, \tag{19}$$

where

$$\eta_i = c_i / m_i \omega_i, \quad \omega_i = \sqrt{(k_i + k_c)/m_i}, \quad \chi = k_c / \sqrt{m_1 m_2}. \tag{20}$$

The exact SEA coupling loss factor for two spring-coupled oscillators is given by [11]

$$\omega\eta_{12} = \frac{\chi^2\;(\omega_1\eta_1 + \omega_2\eta_2)}{(\omega_1^2 - \omega_2^2)^2 + (\omega_1\eta_1 + \omega_2\eta_2)(\omega_1\eta_1\omega_2^2 + \omega_2\eta_2\omega_1^2)}. \tag{21}$$

From this it follows that the right hand side of equation (19), for equal oscillators, is

$$\chi/\eta_1\,\omega_1^2 = \sqrt{2\eta_{12}/\eta_1}. \tag{22}$$

The 'Smith criterion' for coupling strength in SEA is that $\eta_{12}/\eta_1 << 1$ [19]. Thus, it was in [18] concluded that the rise time is very dependent on coupling strength as defined in this manor. In the limits of very strong or very weak coupling it is given by

$$\omega t_m = 2/\eta_1, \ \eta_{12}/\eta_1 << 1; \quad \omega t_m = \pi \omega_1^2/\chi, \ \eta_{12}/\eta_1 >> 1. \tag{23}$$

It is seen that for equal resonators, for very weak coupling the rise time is independent of the strength of the coupling spring while for very strong coupling it is independent of damping.

2.7 Measurements of SEA coupling strength

The observation that the transient response of a secondarily excited system is very dependent on coupling strength is explored by Fahy and James in developing a measuring technique for in-situ assessment of the correct sub-division of a structure into SEA-elements [2, 20, 21]. The method's principle is motivated by the Langley definition of coupling strength: coupling is said to be weak if the Green's function for a system is approximately equal to that for the uncoupled system. It is argued that the quite dramatic qualitative change in the impulse response when the coupling strength is altered is a good indicator of the degree of modification of the Green's function.

In [2] numerical experiments are made considering coupled rods, beams and plates. The physical coupling strength is altered, showing that if there is a reasonable overlap of uncoupled resonances in the connected elements, and if coupling is weak, the rise time is very long while, otherwise, it is only of the order of a few cycles or less. A normalised measure of the rise time, C_s, is introduced as the ratio of the rise time to the time until the energy has decayed to 10% of its peak value.

In [20] measurements on coupled plates and connected reverberation rooms are presented. For the plates the physical coupling strength is altered while for the reverberation rooms the damping is modified. These measurements confirm the trends indicated by the numerical experiments. However, attempts to match the results to the Smith criterion of coupling strength are not entirely successful. Large values of C_s are correlated to the ratio η_{12}/η_1 but not in a systematic manner.

3. ANALYSIS

3.1 WEAK COUPLING IN SEA, TWO ELEMENT STRUCTURE

For the series expansion in equation (17) to be useful, it must converge for all frequencies; hence, also when there is resonance in both connected elements. Combining equations (4), (5), (16) and (18) it is seen that the convergence is certain if

$$\lambda^2 = \left|\frac{k_c^2}{\omega^2} Y_1 Y_2\right| \leq \frac{k_c^2}{\omega^2} \frac{Y_{c1} Y_{c2}}{M_{n1} M_{n2}} = \frac{2}{\pi} \frac{C}{M_1 M_2} = 2\gamma < 1 , \qquad (24)$$

where C is given in equation (10) and where γ is defined by equation (14a). It is the coupling strength measure proposed by Finnveden and Mace [3, 4, 5].

This means that if γ is small the local modes are effective in describing the response of the coupled system: the series in equation (17) converges rapidly. Hence, if γ is small, the first SEA assumption is fulfilled and equation (12) is asymptotically accurate. Consequently, the analysis by Langley proves that the energy balance is governed by the SEA equation written as in [13, equation (6)], with coupling loss factors defined by [13, equation (2)].

Here, it is preferred to use the modal energy, defined by equation (6), as independent variable since the modal density does not in general depend on the bandwidth, and, more important, this scaling renders the matrix in equation (7) non-dimensional. Using this scaling in Langley's work, the energy variable in equation (12), [13, equation (3)] is replaced by the modal energy defined in equation (6) and the matrices $\overline{\mathbf{M}}$ and $\mathbf{q}$, [13, equations (4) and (5)] are evaluated as frequency averages, that is they are divided by Ω. Upon this basis the energy balance for a two-element structure is written as in equation (7).

Thus, the application of SEA for the energy balance between two spring-coupled multi-modal elements is demonstrated. It is found that this application is valid when the non dimensional number γ , defined in equation (14), is small.

3.1.1 *Moderately strong coupling*

Above, results in SEA are derived directly from a deterministic analysis without using any statistical assumptions or, even, without accounting for the smoothing effect of frequency averaging. The results were derived on the assumption that $\lambda << 1$. In equation (17), neglecting all but the leading order terms results in a relative error that is $o(\lambda^2) = o(\gamma)$. Including the coupling terms result in modifications, which are of this order, of the mode shapes, damping and resonance frequencies. However, the frequency- and space- averaged point mobility in equation (3), and thus also equation (12), is hardly altered by such modifications, unless they are large. This observation is supported by the results in [22] showing that for three cascade-coupled rods SEA predicts ensemble averages accurately if $\gamma < 1$, implying that $\gamma << 1$ is much too severe a criterion.

In the present work, we are content to derive results for very weak coupling and to demonstrate a criterion by which the validity of this assumption could be assessed. Quantitative estimates of the errors resulting for moderate or strong coupling, or the quantitative implications of stochastic assumptions in SEA, are not dealt with.

3.1.2 *Other coupling conditions*

It should be noted that the rapid convergence of (17) is only a sufficient condition for weak coupling - not a necessary condition. For instance, consider two large thin flexurally vibrating plates which are connected via a rigid, very heavy, mass. These elements are, from an energetic point-of-view, weakly coupled. However, if the free plate modes are used in (15) to express the solution to the coupled equations of motion, for each frequency a large number of modes are required to fulfil the condition of almost blocked motion at the heavy mass. Thus, the series expansion (17) converges slowly (if at all) even though the elements are weakly coupled. If, instead, equation (15) is formulated with fixed-base modes, having blocked motion at the mass, convergence of (17) is rapid. The implication of this is that when calculating SEA coupling factors and using a weak coupling assumption, as in (10) or in [13, equation (43)], care must be taken so that the uncoupled modes obey appropriate boundary conditions. This, of course, also applies when, as explained above, the effectiveness of the uncoupled modes in describing coupled motion is used as an indicator of SEA coupling strength.

3.1.3 *Transient analysis for weak coupling*

If $\gamma << 1$, it is sufficient to calculate the response of a directly excited element to zero order in coupling strength, neglecting connected elements. Similarly, the response of a second, indirectly excited, element may be found considering only the first order terms in coupling strength, using the calculated response in the first element as a velocity-source at the end of the spring; this, for weak coupling, results in a force-source on the element. This mode of analysis, when $\gamma << 1$, applies for all frequencies, hence it also applies for transient analysis. This approach is used by James and Fahy in investigating theoretically the efficiency of the rise times as an indicator of coupling strength [21]. Here, the criterion $\gamma << 1$ for this application is demonstrated.

3.2 WEAKLY COUPLED OSCILLATORS

The series expansion in equation (17) also applies for sets of spring-coupled oscillators provided that equation (24) is fulfilled. Using the results in Section 2.1

$$M_i = \eta_i \omega n_i, \quad Y_{ci} = \frac{\pi}{2}\frac{n_i}{m_i}; \quad n_i = \frac{N_i}{\Omega}. \tag{25}$$

From this, the conductivity C and the coupling strength measure γ are

$$C = \frac{2}{\pi}\frac{k_c^2}{\omega^2} Y_{c1} Y_{c2} = \frac{\pi}{2}\frac{\chi^2}{\omega^2}\frac{N_1 N_2}{\Omega^2}, \tag{26}$$

$$\gamma = C/(\pi M_1 M_2) = \chi^2/(2\omega^4 \eta_1 \eta_2) ,$$

where χ is defined in equation (20).

3.2.1 *Two coupled oscillators*

The equations above apply also for two coupled oscillators having their resonance frequencies within the frequency band Ω. In this case $N_1 = N_2 = 1$. The expression for the coupling strength measure γ, in equation (26), is independent of the mode count and hence it applies to the two-oscillator problem. Notably, γ is also independent of bandwidth, that is, of the probability that there is resonant coupling.

If two oscillators are weakly coupled according to the criterion $\gamma << 1$, and if their uncoupled resonance frequencies are assumed to be uniformly probable within the frequency band Ω, the energy balance between them is written as in SEA, equation (7), with

$$M_i = \eta_i \omega/\Omega ,$$

$$C = \frac{2\,k_c^2}{\pi\,\omega^2}\,Y_{c1}\,Y_{c2} = \frac{\pi}{2}\frac{\chi^2}{\omega^2\,\Omega^2} . \qquad (27)$$

The generic problem of two statistically defined, weakly coupled, oscillators is the foundation for much of our understanding of the modal approach to SEA. Curiously enough, it appears as if the standard SEA formulation for this problem has not appeared in the literature previously.

3.3 THE RISE TIME AS AN INDICATOR OF THE APPLICABILITY OF SEA

3.2.1 *Undamped single d.o.f. system*

To understand the transient response of a secondarily excited element it is beneficial to study a simple case first - the transient response of an undamped single d.o.f. system. As discussed above, if two sets of spring connected elements are weakly coupled, the motion of the directly excited element is governed by its uncoupled modes. This motion acts via the spring as a force source on the indirectly excited element. If damping is small this force is as a set of harmonic forces with frequencies equal to the uncoupled resonance frequencies. For weak coupling, the response of the indirectly excited element is then governed by its local modes which may be treated as set of independent single d.o.f. oscillators. For times that are small compared to the of the decay time, $\omega t < \eta$: the motion is as if the oscillator had no damping. Hence, the response of one undamped single d.o.f. oscillator describes much of the response of an indirectly excited weakly coupled element.

Considering one such, undamped oscillator, and sinusoidal excitation the equations of motion are

$$\omega_1^2 x + \ddot{x} = f(t)/m_1,$$

$$f(t) = 0, \; t \le 0; \qquad f(t) = f_0 \sin(\omega_2 t), \; t > 0. \tag{28}$$

$$x(0) = \dot{x}(0) = 0$$

The solution is

$$x = \frac{f_0}{m_1 \omega_1} \frac{\omega_1 \sin(\omega_2 t) - \omega_2 \sin(\omega_1 t)}{\omega_1^2 - \omega_2^2}, \quad \omega_1 \ne \omega_2,$$

$$x = \frac{f_0}{2 m_1 \omega_1^2} \left(\sin(\omega_1 t) - \omega_1 t \cos(\omega_1 t) \right), \quad \omega_1 = \omega_2. \tag{29}$$

If $\omega_1 >> \omega_2$ or $\omega_1 << \omega_2$, the amplitude of the oscillator reaches its maximum almost instantaneously. Therefore, for two weakly coupled systems with motion described with their uncoupled modes: if none of the resonances of the two systems are equal, or almost equal, the rise time is very small.

However, if $\omega_1 = \omega_2$, after a few cycles the velocity is

$$v = f_0 / (2 m_1 \omega_1) \; \omega_1 t \sin(\omega_1 t), \tag{30}$$

and the input power, averaged over a cycle, is approximately

$$P_{in} = \frac{1}{T_1} \int_0^{2\pi/\omega_1} f \, v \, \mathrm{d}t = \frac{f_0^2}{4 m_1} t \, ; \quad T_1 = \frac{2\pi}{\omega_1}. \tag{31}$$

Consequently, for an undamped oscillator excited at its resonance frequency, both the velocity and the input power increase linearly with time. This solution is valid also for a weakly damped oscillator as long as $t \ll 1/(\eta_1 \omega_1)$. At larger times, if there is damping a steady state will asymptotically be reached. The amplitude at steady state is given by energy conservation

$$P_{in} = \frac{1}{T_1} \int_0^{2\pi/\omega_1} f\, v \, \mathrm{d}t = P_d = \frac{1}{T_1} \int_0^{2\pi/\omega_1} \eta_1 \, \omega_1 \, m_1 \, v^2 \, \mathrm{d}t. \tag{32}$$

For excitation at the resonance frequency, the force and the velocity are in phase. Assuming this, the steady state velocity is found to be

$$v = f_0/(2 m_1 \eta_1 \omega_1) \sin(\omega_1 t). \tag{33}$$

An estimate of the time to reach steady state is arrived at if the solutions for v in equation (30) and the expression (31) for input power are used in the energy balance equation. Thus

$$P_{in} = \frac{f_0^2}{4 m_1} t = P_d = \frac{1}{T_1} \int_0^{2\pi/\omega_1} \eta_1 \, \omega_1 \, m_1 \, v^2 \, \mathrm{d}t = \eta_1 \, \omega_1 \frac{f_0^2}{8 m_1} t^2 \tag{34}$$

From this, the steady state is reached approximately at a time

$$\omega_1 \, t_m = 2/\eta_1. \tag{35}$$

This, see equation (23), is the rise time for two equal, weakly coupled oscillators. The exact agreement is, of course, a coincidence which perhaps is explained by two counteracting effects. On the one hand, the rise time is underestimated using equation (30) for the velocity which does not account for dissipation, which becomes important in the power balance as steady state is approached. On the other hand, the rise time is overestimated when the exponential decrease in the force excitation, because of damping and coupling loss in the directly excited system, is neglected.

3.3.2 *Two coupled unequal oscillators*

Pinnington and Lednik investigated the transient response of two equal oscillators and found equations (18) and (23) showing that the rise time is dependent on coupling strength [18]. In the previous section it is, for weakly coupled oscillators, indicated that the rise time is also dependent on the frequency spacing between the uncoupled resonance frequencies. Fahy and James define 'modal proximity' to be when spacing between resonances is small enough not to distort the rise time's capability as an indicator of coupling strength [20]. To quantify this and to further our understanding of the relation between rise time and coupling strength the transient response of two unequal oscillators is studied.

For computational ease, damping is described as viscoelastic. Thus, the viscous damping coefficients $c_1 = c_2 = 0$, whereas the relation between force and displacement in the springs are of the form [23]

$$F(t) = k_0 x(t) - \int_0^\infty H(\tau) x(t-\tau)\,\mathrm{d}\tau , \tag{36}$$

The relaxation function H is

$$H(\tau) = \sum_\alpha C_\alpha \, \mathrm{e}^{-\lambda_\alpha \tau} , \tag{37}$$

where C_α and λ_α are parameters governed by the relaxation process in the spring.

For harmonic motion, $x(t) = \tilde{x}\,\mathrm{e}^{-\mathrm{i}\omega t}$, equation (36) simplifies to

$$F = f\,\mathrm{e}^{-\mathrm{i}\omega t} = \tilde{k}\,\tilde{x}\,\mathrm{e}^{-\mathrm{i}\omega t}, \quad \tilde{k} = k(1 - i\eta) = k_0 - \sum_\alpha C_a / (\lambda_\alpha - i\omega) . \tag{38}$$

The equations of motion and the initial conditions for the coupled oscillators are

$$\begin{gathered} F_1 + m_1 \ddot{x}_1 + k_c(x_1 - x_2) = 0, \quad F_2 + m_2 \ddot{x}_2 + k_c(x_2 - x_1) = 0, \\ x_1 = x_2 = \dot{x}_2 = 0, \quad \dot{x}_1 = 1/m_1 \ \text{ at } t = 0. \end{gathered} \tag{39}$$

Harmonic solutions are anticipated and the spring properties, including the loss factors, are assumed to be constant for the considered frequencies, while it is accounted for that the loss factors are odd functions of frequency: $\tilde{k} = k(1 - i\eta\,\mathrm{sign}(\omega))$. Thus, the equations (39) are transformed to a linear eigenvalue problem in ω^2. Upon solving this, and applying the three homogenous initial conditions, the displacement of the second oscillator is

$$x_2 = A_1 \left(\frac{1}{b_1} \mathrm{e}^{-\alpha_1 t} \sin b_1 t - \frac{1}{b_2} \mathrm{e}^{-\alpha_2 t} \sin b_2 t \right) , \tag{40}$$

where the constant A_1 is determined by the non-homogeneous initial condition together with the eigenvectors found, while α_i and b_i are given by minus the imaginary and by the real part of the positive square root of the eigenvalues

$$\begin{aligned} \alpha_{1.2} &= -\operatorname{Im}\sqrt{\tilde{\omega}_s^2 \pm \sqrt{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2/4 + \chi^2}} \;, \\ b_{1.2} &= \operatorname{Re}\sqrt{\tilde{\omega}_s^2 \pm \sqrt{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2/4 + \chi^2}} \;, \end{aligned} \tag{41}$$

where χ is defined in equation (20) and

$$\tilde{\omega}_s = \sqrt{\left(\tilde{\omega}_1^2 + \tilde{\omega}_2^2\right)/2} \;;\; \tilde{\omega}_i^2 = \left(\tilde{k}_i + k_c\right)/m_i = \omega_i^2\left(1 - i\eta_i\right). \tag{42}$$

If damping is small, $\eta_i^2 << 1$, ω_i is as given in equation (20).

The velocity of the oscillator is given by

$$\begin{aligned} \dot{x}_2/A_1 &= \mathrm{e}^{-\alpha_1 t}\left(\cos b_1 t - \alpha_1/b_1 \sin b_1 t\right) - \mathrm{e}^{-\alpha_2 t}\left(\cos b_2 t - \alpha_2/b_2 \sin b_2 t\right) = \\ & -D \sin(\omega_m t + \phi_m)\sin(\omega_p t + \phi_p) + d \cos(\omega_m t + \phi_m)\cos(\omega_p t + \phi_p) \end{aligned} \tag{43}$$

where

$$\omega_p = \left(b_1 + b_2\right)/2, \quad \omega_m = \left(b_1 - b_2\right)/2, \tag{44}$$

Upon assuming $\eta^2 << 1$, it follows that

$$\left(\alpha_1/b_1\right)^2 << 1, \quad \left(\alpha_2/b_2\right)^2 << 1, \tag{45}$$

thus, neglecting these small terms, we have

$$D = \mathrm{e}^{-\alpha_1 t} + \mathrm{e}^{-\alpha_2 t}, \quad d = \mathrm{e}^{-\alpha_1 t} - \mathrm{e}^{-\alpha_2 t}, \tag{46}$$

$$\phi_p = \left(\alpha_1/b_1 + \alpha_2/b_2\right)/2, \quad \phi_m = \left(\alpha_1/b_1 - \alpha_2/b_2\right)/2. \tag{47}$$

To arrive at an interpretable expression for the rise time, simplifying assumptions are made. First of all it is, as above, assumed that damping is small, $\eta^2 << 1$, so that in all expressions only the leading order terms in η need to be retained. Upon this basis, it is also assumed that

$$\frac{\left|\omega_1^2 - \omega_2^2\right|}{\omega_s^2} = o(\varepsilon), \quad \frac{k_c^2}{k_1 k_2} = o(\varepsilon), \quad \varepsilon << 1 \;. \tag{48}$$

For small η and ε, the time scale given by ω_p is much shorter than those given by ω_m and α_i. Then it is justified to calculate the average kinetic energy over one period of ω_p while considering ω_m, D and d constant. Thus

$$\hat{e}_k = \frac{\omega_1}{2\pi} \int_0^{2\pi/\omega_p} e_k(t)\, dt \Big/ \left(A_1^2 m_2 / 2\right) \approx$$

$$D^2/2 \sin^2(\omega_m t + \phi_m) + d^2/2 \cos^2(\omega_m t + \phi_m) = \tag{49}$$

$$e^{-2\alpha_p t}\left(\cosh(2\alpha_m t) - \cos(2\omega_m t + 2\phi_m)\right)$$

where

$$2\alpha_p = \alpha_1 + \alpha_2, \quad 2\alpha_m = \alpha_1 - \alpha_2. \tag{50}$$

From equations (41) and (48)

$$\alpha_p \approx -\mathrm{Im}(\tilde{\omega}_s^2) = \omega_s \frac{\eta_1 + \eta_2}{4} + o(\eta\,\varepsilon), \tag{51a}$$

$$\omega_m \approx \mathrm{Re}\left(\frac{\tilde{\omega}_s}{2}\sqrt{\frac{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2}{4\tilde{\omega}_s^4} + \frac{\chi}{\tilde{\omega}_s^4}}\right) = \frac{\omega_s}{2}\left(\frac{(\omega_1 - \omega_2)^2}{\omega_s^2} + \frac{\chi^2}{\omega_s^4}\right)^{1/2} + o(\varepsilon^3), \tag{51b}$$

$$\alpha_m \approx -\mathrm{Im}\left(\frac{\tilde{\omega}_s}{2}\sqrt{\frac{\left(\tilde{\omega}_1^2 - \tilde{\omega}_2^2\right)^2}{4\tilde{\omega}_s^4} + \frac{\chi}{\tilde{\omega}_s^4}}\right) = \frac{\omega_s}{2}\frac{(\omega_1 - \omega_2)(\eta_1 - \eta_2)}{4\,\omega_m} + o(\eta\,\varepsilon^2) \tag{51c}$$

Also consistent with the assumption (48) is

$$\phi_m = \frac{1}{2}\left(\frac{\alpha_1}{b_1} - \frac{\alpha_2}{b_2}\right) = \frac{1}{4}\frac{(\omega_1 - \omega_2)(\eta_1 - \eta_2)}{\sqrt{(\omega_1 - \omega_2)^2 + \chi^2/\omega_s^2}} + o(\eta\,\varepsilon). \tag{51d}$$

In what follows the small terms are neglected. It is seen, the evolution of the time averaged kinetic energy is governed by three time scales: α_p which is an average damping constant; α_m which is a measure of the difference of the damping constants of the two modes and ω_m which is the beating frequency. To further simplify the expression for the rise time, non dimensional time, T, is introduced

$$T = \alpha_p t = \eta_p \omega_s t/2. \tag{52}$$

This leads to

$$\hat{e}_k(T) = 2e^{-2T}\left(\sinh^2(\mu T) + \sin^2(\kappa T + \phi_m)\right), \tag{53}$$

where

$$\phi_m = \eta_p \mu, \quad \eta_p = (\eta_1 + \eta_2)/2, \quad \mu = \delta \Delta/\kappa$$

$$\delta = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}, \quad \kappa = \sqrt{\Delta^2 + 2\gamma\left(1 - \delta^2\right)}, \quad \Delta = \frac{\omega_1 - \omega_2}{\eta_p \omega_s} \tag{54}$$

and where the coupling strength measure γ is defined in equation (26).

The evolution of the average kinetic energy is thus given by three non-dimensional numbers μ, ϕ_m and κ; μ is a relative measure of the difference in modal damping, ϕ_m is a time 'delay' that occurs if there is a difference in modal damping, and κ is the ratio of the beating frequency to the average damping constant.

3.3.3 *Equal damping*

For equal damping, $\eta_1 = \eta_2$, the expression for the time averaged non dimensional kinetic energy, equation (53), simplifies to

$$\hat{e}_k = 2\, e^{-2T} \sin^2 \kappa T. \tag{55}$$

So, for equal damping, the evolution of the time averaged kinetic energy is described by only one parameter, κ. The non-dimensional rise time, Tm, is determined by equating the time derivative of (55) to zero, resulting in

$$Tm = \operatorname{atan}(\kappa)/\kappa. \tag{56}$$

This function is plotted in Figure 2 where it is seen that Tm is equal to unity for small κ, then at $\kappa \approx 1$ it rapidly decreases to small values.

For equal damping the parameter κ simplifies to

$$\kappa = \sqrt{\Delta^2 + 2\gamma} \tag{57}$$

where γ is the coupling strength measure defined in equation (26) and where the non dimensional number Δ is the ratio of the frequency spacing between uncoupled resonance frequencies to their 3-dB bandwidth.

In Figure 3 the rise time is shown as a function of γ and Δ. For small Δ, the rise time is close to unity if $\gamma \leq 1$ whereas it rapidly decreases for larger γ. For large Δ, the rise time is small regardless off the size of γ. It is seen that if $\Delta = 1$ then $\gamma < 1$ if $Tm > 2/3$ and, put the other way, if $\Delta < 1$ and $Tm > 2/3$ then $\gamma < 1.5$. Similarly, if $\Delta < 2$ and $Tm > 0.5$ then $\gamma < 3$. From the Figure it is concluded that, for equal damping, modal proximity means that the separation between uncoupled resonance frequencies is less than or approximately equal to the 3-dB bandwidth of the resonances. If there is modal proximity, Tm is a good indicator of coupling strength, that is, an indicator of whether γ is smaller or of the order of one, or if it is larger than this.

3.3.4 *Unequal damping*

For unequal damping, in addition to Δ and γ, the parameter δ, specifying the degree of inequality of the modal dampings, is needed to determine the rise time. Also needed is the factor $\phi_m = \eta_p \mu$ which appears in the phase of the beating oscillation. By definition, the magnitude of μ is always less than unity, so ϕ_m is restricted. However, the sign of μ may be just as well positive as negative (dependent on which of the oscillators that has the higher resonance frequency and which of them that has the higher damping). This means that ϕ_m has the annoying property of making the expression for $\hat{e}_k$ unsymmetric with respect to the characteristics of the two oscillators. Clearly, unless $\delta = 0$, ϕ_m is not negligible for small times. Here, we are interested in its significance for the rise time.

For three limiting cases ϕ_m is clearly negligible: 1) If $\kappa >> 1 > |\mu|$, the rise time is approximately given by $\kappa T_m = \pi / 2$ and the small phase shift, $\phi_m < \eta_p$, is of no significance. 2) The same applies, off course, for $\eta_p \to 0$. 3) If $1 > |\mu| >> \kappa$, ϕ_m is negligible since, in this case, for $T << 1/\kappa$

$$\hat{e}_k \approx 2\mathrm{e}^{-2T}\left(\sinh^2 \mu T + (\eta_p \mu)^2\right) + o(\kappa T) \tag{58}$$

and the rise time is

$$Tm \approx \operatorname{atanh}(\mu)/\mu \tag{59}$$

where it is used that, in the present case, $Tm > 1 >> \eta_p^2$.

For all cases but the three discussed above, it appears that numerical investigations have to be made to find the relative importance on the rise time of the parameters ϕ_m, μ and κ. Now, the rise time is determined when the derivative of equation (52) is equated to zero

$$\left[\mu \sinh(\mu\, Tm)\cosh(\mu\, Tm) \;+\; \kappa \sin(\kappa\, Tm+\phi_m)\cos(\kappa\, Tm+\phi_m)\right] -$$

$$\left[\sinh^2(\mu\, Tm) \;+\; \sin^2(\kappa\, Tm+\phi_m)\right] \;=\; 0 \tag{60}$$

This equation is solved numerically. In Figure 4, Tm is shown as function of μ and κ for $\phi_m = \eta_p = 0$. In the figure it is seen that if μ is close to unity while κ is small, Tm is large as stated in equation (59). For all other cases, Tm is virtually independent of μ. That is, if $\gamma << \Delta < 1$, the rise time may become larger than that given by equation (56), otherwise this equation is also valid for very small but unequal dampings.

For $\eta_p > 0$, Tm depends on ϕ_m and it is not a symmetric function of μ. For $\eta_p = 0.01$ and $\eta_p = 0.1$, in Figures 5a and 5b, Tm is shown as a function of $|\mu|$ and κ. In the calculations the sign of μ have been chosen at random. In this manor the irregularity of Figures 5 expresses the errors resulting when ϕ_m is neglected in equation (60). For engineering structures η_p is rarely larger than 0.1. Therefore, from the figures, it is concluded that the rise time is determined with sufficient accuracy also when ϕ_m is neglected.

Neglecting ϕ_m, the rise time is determined by the non dimensional numbers $|\delta|$, $|\Delta|$ and γ. In Figure 3, Tm was shown as a function of $|\Delta|$ and γ for $\delta = 0$ while in Figures 6 it is shown for $\delta = 0.98$. This value should be of the order of the maximum possible value since, in engineering structures, the loss factor is in general limited to $10^{-3} < \eta < 10^{-1}$. Comparing Figures 3 and 6, it appears that the rise time is larger for large δ, especially so when $\gamma << \Delta << 1$. The question, however, is whether small values of the rise time are, ultimately, an indicator of large values of γ.

For various values of Δ, in Figures 7a, 7b and 7c for $\delta = 0, 0.75$ and 0.98, the rise time is shown as a function of γ. In all cases, if Δ is not large then if γ is small the rise time is large. Also, if γ is large the rise time is small. However, if δ is large, the transition between these two states is not at $\gamma \approx 1$ but at larger values.

In equation (54) κ is defined as a function of Δ and

$$\gamma_a \;=\; \gamma\left(1-\delta^2\right) \;=\; \frac{C/\pi}{\left(M_1+M_2\right)^2/4}. \tag{61}$$

In Figure 8 the rise time is shown for $\delta = 0.98$ as a function of γ_a. It is seen, the transition between large and small values of the rise time is at $\gamma_a \approx 1$.

For ensemble averages in one-dimensional structures the most natural descriptor of coupling strength is

$$\gamma = \frac{C/\pi}{M_1 M_2}. \tag{14}$$

The same applies for the deterministic analysis in Section 3.1. However, the coupling strength measure that is indicated by the rise time is apparently γ_a, equation (61). When the dampings of connected elements are of the same order, $\gamma_a \approx \gamma$ and the analysis made demonstrates the link between coupling strength in SEA and the rise time. But, for largely unequally damped elements, there is yet more to discover.

CONCLUSIONS

The response of structures with two spring-coupled elements is investigated in an attempt to develop a unifying approach to the weak coupling criterion for applying SEA. The study is mainly a compilation of previous results while some new results for the transient response of two coupled oscillators are derived. Also, in an intermediate step of the analysis, the "standard" SEA equations for the energy flow between two weakly coupled, statistically defined, single d.o.f. elements are formulated, it is believed, for the first time.

The responses of the elements are described by their 'local' uncoupled modes. These modes are a complete set and may be used to formulate the equations of motion for the coupled structure, as was made by Bessac et. al. [1]. The equations of motion were then solved by a series expansion which is valid for weakly coupled elements and the authors found a criterion for this. Coupling becomes stronger for frequencies and/or systems for which there is resonance in the elements. Here, by using Skudrzyks result for the mobility at resonance [8] (the characteristic mobility divided by the modal overlap) it is found that the rate of convergence of the series is governed by the ratio of the modal energy conductivity to the product of the modal overlaps in the connected elements. This is the parameter γ that previously was found to govern the application of SEA to predict ensemble averages in one dimensional structures [3-5].

If coupling is very weak, the series expansion converges immediately and the space- and frequency- averaged input mobility in an element can be calculated without considering connected elements. This is the assumption that Langley used to derive the SEA equations from exact equations for the energy balance in structures built up of conservatively coupled rain-on-the-roof excited elements [7, 13]. Thus, it is here found that if γ is small, the local modes are effective in describing the response of a coupled structure. It is also found that the criterion for Langley's exact equations for the energy balance to transform to the SEA equations is that there are resonances in the structure and that $\gamma << 1$.

Fahy and James developed a method for *in situ* measurements to determine whether a structure is subdivided according to the weak coupling criterion [2]. This is vital when using SEA in the predictive mode and, even more important, when SEA is used in the inverse mode, for when the elements are strongly coupled, the equations are badly conditioned and measurements of SEA parameters become impossible.

It is first noticed that if $\gamma << 1$, the response of two coupled elements is determined by their local modes. Thus, if one element is excited by an impulse force, the response of this element is given by its modes, acting via the spring as a set of harmonic forces on the other element. The response of this element is governed by its local modes which can be described as a set of single d.o.f. elements. Considering one such element excited by a harmonic force, which started at $t = 0$, it is found that if the excitation frequency is close to the oscillator frequency then the rise time, the time to reach maximum vibration energy, is large. If, however, the frequencies are different, the rise time is small.

Fahy and James define modal proximity to be when the frequency spacing is small enough not to distort the rise time's validity as an indicator of coupling strength [20]. To quantify this, and to find the dependence of coupling strength on the rise time, the transient response of two coupled oscillators is analysed. It is found that, in effect, the coupling strength quantity that determines the rise time is γ . Also it is found that there is modal proximity if the separation between uncoupled oscillator frequencies is of the order of their 3 dB bandwidth, or less. If there is modal proximity, the rise time is large if $\gamma \leq 1$ while it rapidly decreases for larger γ .

It is believed that SEA applies if: 1) the response in each element is governed by its resonances, i.e., by its local modes. 2) Coupling Power Proportionality holds. In this study it is shown that, for spring coupled two-element structures, the second criterion is a consequence of the first. It is speculated: if the first assumption holds, the response of each element is given by its local solutions to the equations of motion (unless the forcing or coupling apply restraints - as, e.g., in periodic structures). The local solutions act independently and interact weakly, otherwise the first assumption would not hold. If, in addition, the local solutions are statistically defined, the interaction is of a random nature and the situation resembles the one treated in statistical mechanics. Hence, the heat conduction analogy for SEA, i.e., Coupling Power Proportionality.

To sum up: it is demonstrated that a deterministic approach to the SEA equations is valid if $\gamma << 1$. It is found that modal proximity, defined by Fahy and James [20], is measured by the separation between uncoupled resonance frequencies divided by their 3 dB bandwidth. If there is modal proximity, the rise time is an indicator of whether $\gamma \leq 1$ or not. Previously, it was found that the standard SEA equations predict the ensemble averaged vibration energies in one dimensional structures if $\gamma \leq 1$ [4, 5]. The coupling strength measure that characterises the solutions to the SEA equations is the ratio of the conductivity to the modal overlap [19]. It is argued, the coupling strength criterion that indicates the applicability of these equations is not based on this ratio but on γ , the ratio of the conductivity to the product of the modal overlaps in the connected elements.

Obviously, there is a gap between the criterion for the deterministic result ($\gamma << 1$) and the one for the stochastic ensemble average result and for the threshold level for the rise time ($\gamma \leq 1$). Perhaps this gap will be bridged, also for other structures than rods and beams, if frequency and ensemble averages are considered.

ACKNOWLEDGEMENT

This study was provoked by, and thrived on, discussions with colleagues at the University of Southampton. Special thanks are due to Dr. Roger Pinnington and Professor Frank Fahy.

REFERENCES

1. F. Bessac, L. Gagliardini and J. -L. Gyader 1996 *Journal of Sound and Vibration* **191**, 881-899. Coupling eigenvalues and eigenvectors: a tool for investigating the vibroacoustic behaviour of coupled vibrating systems.
2. F. J. Fahy and P. P. James 1996 *Journal of Sound and Vibration* **190**, 363-386. A study of the kinetic energy impulse as an indicator of the strength of coupling between SEA subsystems.
3. S. Finnveden 1990 *Proceedings of Inter-Noise* **90,** 945-948. Energy flows within a three element structure with a statistical description of the design parameters.
4. B. R. Mace 1993 *Journal of Sound and Vibration* **166**, 429-461. The statistical energy analysis of two continuous one-dimensional subsystems.
5. S. Finnveden 1995 *Journal of Sound and Vibration* **187,** 495-529. Ensemble averaged vibration energy flows in a three element structure.
6. T. D. Scharton and R. H. Lyon 1967 *Journal of the Acoustical Society of America* **43,** 1332-1343. Power flow and energy sharing in random vibration.
7. R. S. Langley 1989 *Journal of Sound and Vibration* **135**, 499-508. A general derivation of the statistical energy analysis equations for coupled dynamic systems.
8. E. Skudrzyk *Journal of the Acoustical Society of America* **67**, 1105-1135. The mean-value method of predicting the dynamic response of complex vibrators.
9. Cremer L. Heckl M. and Ungar E. E. 1988 *Structure-borne sound.* Springer-Verlag, 2 edition.
10. J. E. Manning 1994 *Phil. Transaction of the Royal Society of London A* **346**, 477-488. Formulation of SEA parameters using mobility functions.
11. R. H. Lyon and R. G. DeJong 1995 *Theory and application of SEA.* Butterworth-Heinemann.
12. C. H. Hodges, J. Woodhouse 1986 *Rep. Prog. Phys.* **49**, 107-170. Theories of noise and vibration transmission in complex structures.
13. R. S. Langley 1990 *Journal of Sound and Vibration* **141**, 207-219. A derivation of the coupling loss factors used in statistical energy analysis.
14. F. F. Yap and J. Woodhouse 1996 *Journal of Sound and Vibration* **197**, 351-371. Investigation of damping effects on statistical energy analysis of coupled structures.
15. B. R. Mace 1992 *Journal of Sound and Vibration* **154**, 321-341. The statistics of power flow between two continuos one-dimensional subsystems.
16. B. R. Mace 1997 *Journal of Sound and Vibration* **199**, 369-380. Wave coherence , coupling power and statistical energy analysis.
17. D. E. Newland 1966 *Journal of Sound and Vibration* **3**, 262-276. Calculation of power flow between coupled oscillators.
18. R. J. Pinnington and D. Lednik 1996 *Journal of Sound and Vibration* **189**, 249-264. Transient statistical energy analysis of an impulsively excited two oscillator system.
19. P. W. Smith Jr. 1979 *Journal of the Acoustical Society of America* **65**, 695-698. Statistical models of coupled dynamical systems and the transition from weak to strong coupling.

20. P. P. James and F. J. Fahy 1995 *ISVR Technical Report No.* 242. A technique for the evaluation of strength of coupling between SEA subsystems: experiments with two coupled plates and two coupled rooms.
21. P. P. James and F. J. Fahy 1995 *ISVR Technical Report No.* 243. Evolution of the energy impulse response in the case of two very weakly coupled subsystems: a mathematical model
22. S. Finnveden 1992 *Proc. 14th Int. Conf. on Acoustics* **D3-7**. Coupling strength estimates based on exact statistical ensemble averages.
23. Y. C. Fung 1965 *Foundations of solid mechanics*. Prentice-Hall.

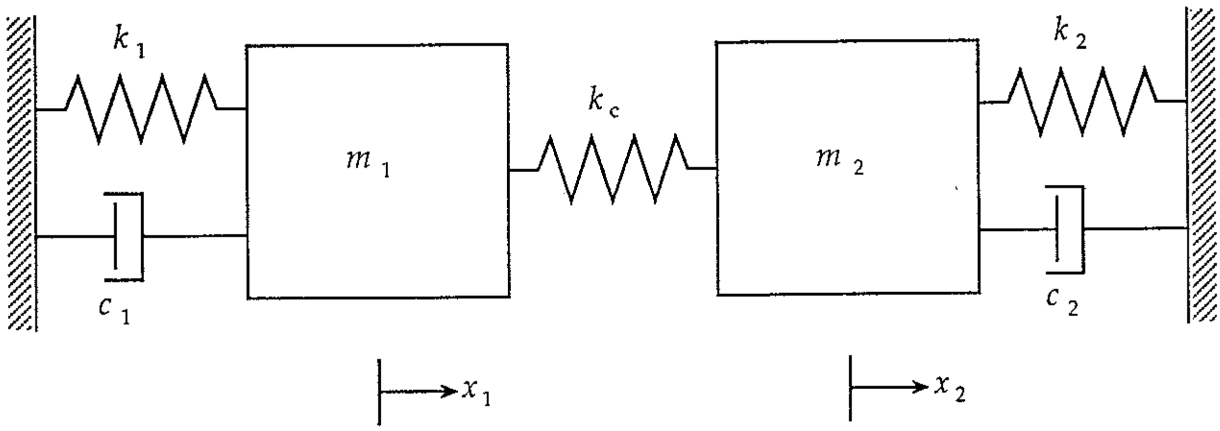

Figure 1. Two spring-coupled oscillators.

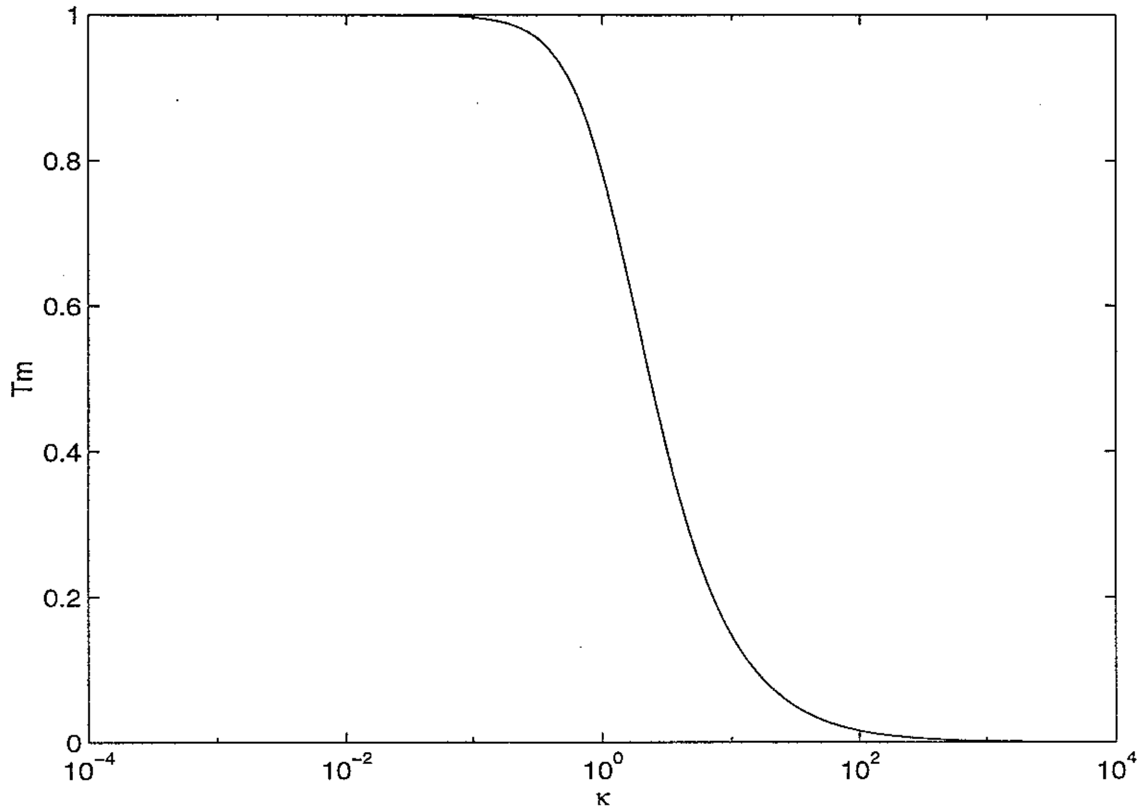

Figure 2. The rise time as a function of κ , equation (56).

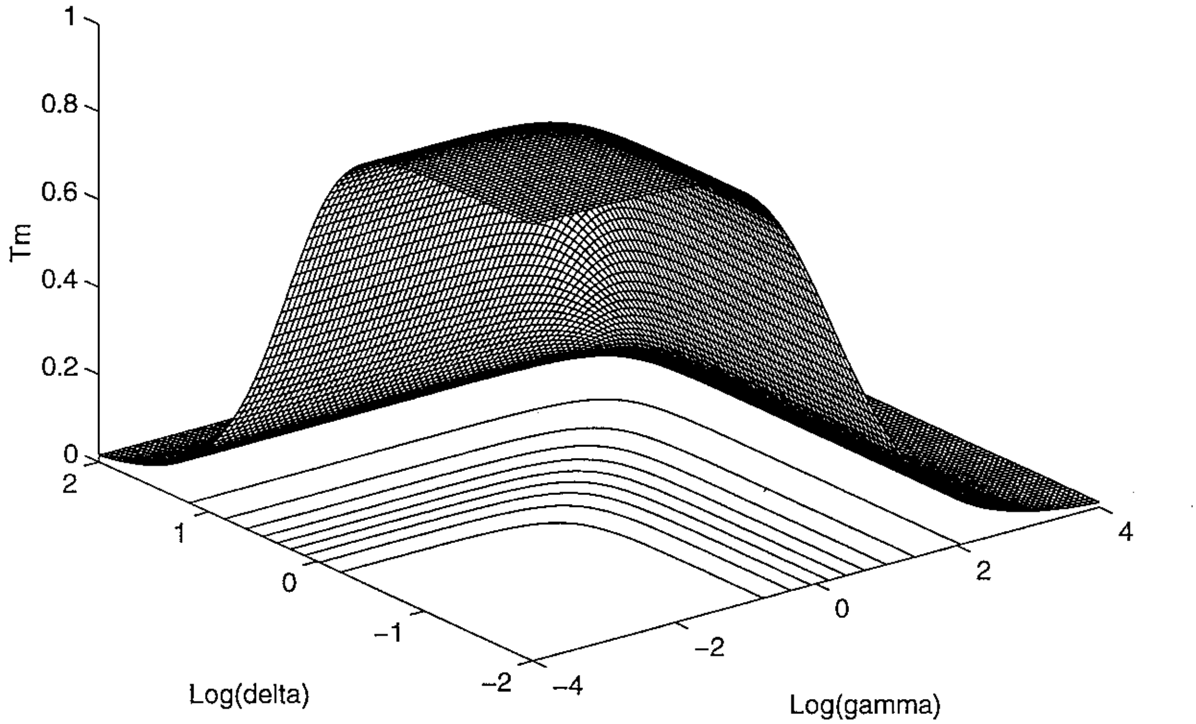

Figure 3. The rise time as a function of Δ and γ for $\delta = 0$. The contour curves shows, roughly, where $Tm = 0.1, 0.2, \ldots, 0.9$.

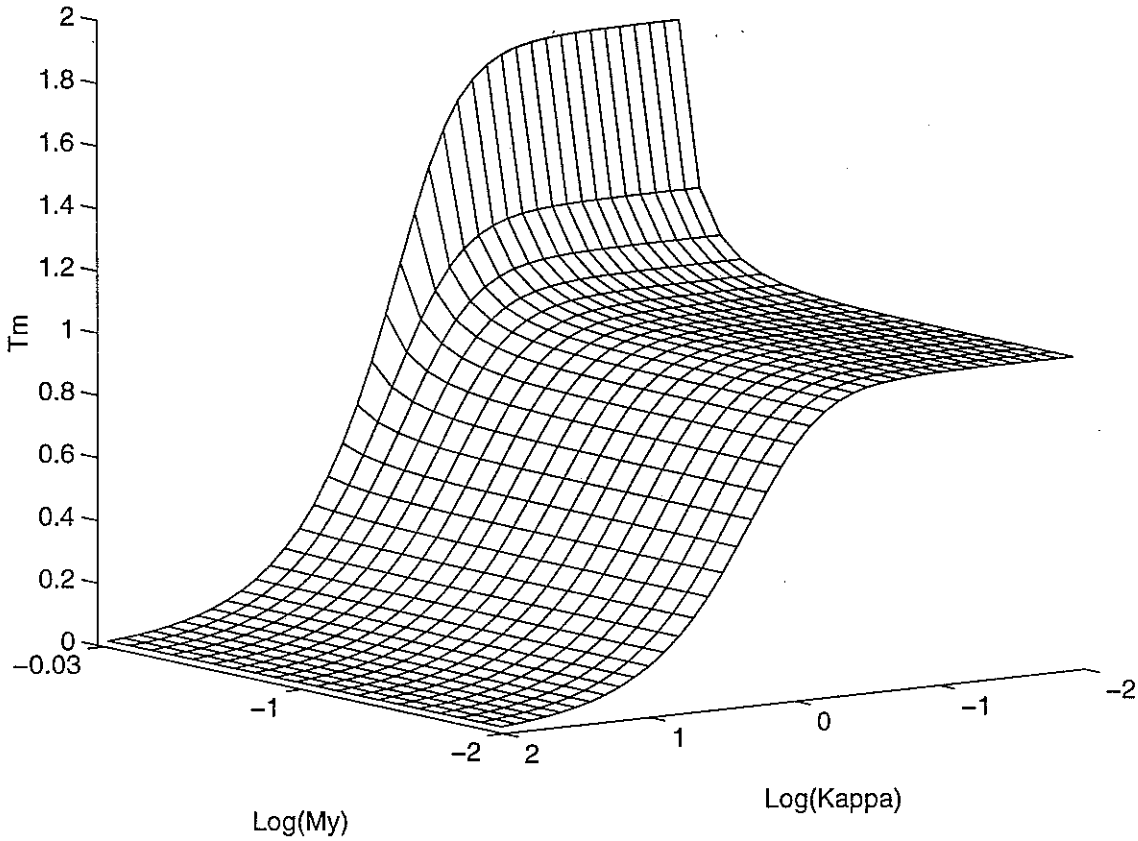

Figure 4. The rise time as a function of μ and κ for $\eta_p = 0$.

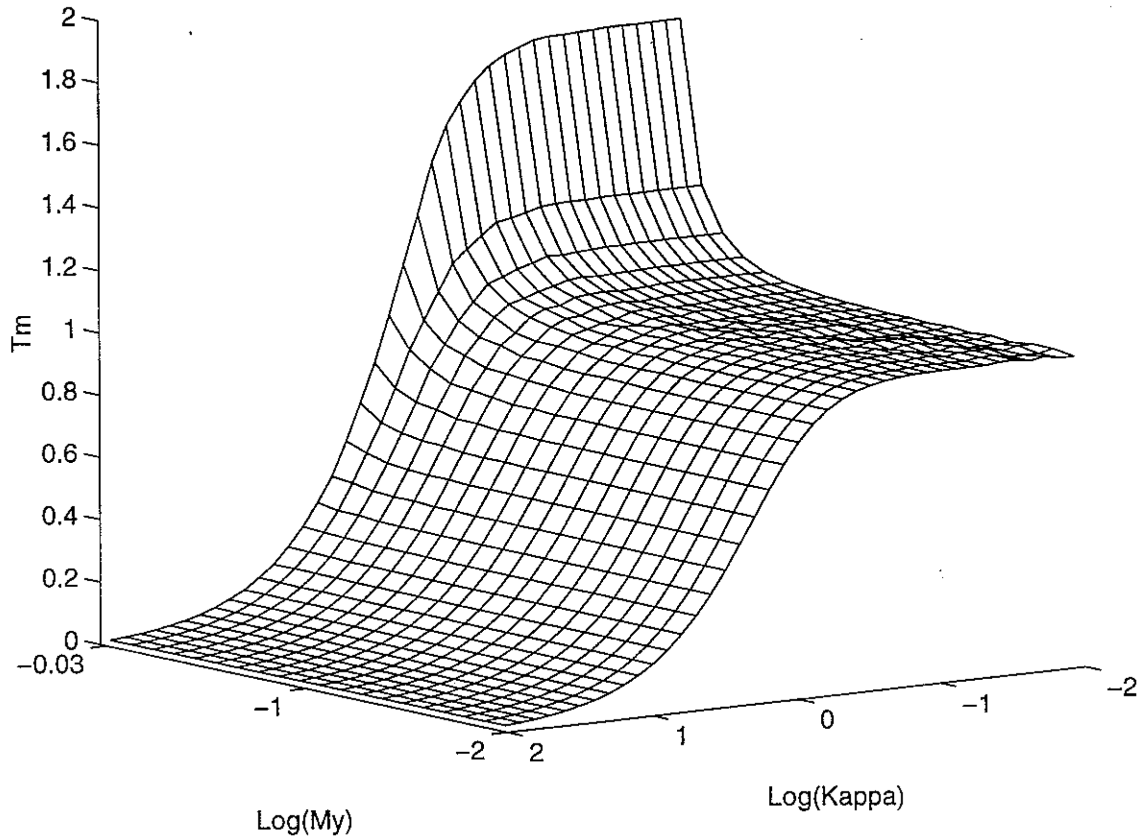

Figure 5a. The rise time as a function of κ and μ for $\eta_p = 0.01$. Note, the sign of μ is chosen at random, see text.

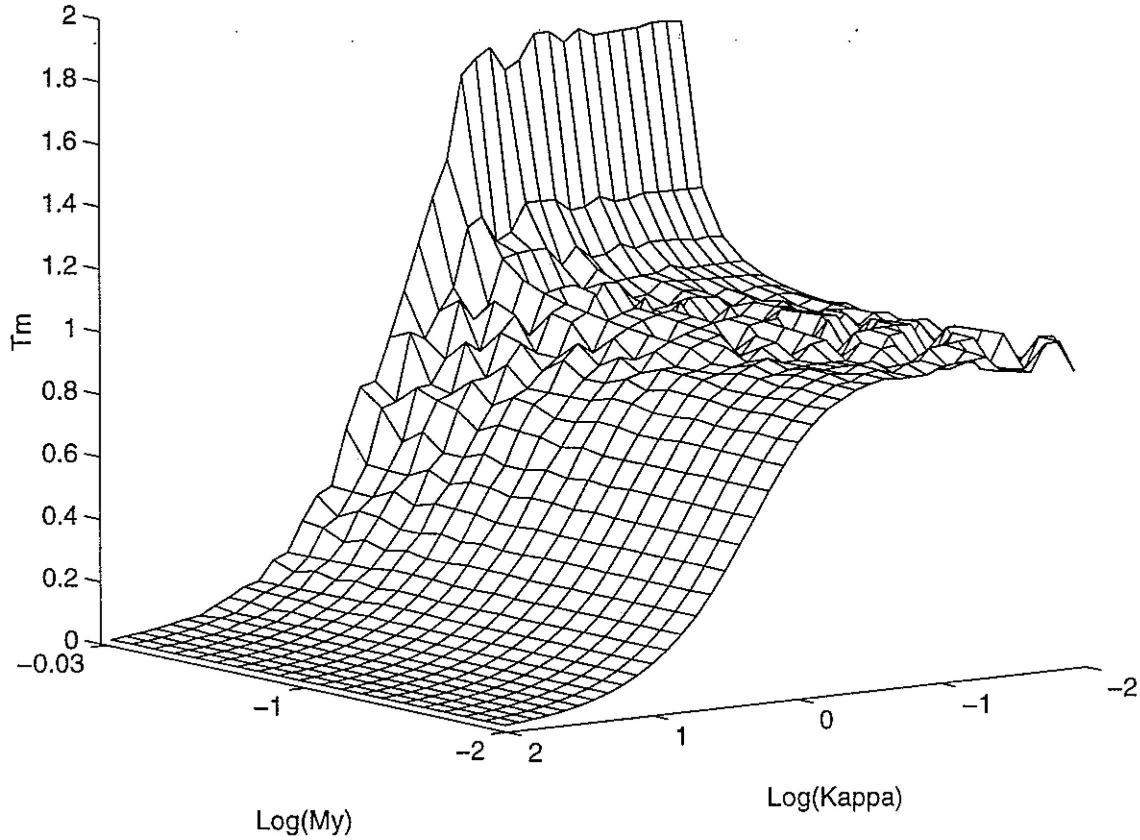

Figure 5b, as Figure 5a but $\eta_p = 0.1$

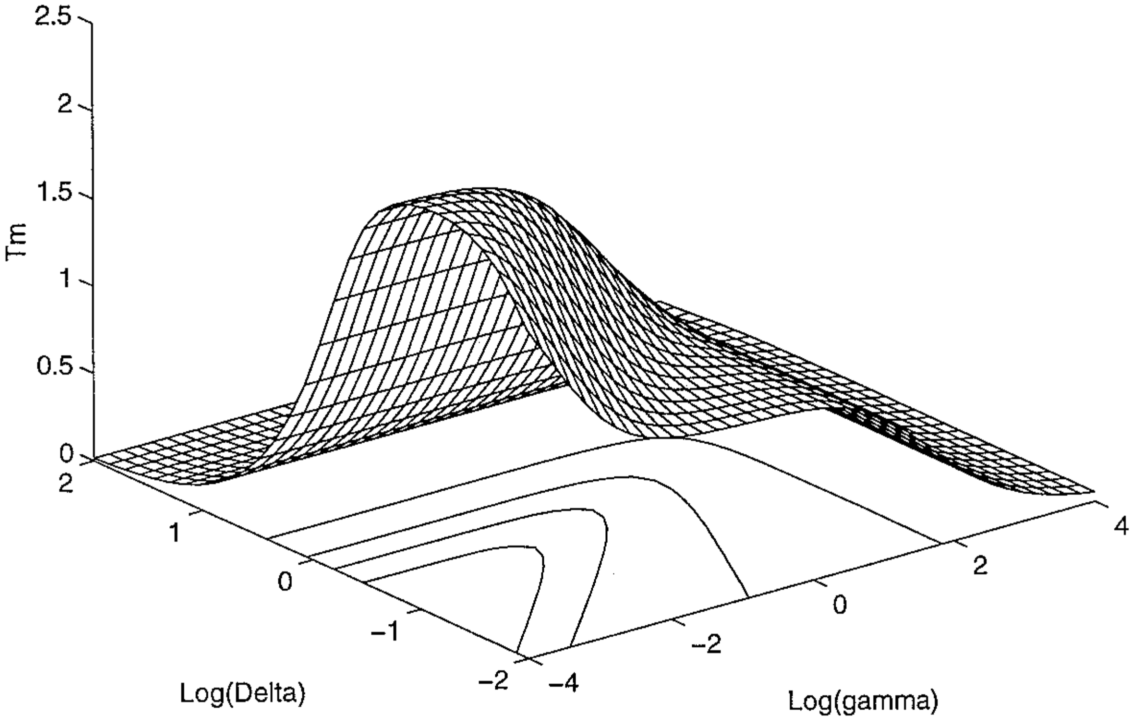

Figure 6. As Figure 4 but $\delta = 0.98$. The contour lines are, roughly, where $Tm = 0.5, 1, 1.5$ and 2 .

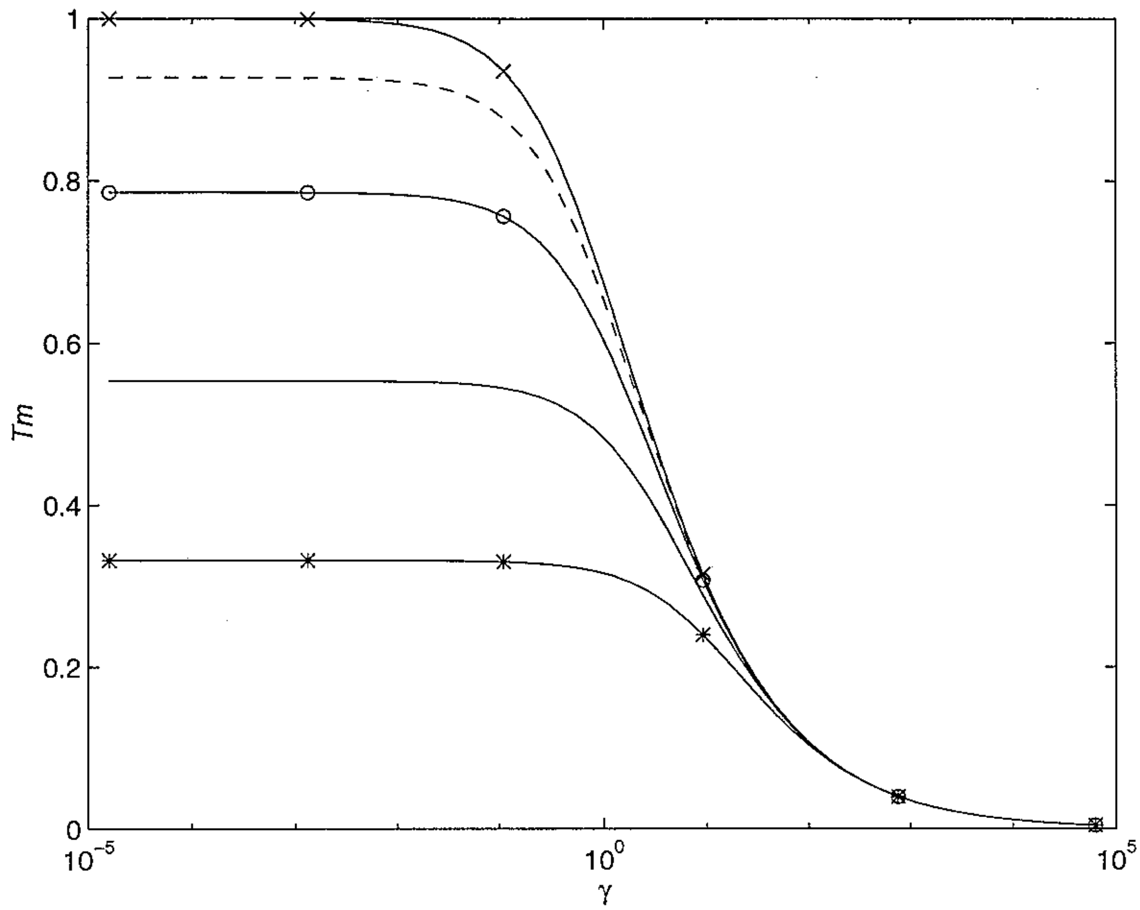

Figure 7a. The rise time as a function of γ for $\delta = 0$; —x—, $\Delta = 0$; – – –, $\Delta = 0.5$; —o—, $\Delta = 1$; ——, $\Delta = 2$; –*–, $\Delta = 4$.

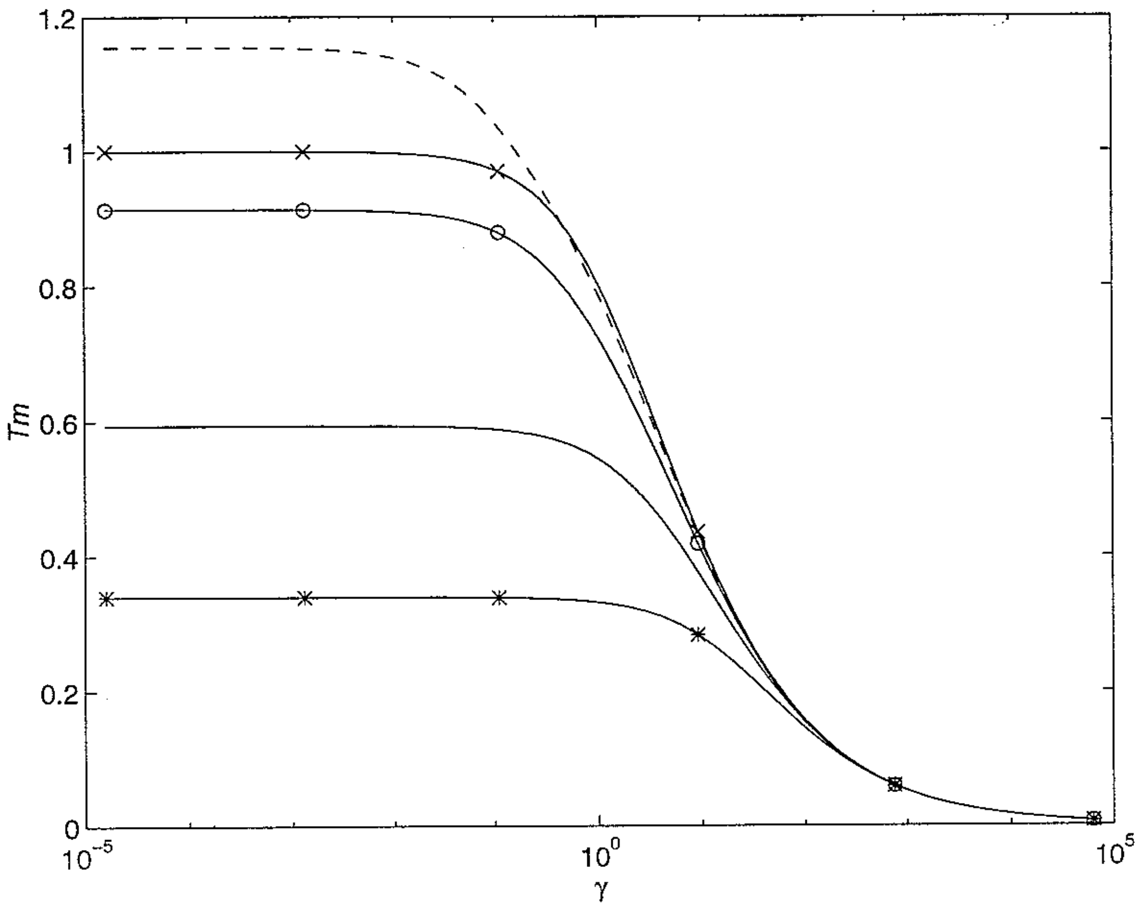

Figure 7b. As Figure 7a but $\delta = 3/4$.

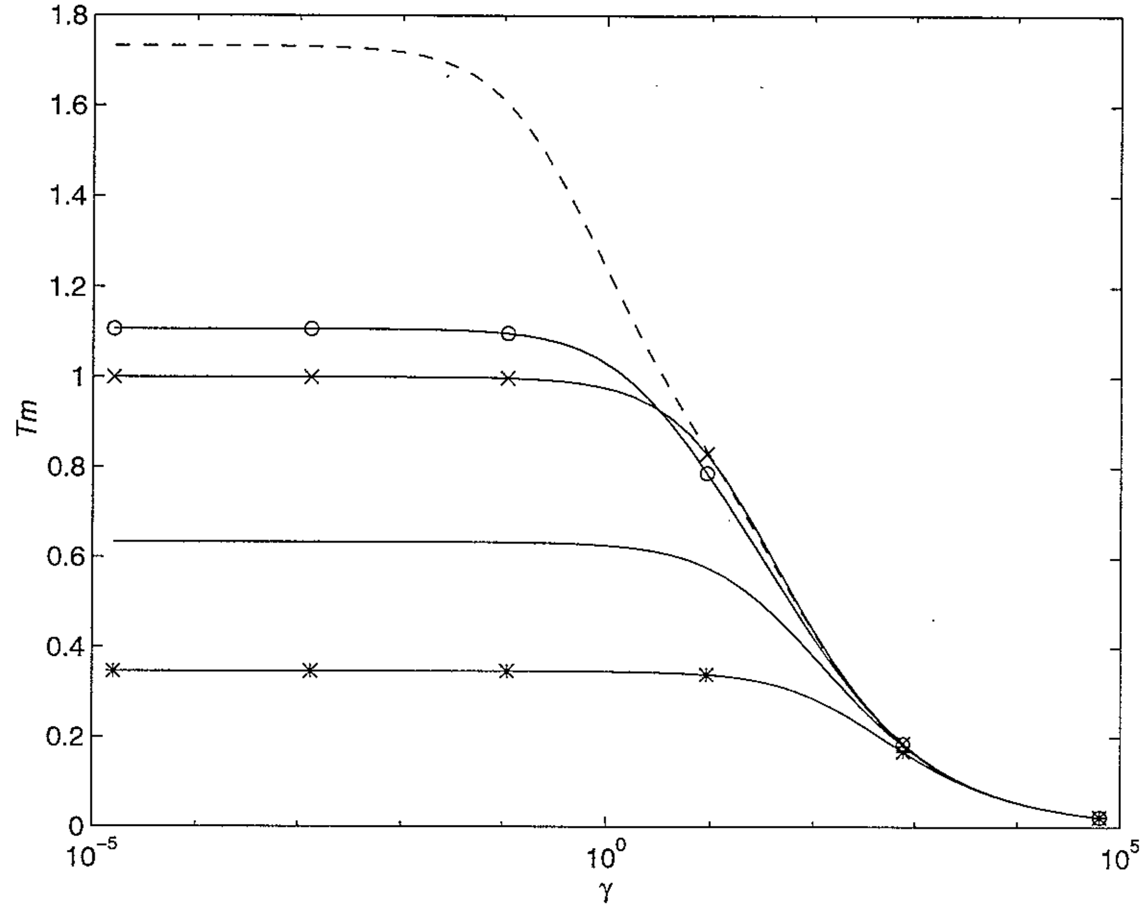

Figure 7c. As Figure 7a but $\delta = 0.98$.

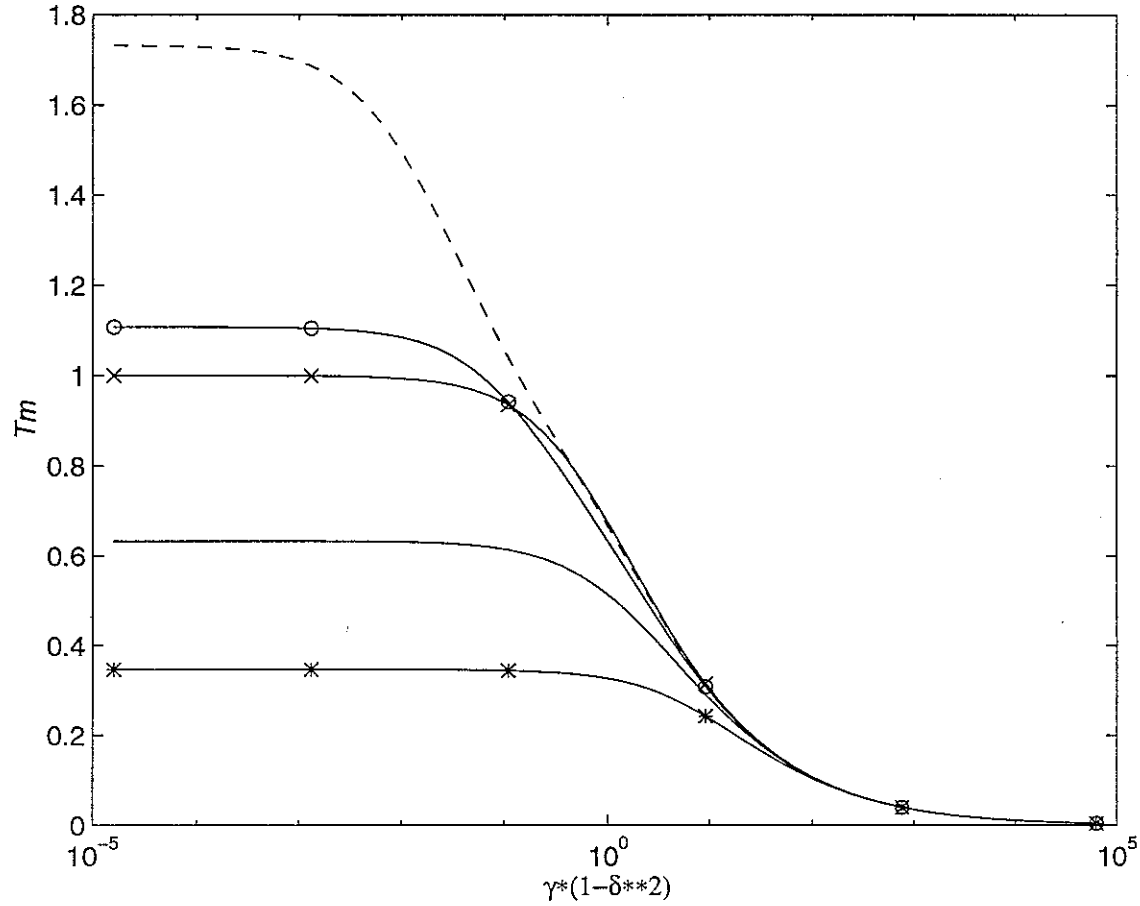

Figure 8. As 7c but as a function of the coupling factor divided by the square of the arithmetic mean modal overlap instead of the square of the geometric mean modal overlap.