isvr Institute of Sound and Vibration Research

Active Noise Control Using Feedback. Fixed and Adaptive Controllers

M. Pawelczyk, S.J. Elliott and B. Rafaely

ISVR Technical Memorandum 822

December 1997

University of Southampton

SCIENTIFIC PUBLICATIONS BY THE ISVR

Technical Reports are published to promote timely dissemination of research results by ISVR personnel. This medium permits more detailed presentation than is usually acceptable for scientific journals. Responsibility for both the content and any opinions expressed rests entirely with the author(s).

Technical Memoranda are produced to enable the early or preliminary release of information by ISVR personnel where such release is deemed to the appropriate. Information contained in these memoranda may be incomplete, or form part of a continuing programme; this should be borne in mind when using or quoting from these documents.

Contract Reports are produced to record the results of scientific work carried out for sponsors, under contract. The ISVR treats these reports as confidential to sponsors and does not make them available for general circulation. Individual sponsors may, however, authorize subsequent release of the material.

UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

SIGNAL PROCESSING & CONTROL GROUP

Active Noise Control Using Feedback.

Fixed and Adaptive Controllers

by

M.Pawelczyk, S.J.Elliott and B.Rafaely

ISVR Technical Memorandum No. 822

December 1997

Authorised for issue by
Prof S J Elliott
Group Chairman

ACKNOWLEDGEMENTS

The primary author, Marek Pawelczyk*, would like to acknowledge institutions which significantly supported him during the research reported in this memorandum:

1. *The British Council* for funding the Fellowship which allowed the author to spend over three months at ISVR;

2. *Foundation for Polish Science (FNP)* for funding the Domestic Fellowship for Young Scientists;

3. *Polish Committee of Scientific Research (KBN)* that also partially supported the research under Grant 8 T11A 006 13.

*On leave from:
Institute of Automation, Silesian Technical University
ul. Akademicka 16, 44-101 Gliwice, POLAND
e-mail: mpawelczyk@ia.polsl.gliwice.pl

CONTENTS

1 INTRODUCTION

The purpose of this report is to discuss the application of feedback control methods in the active reduction of sound. Thoroughly examined in the literature single, or multi-channel feedforward control systems fail for problems that involve random disturbances where no good reference for the primary soundfield exists. Therefore recently, there has been a growing interest in feedback control systems that do not require any separate reference but are only based on the residual error.

In chapter 2 the plant and disturbance are introduced and some necessary assumptions are made. Both the plant and disturbance are described in the Z - transform domain and the latter is split into two separate parts by the Diophantine equation.

Chapter 3 deals with the design of optimal controllers. In section 3.1 two structures of control system are presented that will further aim at minimising sound energy at the residual microphone. The first is a classical feedback referred to as Minimum Variance Controller and the second includes a model of the plant. The model, when it is perfect, cancels the contribution of the controller output back to its input thus turning the system feedforward. It is proved (sections 3.2 and 3.3) that, in this case, both the strategies are equivalent and will be further treated together. However, it turns out (in section 3.2) that when the plant is non-minimum phase the closed-loop is unstable and a modification of the control goal is required, which is the subject of section 3.3. Special attention is paid to a modification in which the control effort is included in the cost function (section 3.3.2). Section 3.4 continues to deal with optimal IMC, but using a different approach as to that presented so far. It is based on signal processing analysis which leads to the feedforward Wiener Filter applicable for both minimum- and non-minimum phase plants. In section 3.5 robust stability of a feedback control system is discussed and its relation to optimal performance is emphasised. The effects of measurement noise and plant delay are the subjects of sections 3.6 and 3.7, respectively.

Chapter 4 is devoted to adaptive control and it is implicitly divided into two parts: the Z - transform domain approach (sections 4.1 and 4.2) and the signal processing approach (section 4.3). All adaptive algorithms derived in this chapter are based on the optimal solutions presented in chapter 3. In section 4.1 the mathematical description of the plant is rewritten to contain polynomials of the controller in a linear way, and is referred to as the predictive model of the plant. RHS of the model is considered as the identification error. Section 4.2 leads to update equations of controller parameters via RLS and LMS identification procedures. In section 4.3, the Filtered-x LMS algorithm for IMC is presented. When the model of the plant does not accurately represent the plant itself the control system can become unstable. Therefore two solution to that problem are further discussed: one is based on the on-line identification of the model (section 4.3.2) and the next - on robust stability included in the adaptation procedure (section 4.3.3).

In chapter 5 both the optimal (section 5.2) and adaptive control strategies (section 5.3) are tested in simulations for an assumed plant and disturbance presented in section 5.1.

The final chapter of this memorandum, chapter 6, summarises, concludes and indicates main contributions of this work.

2 THE PLANT AND DISTURBANCE

A real digital feedback active noise control system can be represented by the following block diagram:

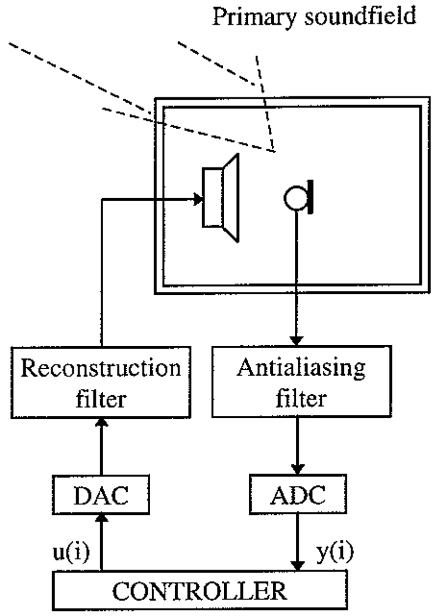


Fig. 1 A feedback noise control system.

Thus, the physical system includes not only the acoustic (path between the loudspeaker and residual microphone) and electroacoustic (the loudspeaker and microphone) elements but also the analogue anti-aliasing and reconstruction filters generally used before and after sampling analogue signals, as shown on Fig. 1. Obviously, the analogue-to-digital (ADC) and digital-to-analogue (DAC) converters are present in the system. All the elements between the output of the controller $u(i)$ and its input $y(i)$ constitute the plant of control and contribute properties of the plant. The signal measured by the residual microphone, when the loudspeaker (the secondary source) is not driven, is referred to as the acoustic disturbance (noise, sound) to be controlled and $y(i)$ is the disturbance. When the primary soundfield is random the disturbance can be modelled as a white noise passing by a filter which is called the disturbance shaping filter.

It is now assumed that the plant and disturbance shaping filter are linear (for frequencies of the sound and its levels we are dealing with) and no aliasing is caused by the sampling process. The presence of aliasing (due to, for example, inadequate anti-aliasing filters) causes frequency components above half the sampling rate in the continuous waveform $y(t)$ to appear as components below half the sampling rate in the sampled signal $y(i)$. This would then constitute a form of non-linearity in the loop.

In general, for active noise control applications the following structure of the plant and acoustic disturbance can be considered:

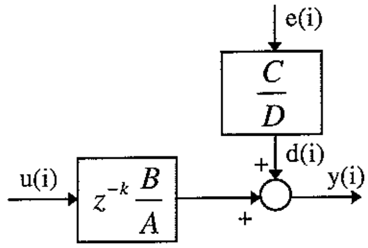


Fig. 2 General structure of electro-acoustic plant and acoustic disturbance shaping filter.

where:
A, B, C, D - polynomials of z^{-1} of orders: dim A, dim B, dim C, dim D, respectively;
k - the discrete time delay;
$u(i)$ - the control signal;
$e(i)$ - the white noise of variance λ^2;
$y(i)$ - the output - cancellation effect;
$d(i)$ - the acoustic disturbance.

The mathematical description is as follows:

$$y(i) = \underbrace{z^{-k}\frac{B}{A}}_{P}u(i) + \frac{C}{D}e(i)\,. \tag{1}$$

The measurement noise is neglected here. This is justified due to its very low level comparing to the level of the acoustic disturbance. Nevertheless, its presence will be the subject of section 3.6. The disturbance shaping filter can be split using the following Diophantine equation:

$$\frac{C}{D} = F + z^{-k}\frac{G}{D}, \quad \dim F = k-1, \quad \dim G = \max(\dim D - 1,\, \dim C - k)\,. \tag{2}$$

However, all calculations are simplified without loss of generality if the following structure is considered:

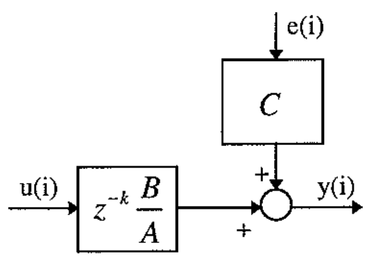


Fig. 3 Simplified structure of electro-acoustic plant and acoustic disturbance shaping filter.

for which the output can be expressed as follows:

$$y(i) = \underbrace{z^{-k}\frac{B}{A}}_{P}u(i) + Ce(i)\,. \tag{3}$$

It will be shown later that this description (the disturbance shaping filter being an FIR filter) has a lot of other advantages.
Then the Diophantine equation is of the form:

$$C = F + z^{-k}G, \quad \dim F = k - 1, \quad \dim G = \dim C - k \tag{4}$$

It is assumed that polynomial C is monic ($c_0 = 1$). It does not constraint our considerations because otherwise the value of its first coefficient can be included into the variance of the white noise, $e(i)$. Similarly, to have a unique plant representation, polynomial A is also assumed to be monic, which means that $a_0 = 1$. In general, both the plant and the disturbance can be time-dependant (this refers to sections 4 and 5). Polynomials A, B, C should be then written as $A(i)$, $B(i)$, $C(i)$. To simplify the notation the time variable, i, will be omitted, further. However, one should bear this in mind.

3 OPTIMAL (FIXED) CONTROL

The aim of this section is to design optimal (fixed) controllers which minimise energy at the output of an active noise control system. However, some features of the plant, stability conditions, and nonstationarities require a modification of the former performance criterion. In section 3.1 two different control structures are presented. Section 3.2 deals with noise control for minimum phase plants (see also Elliott, 1994) whilst section 3.3 considers more general case - non-minimum phase plant (see also Niederlinski et al., 1995). To cope with the problem different approaches are suggested (sections 3.3.1 and 3.3.2). Section 3.4 presents the design of optimum controllers using signal processing approach (Rafaely, 1997). Robust stability is the subject of section 3.5 (Elliott, 1994). The remainder of chapter 3 points out limitations of performance due to the presence of measurement noise (sec. 3.6) and plant delay (sec. 3.7), and a relation of the measurement noise to robust control is developed.

3.1 MVC AND IMC FEEDBACK STRUCTURES

In this work two different structures of the control system, which minimises energy at the output, are considered. The first one is a regulator with classical feedback and will be further referred to as Minimum Variance Controller (MVC):

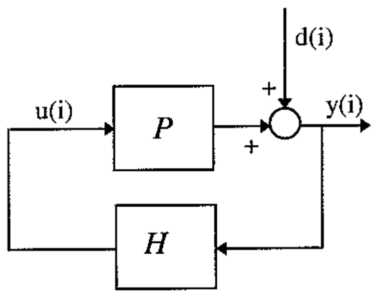


Fig. 4 Classical feedback structure - MVC.

In the second one the controller consists of two parts: W and $\hat{P}$, where $\hat{P}$ is a model of the plant P. Hence, the control structure is called Internal Model Control (IMC).

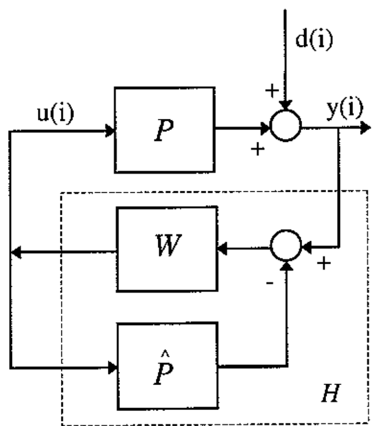


Fig. 5 IMC structure.

The overall IMC controller (see Fig. 5) is of the form:

$$H = \frac{W}{1 + \hat{P} W}. \tag{5}$$

The graph from Fig. 5 can be redrawn as shown on Fig. 6:

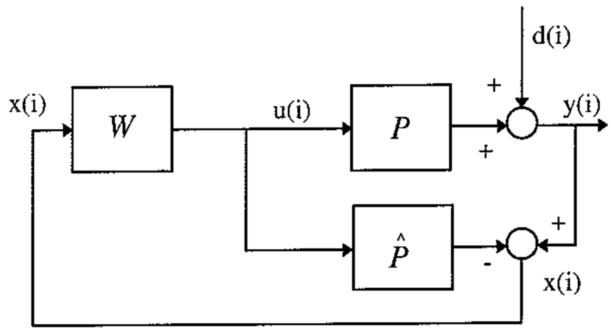


Fig. 6 Redrawn IMC structure.

Now it is clearly seen that if the plant model is perfect, $\hat{P} = P$, then the input to W becomes the disturbance, $d(i)$, which was before not allowable, and the system becomes purely feedforward:

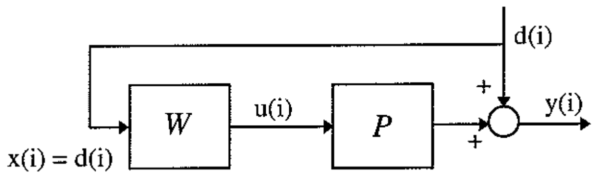


Fig. 7 IMC reduced to feedforward control for $\hat{P} = P$.

Further in the work filter, the control filter, W, will be referred to as the IMC filter.

3.2 NOISE CONTROL IN CASE OF MINIMUM PHASE PLANT

Let us now define the cost function $L(i+k)$ which we are going to minimise:

$$L(i+k) = E\{y^2(i+k)\}. \tag{6}$$

The optimum MVC is given by the following equation (see Eq. (A.14)):

$$H_{opt} = -\frac{A}{B}\frac{G}{F}. \tag{7}$$

The optimum IMC filter (IIR type) is given by (see Eq. (A.85)):

$$W_{opt} = -\frac{A}{B}\frac{G}{C}. \tag{8}$$

Substituting Eq. (8) for W in Eq. (5) and using Eq. (3) leads to the form of optimal IMC for $\hat{P} = P$:

$$H_{opt} = \frac{-\frac{AG}{BC}}{1 - \frac{AG}{BC} z^{-k} \frac{B}{A}} = \frac{-\frac{AG}{BC}}{C - z^{-k} G}, \tag{9}$$

which after using DE (4) can be reduced to:

$$H_{opt} = -\frac{A}{B}\frac{G}{F}. \tag{10}$$

This is exactly the same as for MVC (see Eq. (7)).
In general, no matter how the plant and disturbance is modelled, IMC with perfect model of the plant (optimal feedforward IMC filter) is always equivalent to optimal MVC.

Under optimal control the output value is an MA process (see Eq. (A.15)):

$$y_{opt}(i) = F e(i) \tag{11}$$

and the minimum cost function (6), which is also the minimum energy at the output (see Eqs. (A.17), (A.18), (A.86), (A.87)):

$$\min L(i+k) = \min E\{y^2(i+k)\} = E\{(Fe(i+k))^2\} = \lambda^2 \sum_{l=0}^{k-1} c_l^2 . \tag{12}$$

It is clear that only in the particular case when $k = 1$, will white noise be left at the output: $y_{opt,k=1}(i) = e(i)$ (see Eq. (11)) because C is monic and minimum variance is the variance of white noise λ^2 (see Eq. (12)). Otherwise, the form of the output can be deduced prior to control, only on the basis of the disturbance shaping filter and plant delay.

It has been proved (see appendix A) that for control goal given by minimising (6) the characteristic equation of the optimal closed loop system (or the condition for stable optimal feedforward filter) is of the form (see Eqs. (A.22), (A.90)):

$$A \cdot B \cdot C = 0. \tag{13}$$

Thus to have the closed loop system stable with the optimal controller in operation the following requirements must be fulfilled:
a). the plant is stable;
b). the plant is minimum phase apart from delay;
c). the polynomial modelling the shaping filter is minimum phase.

Because no prior knowledge of the disturbance is used in any of these algorithms and only magnitude of its frequency response is required (the phase is not important) spectral factorisation over polynomial C can be made turning it into $\bar{C}$ which has only minimum phase parts. Then the characteristic equation can be not formally transformed to the following form:

$$A \cdot B = 0\,. \tag{14}$$

For minimum phase plant the best possible attenuation of noise at the output that can be achieved depends only on the disturbance shaping filter and is expressed by:

$$J = 10\log_{10}\left(\frac{\sum_{i=0}^{k-1} c_i^2}{\sum_{i=0}^{\dim(C)} c_i^2} \right) [\mathrm{dB}] \tag{15}$$

provided that polynomial C is minimum phase. Otherwise, firstly spectral factorisation over non-minimum phase parts of this polynomial should be made and then the optimum performance can be calculated according to formula (15). From Eqs. (15) and (4) follows that when C = 1 (the disturbance is a white noise) then J = 0 [dB]. Similarly, when the band of the acoustic disturbance is comparable to Nyquist frequency ($f_N = f_s/2$) then there is no sense to employ any algorithm minimising cost function given by Eq. (6).

Looking at Eq. (14) it is clear that constraints are imposed only on the plant which has to be stable and minimum phase. All acoustic continuous space plants are stable and minimum phase for low frequencies, which we here consider (for high frequencies higher modes appear due to e.g. vibration of the membrane of the loudspeaker). But their discrete model can become non-minimum phase. This can happen e.g. because their continuous time delay is usually not an integer multiple of the sampling period. In that case, at least one zero of the plant lays outside the unit circle on the left half of the *Z*-plane. If the continuous plant was non-minimum phase, in its discrete representation at least one zero would lay outside the unit circle on the right half of the *Z*-plane, that for acoustic plants is impossible. To try to avoid this non-minimum phase discrete representation, a higher sampling rate should be chosen or a bigger discrete time delay in the identification procedure should be imposed.

3.3 NOISE CONTROL IN CASE OF NON-MINIMUM PHASE PLANT

For non-minimum phase plants the optimum control law (10) derived in section 3.2 is not valid because according to Eq. (14) the closed loop system is unstable. As a consequence, the attenuation cannot be anymore calculated according to Eq. (15). However, as will be shown in this section, it is possible to obtain stable controllers even for non-minimum phase plants although it will influence other features of the whole system.

3.3.1 Splitting polynomial *B*

One of the possible solutions to cope with the problem of instability for non-minimum phase plants is to split polynomial B into two parts:

$$B = B^{+} B^{-} \tag{16}$$

where B^{+} includes only the minimum phase part and B^{-} consists only of the non-minimum phase part of B. It further requires reformulations of the cost function and the Diophantine equation. As the result, the controller compensates only minimum phase polynomial B^{+} allowing stable behaviour, but the output signal is now an ARMA process and its variance is increased. This approach is not presented in the report. An interested reader is referred to Niederlinski et al., 1995 where this strategy is discussed in details for the particular case of the disturbance shaping filter having exactly the same poles as the plant itself (an ARMAX plant).

3.3.2 Control effort in the cost function

Another approach that can be used to obtain the control system stable for non-minimum phase plant is to modify the cost function (6) to be minimised to the following one:

$$L_q(i+k) = E\{y^2(i+k) + qu^2(i)\}, \tag{17}$$

where q is a penalty imposed on an excess control effort. This modification for classical feedback can be called Weighted Minimum Variance Control - WMVC (although as it is shown later, see Eqs. (24) and (25), the variance at the output is no longer the minimum one).
The optimum WMVC has the form (see Eq. (A.44)):

$$H_{opt} = -\frac{AG}{BF + q'AC} \tag{18}$$

and optimal feedforward filter for the cost function given by (17) (see Eq. (A.110)):

$$W_{opt} = -\frac{GA}{C(B + q'A)}, \tag{19}$$

where

$$q' = \frac{q}{b_0}. \tag{20}$$

It can be easily shown that, as it was done in section 3.2 for the case without control effort (follow Eqs. (8) through (10)), the whole IMC feedback controller (for $\hat{P} = P$) is expressed by Eq. (18) and is equivalent to WMVC. Hence, the remainder of this section concerns both MVC and IMC controllers.
The stability condition is given by the following equation (see Eq. (A.49)):

$$AC(B + q'A) = 0 \tag{21}$$

or after excluding C (refer to the explanation to Eqs. (13) and (14)) by:

$$A(B + q'A) = 0. \tag{22}$$

From this equation follows that choosing proper value of q' it is possible to have the control system stable. When A and B are of the same order, it is easy to precisely calculate the optimum q'. This method can be also used in case of minimum phase polynomial B, for which some of its zeros are very close to the unit circle, to suppress an excess variance of control signal which can be difficult to stand by some actuators (mainly in vibration control).
But, as one can expect, this approach influence other features of the algorithm. The output is no longer an MA process but is closely related to control effort and takes the form (see Eq. (A.58)):

$$y(i) + q^{'} u(i-k) = Fe(i) \tag{23}$$

and, what is mostly important, its variance is increased (see Eqs. (A.60) and (A.61))):

$$E\left\{y^2(i+k)\right\} = E\left\{\left[q^{'} \frac{AG}{B+q^{'}A} e(i)\right]^2\right\} + E\left\{\left[Fe(i+k)\right]^2\right\}, \tag{24}$$

or in other form:

$$E\left\{y^2(i+k)\right\} = \left(q^{'}\right)^2 E\left\{\left[u(i)\right]^2\right\} + E\left\{\left[Fe(i+k)\right]^2\right\}. \tag{25}$$

Eq. (25) shows that the variances of the output and control effort are strictly related and when one is reduced, the second has to be increased because the second term on the RHS of Eq. (25) can be regarded as a constant for a given disturbance shaping filter. Thus the choice of $q^{'}$ is a trade-off between sufficient stability margin (and in the same time, suppression of the variance of the control signal) and increasing the output variance (and as the result - decreasing the attenuation).

The minimum value of the cost function (17) is of the form (see Eq. (A.52)):

$$\min L_q(i+k) = \left(q^{'}\right)^2 \left(1+b_0^2\right) E\left\{\left[\left(\frac{GA}{B+q^{'}A}\right) e(i)\right]^2\right\} + E\left\{\left[Fe(i+k)\right]^2\right\}. \tag{26}$$

The control effort introduced in the cost function (17) influences also other features of the algorithm like increases the robustness of the system and decreases the sensitivity of the output to non-stationarities or nonlinearities usually met in real plants.

3.4 SIGNAL PROCESSING APPROACH - WIENER FILTER

Control algorithms presented in sections 3.2 and 3.3 (both MVC and IMC, with- and without control effort) where based on Z-transform domain approach and led to forms of controllers expressed in terms of plant polynomials and polynomials defined by the Diophantine equation. Another way of calculation of optimal feedforward filter (and thus IMC design) is based on the signal processing approach and particularly - on the correlation analysis. Assuming that both the plant P and the filter W are time invariant their order can be commuted:

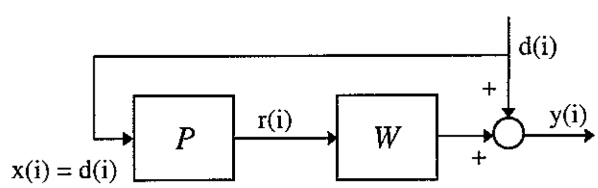


Fig. 8 IMC reduced to feedforward control for $\hat{P} = P$.

An assumption is also made that both the plant and the filter have finite impulse response and can be regarded as FIR filters of orders N.

Let us first define (vertical) vectors expressing:

the reference signal (equivalent here to the disturbance signal):

$$\mathbf{x}^T(i) = [x(i), x(i-1), \ldots, x(i-N+1)], \tag{27}$$

the reference signal filtered by the plant:

$$\mathbf{r}^T(i) = [r(i), r(i-1), \ldots, r(i-N+1)], \tag{28}$$

the impulse response of the plant:

$$\mathbf{p}^T(i) = [p_o, p_1, \ldots, p_{N-1}], \tag{29}$$

and the impulse response of the control filter:

$$\mathbf{w}^T(i) = [w_o, w_1, \ldots, w_{N-1}]. \tag{30}$$

The input to the controller in time domain can be written on the basis of the reference signal x(i) (equal in this case to the disturbance) and plant impulse response **p**:

$$r(i) = \mathbf{p}^T\mathbf{x}(i). \tag{31}$$

The output signal can now be written as:

$$y(i) = d(i) + \mathbf{w}^T\mathbf{r}(i). \tag{32}$$

The discrete plant delay is now hidden in vector **p** and that is why the cost function $L(i)$ instead of $L(i+k)$ (see Eq. (6)) will be further considered.
The cost function can now be written as:

$$L(i) = E\{y(i)y^T(i)\} = E\{[d(i) + \mathbf{w}^T\mathbf{r}(i)][d(i) + \mathbf{w}^T\mathbf{r}(i)]^T\}$$

$$= E\{\mathbf{w}^T\mathbf{r}(i)\mathbf{r}^T(i)\mathbf{w} + d(i)\mathbf{r}^T(i)\mathbf{w} + d(i)\mathbf{w}^T\mathbf{r}(i) + d^2(i)\}$$

$$= \mathbf{w}^T E\{\mathbf{r}(i)\mathbf{r}^T(i)\}\mathbf{w} + 2\mathbf{w}^T E\{\mathbf{r}(i)d(i)\} + E\{d^2(i)\} \tag{33}$$

or, in more comprehensive form:

$$L(i) = \mathbf{w}^T\mathbf{A}\mathbf{w} + 2\mathbf{w}^T\mathbf{b} + c, \tag{34}$$

where:

$$c = E\{d^2(i)\} \tag{35}$$

is a scalar.
b is a vector of cross-correlation:

$$\mathbf{b}^T = \left[R_{rd}(0), R_{rd}(1), \ldots, R_{rd}(N-1)\right], \tag{36}$$

where:

$$R_{rd}(l) = E\{r(i-l)d(i)\}. \tag{37}$$

A is a Toeplitz matrix of autocorrelation values of **r**:

$$\mathbf{A} = \begin{bmatrix} R_{rr}(0) & R_{rr}(1) & \cdots & R_{rr}(N-1) \\ R_{rr}(1) & R_{rr}(0) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ R_{rr}(N-1) & \cdots & \cdots & R_{rr}(0) \end{bmatrix}. \tag{38}$$

Since Eq. (36) is a quadratic one, the minimum of the mean square error with respect to **w** can be easily found:

$$\frac{\partial E\{y^2(i)\}}{\partial \mathbf{w}} = 2\mathbf{w}^T\mathbf{A} + 2\mathbf{b} = 0. \tag{39}$$

Thus the optimal controller is expressed by:

$$\mathbf{w}_{opt} = -\mathbf{A}^{-1}\mathbf{b} \tag{40}$$

and is called Wiener Filter.

The Wiener Filter can be also derived starting from the orthogonality principle (Papoulis, 1984) which states that minimisation of Eq. (6) is equivalent to removing all components of the error (output) which are correlated with $r(i)$ over the time-scale of the filter W (see Fig. 8):

$$\underset{0 \le l \le N-1}{\forall} E\{y(i)r(i-l)\} = 0. \tag{41}$$

Then, using Eq. (32) for the output, Eq. (41) leads straightforward to the optimal solution given by Eq. (40).
Substituting Eq. (40) for **w** in Eq. (34) yields the minimum cost function, which in this case is equal to minimum energy at the output, as:

$$\min L(i) = E\{d^2(i)\} + \mathbf{b}^T\mathbf{w}, \tag{42}$$

where the first term on the RHS of Eq. (42) represents the energy at the output without control ($\mathbf{w} = 0$).
Sometimes the autocorrelation matrix **A** can be ill-conditioned e.g. due to not sufficient excitation. To allow the inverse operation in that case a component $\beta \mathbf{I}_{N \times N}$ can be introduced to Eq. (40):

$$\mathbf{w}_{opt} = -\Big(\underbrace{\mathbf{A} + \beta \mathbf{I}_{N\times N}}_{\mathbf{A}'}\Big)^{-1} \mathbf{b}\,, \tag{43}$$

which is equivalent to modification of the cost function minimised, (6) or (33), to:

$$L(i) = E\{y^2(i) + \beta \mathbf{w}^T \mathbf{w}\} \tag{44}$$

where β is a weighting coefficient and $\mathbf{I}_{N\times N}$ denotes unity matrix of dimension $\dim N \times \dim N$.
The weighting coefficient acts in the same way as Tikhonov regularization and improves the conditioning of matrix **A'** (see Elliott and Sutton, 1996).
Because since invention of FFT algorithm it is much faster (for long filters) to calculate auto- and cross-correlations as inverse Fourier transforms of auto- and cross-spectrum densities Wiener filter coefficients are usually computed according to this time - frequency approach.

3.5 ROBUST CONTROL

Acoustic plants are usually subject to changes. The effect of plant changes on a feedback control system is two-fold. First, the performance is generally degraded because the optimal design of a controller is a function of the plant response. Second, and potentially more spectacularly, it may make the system unstable with the controller designed for the original (nominal) plant.
The goal of this section is to design a feedback controller which is robust, in terms of stability, to arbitrarily assumed group of plant changes. It should be emphasised that the design of robust feedback controllers outlined below assumes that their characteristics are fixed at the design stage, or in other words - they are not adaptive.

Let us write the cost function given by Eq. (6) in terms of the sensitivity function of the control system $S(j\omega)$ (frequency response from the disturbance $d(i)$ to the output $y(i)$) in H_2 norm notation:

$$L(j\omega) = \|S(j\omega) D(j\omega)\|_2^2\,, \tag{45}$$

where $D(j\omega)$ is the amplitude of the Fourier Series of periodic disturbances or the square root of the power spectral density of random disturbances (in our case: $D(j\omega) = \lambda\, C(j\omega)$).
Consider a plant with a multiplicative uncertainty described by Doyle, 1992, as:

$$P = P_o(1 + \Delta_p)\,, \tag{46}$$

where P_o is the nominal plant and the plant uncertainties Δ_p are bounded as follows:

$$|\Delta_p(j\omega)| < W_2(\omega) \tag{47}$$

(a fractional uncertainty of $W_2(\omega) = 0.1$, for example, corresponds to an uncertainty in the level of about 1 [dB] and about 0.105 [rad] in phase)
The robustness condition becomes then (as commonly used in H_∞ control theory):

$$\|T_o(j\omega)W_2(j\omega)\|_\infty < 1, \tag{48}$$

where $T_o(j\omega)$ is the complementary sensitivity function with the nominal plant. To simplify the notation, the term '$j\omega$' will be omitted further. Eq. (48) constitutes a constraint to minimisation of the cost function (45). Eqs. (45) and (48) can be combined into a single cost function (see Rafaely and Elliott, 1996a):

$$L = \|SD\|_2^2 + \|T_0 W_2\|_\infty^\infty. \tag{49}$$

This cost function, which represents an H_2 control problem with an H_∞ constraint, has no simple solution required to be implemented in an adaptive system. The simplest approximation is to replace $\|T_o W_2\|_\infty^\infty$ with a weighted $\|T_o W_2\|_2^2$ which results in the following cost function:

$$L = \|SD\|_2^2 + \beta\|T_o W_2\|_2^2 \tag{50}$$

This equation also defines the trade-off between optimal performance and robust stability of a feedback control system.
In IMC structure for perfect model of the plant the complementary sensitivity function simplifies to:

$$T_0 = WP_0 = W\hat{P}. \tag{51}$$

Thus the cost function (50) takes the form:

$$L = \|SD\|_2^2 + \beta\left\|W\hat{P}W_2\right\|_2^2. \tag{52}$$

The relation between the sensitivity function and the complementary sensitivity function, for the nominal plant, is defined by the following equation:

$$T = 1 - S. \tag{53}$$

Eq. (53) defines the compromise between performance and stability in the design of feedback controllers. If the control system is designed for maximum performance, so that $S \approx 0$, then $T \approx 1$ and if the upper bound of the multiplicative uncertainty is greater than one, the system will not be robustly stable. From Eq. (46) the multiplicative uncertainty can be expressed as follows:

$$\Delta_p = \frac{P - P_0}{P_0}. \tag{54}$$

As the nominal response of the plant, P_0, becomes very small at certain frequencies, the uncertainty can become large. Then W_2 (47) is greater than one and the system cannot be stable. Fortunately, this case is more likely to happen at high frequencies whilst good performance is required for low frequencies in active noise control.

3.6 THE EFFECTS OF MEASUREMENT NOISE

Let us now relax the assumption about absence of the measurement noise made in section 2. Now the closed-loop system can be represented by the following graph:

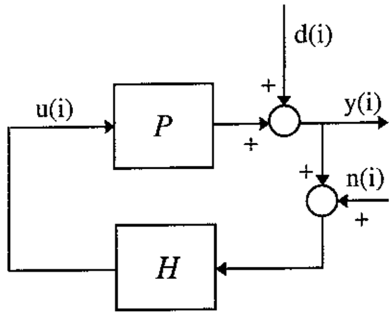


Fig. 9 A closed-loop system with the measurement noise.

n(i) is the measurement noise which is assumed to be uncorrelated with *d(i)*. This assumption is very well justified because *d(i)* is an acoustic disturbance whilst *n(i)* is a random noise introduced by the measurement path: sensor (microphone), wires, A / D converter. No other assumptions concerning its spectral features are made.
The system output can be now expressed as:

$$y(i) = \frac{1}{1-HP}d(i) + \frac{HP}{1-HP}n(i). \tag{55}$$

The first term represents the sensitivity function and the second term - the complementary sensitivity function. Thus the equation can be rewritten as:

$$y(i) = Sd(i) + Tn(i). \tag{56}$$

Because of the assumption of orthogonality of *n(i)* to *d*(i) the cost function (6) takes the form:

$$L(i) = E\{y^2(i)\} = E\{[Sd(i)]^2\} + E\{[Tn(i)]^2\} \tag{57}$$

or in H_2 norm notation:

$$L = \|SD\|_2^2 + \|TN\|_2^2. \tag{58}$$

If we now model *n(i)* as a white noise of variance β_1 shaped by filter W_3, then:

$$L = \|SD\|_2^2 + \beta_1\|TW_3\|_2^2. \tag{59}$$

This is exactly the same as the cost function representing nominal performance and robust stability, given by Eq. (50), when:

$$\begin{cases} \beta_1 = \beta, \\ W_3 = W_2, \\ T = T_0. \end{cases} \tag{60}$$

Thus the control system with the measurement noise can be considered robust to plant variations defined by the multiplicative uncertainty Δ_p (46) bounded by the spectrum of the noise itself.

3.7 INFLUENCE OF PLANT DELAY ON PERFORMANCE

The best possible attenuation of the acoustic disturbance that can be achieved at the output of active control system depends upon many factors. The influence of the form of the cost function minimised (and in this way, other requirements imposed on the system) was widely discussed throughout the whole work. The dependence upon the disturbance to be cancelled (or the disturbance shaping filter) was considered at the end of section 3.2.
Let us now assume that polynomials *A*, *B*, and *C* are minimum phase, the system is free of the measurement noise and the cost function is defined by Eq. (6). Then the optimal performance is defined by Eq. (15). It is strongly dependant on the discrete time delay of the plant. Figure 10 shows graphically the influence of this delay on the optimal performance for a given plant and disturbance (refer to sec. 5.1 and appendix B for details).

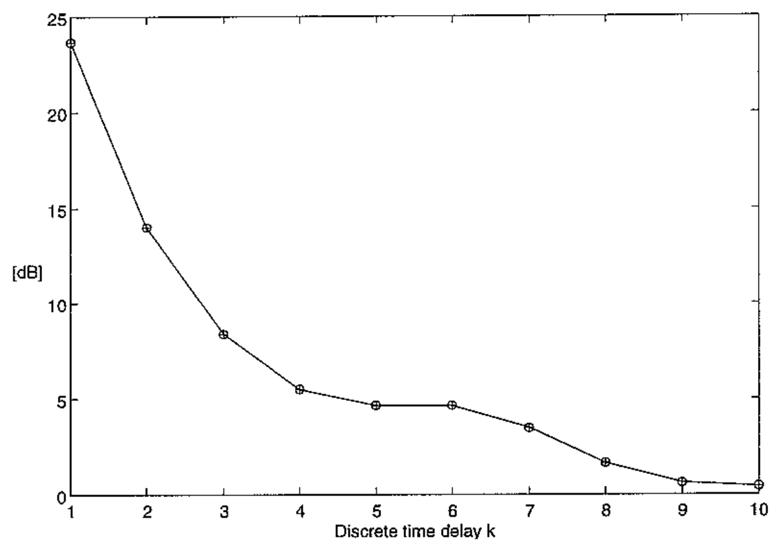


Fig. 10 Attenuation vs. discrete time delay of the plant for optimal feedback controller.

It is clear that increasing the time delay from 1 sample to 2 samples reduces the attenuation of about 10 [dB] which means 40 [%], here. Moreover, there is no sense to use the feedback control for delays more than 7. Therefore every effort should be done to reduce the delay.

The delay consists of three different components:

1. The plant under consideration (Fig. 1) includes an acoustic path, from the output of the loudspeaker (the secondary source) to the input of the error microphone, of length *l*. Making an assumption that the propagating wave is a plane one, the delay associated with this path is equal to:

$$\tau_{acoustic} = \frac{l}{c_o}, \tag{61}$$

where c_o is the speed of sound.

2. A significant part of the plant constitute analogue filters (the anti-aliasing and reconstruction filters). The delay through the analogue filters may be approximately evaluated by assuming that each pole of each filter contributes $\pi/4$ of phase lag, or $1/8$ cycle of phase delay, at its cut-off frequency, f_c. The total delay through the two analogue filters, which are assumed to have a total of m poles, is thus approximately $m/(8f_c)$. The cut-off frequency is typically one-third the sample rate ($f_s = 1/T_s$) so that $f_c \approx f_s/3 = 1/(3T_s)$. Thus (Nelson and Elliott, 1994):

$$\tau_{analogue} \approx \frac{3m}{8} T_s . \tag{62}$$

The discrete delay introduced by the analogue filters is then independent upon the sampling rate.

3. A one sample delay is implied by data converters and computing the response of the digital filter:

$$\tau_{digital} = T_s . \tag{63}$$

Summing up, the total delay of the plant is about:

$$\tau_{plant} = T_s \left(1 + \frac{3m}{8} + \frac{l}{c_o T_s} \right) \tag{64}$$

and the discrete time delay is an integer number of the expression in brackets (measured in samples).

Example

Let us now assume that the plant presented on Fig. 1 is an active headset. Then, the distance between the secondary source and the residual microphone is about 1 [cm]. Thus, for f_s = 2[kHz] and c_o = 343 [m/s], the acoustic path introduces the delay of about 0.058 sample.

To well protect the controller against the aliasing effect and well reconstruct the signal driving the loudspeaker, the analogue filters have usually a high order, e.g. 8th - each of them. Then, according to Eq. (62) they introduce 6 samples delay. Obviously, still remains one sample delay due to data converters and computing the response of the digital filter. It is now clearly seen that the total discrete time delay is approximately 7 samples where for 6 of them the analogue filters are responsible. Therefore, if it is only possible, the order of these filters should be reduced or even they should be excluded from the system. This can be justified when the maximum frequency of the acoustic disturbance is much smaller than the Nyquist frequency ($f_s/2$) what can be achieved by increasing the sampling rate. On the other hand, for plants like active headsets, when they are properly designed, higher frequencies are well attenuated by the passive barrier they contain and only low-frequency noise remains to be dealt with.

For large dimension plants the delay due to the acoustic path can be much larger than those considered above (is equal for 1.2 [m] distance between the loudspeaker and microphone) and then other strategies should be employed.

4 ADAPTIVE CONTROL

Real control system should ensure satisfactory attenuation and have sufficient stability margin for some nonstationarity of the plant and disturbance, which are likely to appear. In IMC design for real applications the controller is usually made robust for significant plant parameters changes (which is considered in the cost function being minimised) and adapts to disturbance nonstationarity and small variations in the plant (Rafaely and Elliott, 1996). In turns, MVC controller cannot tell the difference between nonstationarities of the plant and disturbance, and usually adaptive version is used for both purposes (Niederlinski et al., 1995, and Pawelczyk, 1995). However, no disturbance nonstationarity, even leading to non-minimum phase disturbance shaping filter, can cause instability of the whole adaptive system because while adaptation some kind of spectral factorisation over polynomial C is made implicitly.

4.1 PREDICTIVE MODEL OF THE PLANT

As seen from Eqs. (7), (8), (18), and (19) both optimal MVC and IMC filters (with, and without control effort) are expressed in terms of plant parameters (polynomials A and B). Thus the most obvious method to identify the controllers is to first identify a model of the plant and on its basis calculate the controllers. The main advantage of this approach is that it allows to have a look inside the algorithm and easily interpret it. However, the controller synthesis should be done on-line (usually in every sampling period). It involves a great deal of non-linear computations and is difficult to perform in fast systems like acoustic ones, which require high sampling rate.

Another approach that can be used here is to modify "the plant equation" in such a way that it includes parameters of the controller (to be estimated) in a linear way and, under optimal control, satisfies the right form of the output of the system. This type of plant equation is called the predictive model of the plant. For both MVC and IMC filter (with, and without control effort included in the cost function) the predictive model takes the following form (see appendix A for details, mainly: Eqs. (A.30), (A.31), (A.38), (A.69), (A.71), (A.78), (A.98), (A.99), (A.106), (A.115), (A.122)):

$$\gamma(i) - \hat{R}(i)u(i-k) - \hat{S}(i)x(i-k) = (AC-1)[Fe(i) - \gamma(i)] + Fe(i) = \varepsilon(i), \tag{65}$$

or in vector form:

$$\gamma(i) - (\hat{r}_0)u(i-k) - \varphi^T(i-k)\hat{\theta}(i) = \varepsilon(i). \tag{66}$$

Then, the control law has the form:

$$u(i) = \frac{\hat{S}(i)}{\hat{R}(i)}x(i). \tag{67}$$

Further notations are as follows:

$$x(i) = \begin{cases} y(i) & \textit{for MVC} \\ d(i) & \textit{for IMC filter} \end{cases} \tag{68}$$

is the general controller input (for versions with- and without control effort);

$$\gamma(i)=\begin{cases} y(i) & \text{for MVC and IMC} \\ y(i)+\frac{q}{r_0}u(i-k) & \text{for WMVC and IMC with control effort} \end{cases} \tag{69}$$

is the generalised output;

$$\hat{\theta}^T(i)=\left[s_0(i), s_1(i), \ldots, s_{\dim S}(i), r_1(i), r_2(i), \ldots, r_{\dim R}(i)\right] \tag{70}$$

is the identified controller parameter vector;

$$\varphi^T(i-k)=\left[x(i-k), \ldots, x(i-k-\dim S), u(i-k-1), \ldots, u(i-k-\dim R)\right] \tag{71}$$

is the data vector, or in other words - the vector of regressors.

4.2 IDENTIFICATION OF THE CONTROLLER

In the last section a predictive model of the plant was derived (see Eqs. (65) and (66)). Its RHS can be regarded as an error which linearly depends upon controller parameters. Thus, if the parameters are searched as:

$$\hat{\theta}(i)=Arg\min_{\hat{\theta}(i)} E\left\{\varepsilon^2(i)\right\}, \tag{72}$$

the error surface is quadratic. Having this estimate, the control value is calculated according to the following formula:

$$u(i)=-\frac{1}{\hat{r}_0}\varphi^T(i)\hat{\theta}(i). \tag{73}$$

Figures 11 and 12 depict adaptive control configurations of MVC and IMC filter, respectively, and indicate signals required in the identification procedure:

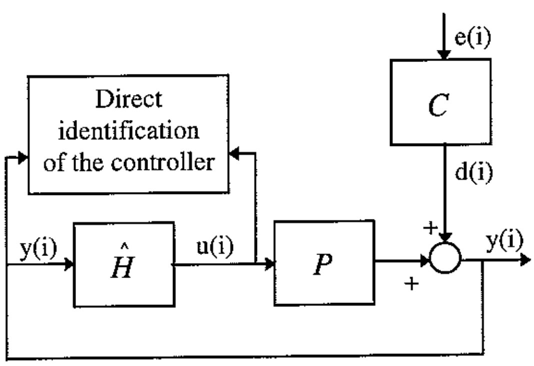


Fig. 11 Adaptive direct MVC.

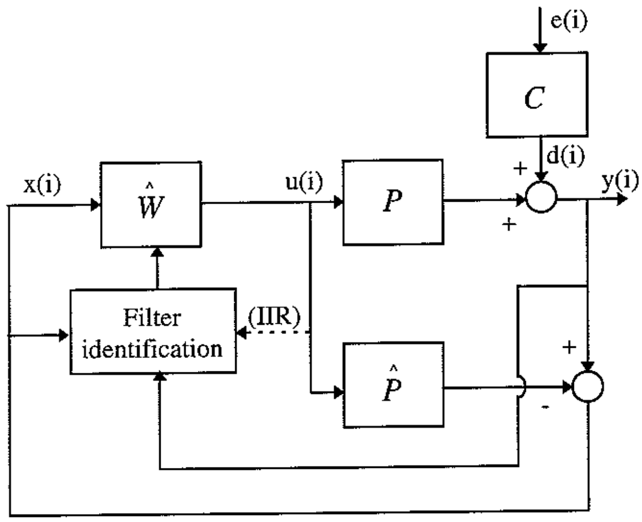


Fig. 12 Adaptive IMC.

4.2.1 RLS

Because the surface to be minimised is quadratic (see Eqs. (72) and (66)), Recursive Least Squares (RLS) algorithm can be then employed to identify the parameters on-line. The expectation operator E is approximated by the sum operator, and usually additional forgetting term is included:

$$\hat{\theta}(i) = Arg \min_{\hat{\theta}(i)} \sum_{l=0}^{i} \left\{ \alpha^{i-l} \varepsilon^2(i) \right\}, \tag{74}$$

where:

$$0 << \alpha \leq 1 \text{ or in practice: } 0.97 \leq \alpha \leq 1. \tag{75}$$

This can be interpreted as the forgetting of old datum (errors), on the present identification, with time constant:

$$\tau = \frac{T_p}{1-\alpha}. \tag{76}$$

RLS algorithm with forgetting is given by following equations:

$$\hat{\theta}(i) = \hat{\theta}(i-1) + \mathbf{k}(i) \left[\underbrace{\gamma(i) - \bar{r}_0\, u(i-k) - \varphi^T(i-k)\hat{\theta}(i-1)}_{\varepsilon(i)} \right], \tag{77}$$

where the amplification vector is given by:

$$\mathbf{k}(i) = \mathbf{P}(i)\varphi(i-k) = \frac{\mathbf{P}(i-1)\varphi(i-k)}{\alpha + \varphi^T(i-k)\mathbf{P}(i-1)\varphi(i-k)}, \tag{78}$$

and the matrix $\mathbf{P}$ is of the form:

$$P(i)=\frac{1}{\alpha}\left[P(i-1)-\frac{P(i-1)\varphi(i-k)\varphi^T(i-k)P(i-1)}{\alpha+\varphi^T(i-k)P(i-1)\varphi(i-k)}\right] \tag{79}$$

with the initial condition:

$$\mathbf{P}(0)=\zeta\,\mathbf{I}. \tag{80}$$

ζ is a positive value assumed arbitrarily (e.g. ζ = 100).

Matrix **P** is called the covariance matrix because if multiplied by the variance of white noise $e(i)$ it has variances of parameters of the identified vector on its diagonal, and beyond - covariances (without forgetting: α = 1.0). Trace of matrix **P** can then serve as an assessment of convergence for RLS without forgetting. For α < 1.0, and stationary plant and disturbance, care must be taken of the conditioning of the matrix (D-U decomposition, constant trace, varying forgetting or other - well developed methods widely presented in the literature on process identification, e.g. Ljung and Soderstrom, 1983, Stoica and Soderstrom, 1989, or Zarrop and Wellstead, 1991).

When the disturbance shaping filter (being already minimum phase) can be factorised as:

$$C=A\,C^* \tag{81}$$

and C^* satisfies the positive reality condition:

$$\underset{0\le\omega T_s\le\pi}{\forall}\ \mathrm{Re}\left\{\frac{1}{C^*}-\frac{1}{2}\right\}\Bigg|_{z^{-1}=e^{-j\omega T_s}}>0 \tag{82}$$

then, after Ljung, 1977:

$$\hat{\theta}(i)\rightarrow\theta_{opt} \tag{83}$$

with unit probability provided the disturbance is sufficiently reach (the plant is sufficiently excited) and orders of polynomials of the controller satisfy adequate conditions defined in appendix A (otherwise, the estimate $\hat{\theta}(i)$ is biased which does not lead to instability, yet). Further, if k = 1 then, according to Ljung, 1977, the estimate $\hat{\theta}(i)$ is consistent.
Condition (81) is justified when $\dim C >> \dim A$ because in this case dim A roots of polynomial C which are close enough to roots of polynomial A are likely to appear. Moreover, as seen on Fig. 1, both the plant and the disturbance shaping filter contain the same measurement path (the residual microphone, anti-aliasing filter and analogue-to-digital converter).

4.2.2 LMS

Another algorithm that can be employed to identify the parameters of the controller (74) on-line is based on the steepest descent search. In such an iterative process, controller parameters are updated in the following way:

$$\hat{\theta}(i+1)=\hat{\theta}(i)-\frac{\mu}{2}\frac{\partial E\{\varepsilon^2(i)\}}{\partial\hat{\theta}(i)}, \tag{84}$$

where μ is the convergence coefficient. Alternatively, the gradient can be written in terms of the error and reference signal as:

$$\frac{\partial E\{\varepsilon^2(i)\}}{\partial\hat{\theta}(i)}=E\left\{2\varepsilon(i)\frac{\partial E\{\varepsilon(i)\}}{\partial\hat{\theta}(i)}\right\}. \tag{85}$$

Substituting for $\varepsilon(i)$ from Eq. (66), Eq. (85) takes the form:

$$\frac{\partial E\{\varepsilon^2(i)\}}{\partial\hat{\theta}(i)}=-2E\{\varepsilon(i)\varphi_{i-k}\}. \tag{86}$$

Since $\hat{\theta}(i)$ is updated in the opposite direction to the gradient of the error surface, it will descent on the error surface towards its minimum. However, estimation of the cross-correlation term $E\{\varepsilon(i)\varphi(i-k)\}$ would require a lengthy averaging procedure which is difficult to perform in on-line procedure. Therefore, a simplified alternative is found in the form of the Least Mean Square (LMS) algorithm (Widrow and Stearns, 1985) where the gradient is estimated by the instantaneous value $\varepsilon(i)\varphi(i-k)$. Then, the update equation becomes (substituting for $\varepsilon(i)$):

$$\hat{\theta}(i+1)=\hat{\theta}(i)+\mu\varphi(i-k)\underbrace{\left(\gamma(i)-\hat{r}_0\,u(i-k)-\varphi^T(i-k)\hat{\theta}(i)\right)}_{\varepsilon(i)}. \tag{87}$$

Control value is calculated according to (73).

Some features of LMS as well as some modifications of this algorithm are discussed further in section 4.3.1.

4.3 SIGNAL PROCESSING APPROACH

In section 4.2.2 the application of LMS algorithm to identify parameters of both MVC and IMC, FIR or IIR, on the basis of the predictive model of the plant was presented. However, originally LMS was designed to search optimal Wiener solution - filter W given by Eq. (40) (this is obviously adopted to seek IMC filter) as:

$$\hat{W}(i)=Arg\min_{\hat{W}(i)}E\{y^2(i)\}. \tag{88}$$

4.3.1 IMC with FXLMS

Recalling Eq. (32) and following the reasoning presented in section 4.2.2, this leads to the LMS update equation:

$$\hat{W}(i+1) = \hat{W}(i) + \mu \mathbf{r}(i) y(i). \tag{89}$$

If applied to noise cancellation problem considered in this research, and particularly to the IMC structure for perfect model of the plant, the input signal to the adaptive filter is expressed in terms of the reference signal (follow Figs. 6, 7, 8, and Eq. (31)):

$$r(i) = \mathbf{p}^T \mathbf{x}(i), \tag{90}$$

where $\mathbf{p}$ is defined by Eq. (29) and $\mathbf{x}$ - by Eq. (27) (see Fig. 8). Because the plant impulse response $\mathbf{p}$ is unknown, the impulse response of its model $\hat{\mathbf{p}}$ is used instead of it. Thus, the parameter update equation can be written together as:

$$\begin{cases} \hat{W}(i+1) = \hat{W}(i) + \mu \hat{\mathbf{r}}(i) y(i) \\ \hat{r}(i) = \hat{\mathbf{p}}^T \mathbf{x}(i). \end{cases} \tag{91}$$

This form of LMS is referred to as Filtered-x LMS (FXLMS) (see Fig. 13) and is rather a feedforward control algorithm then an identification method.

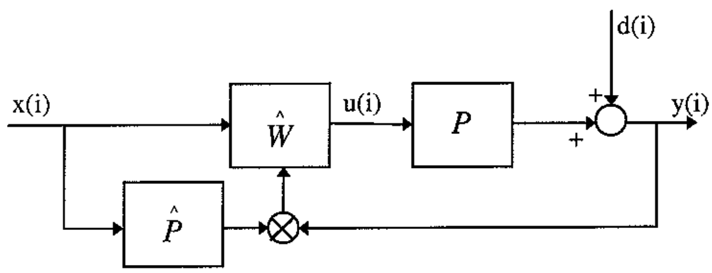


Fig. 13 Filtered-x LMS.

It must be emphasised here that it has been derived after commuting the order of the plant and the control filter (see Figs. 7 and 8) which is only true under assumption that both of them are time-invariant. In practice the controller is adaptive and time-varying, while the plant can also be time-dependant, so Fig. 7 and Fig. 8 are not equivalent. However, when they are slowly varying in time comparing to delays involved in the plant and the controller, this is a reasonable assumption. The accuracy of the plant model affects the convergence of FXLMS and the whole IMC system. It has been found, however, (Burgess, 1981, Morgan, 1980, Elliott and Nelson, 1993) that this algorithm will converge as long as the cross-correlation matrix between $\mathbf{r}$ and $\hat{\mathbf{r}}$ has positive eigenvalues. For tonal disturbances this condition implies that for convergence the phase mismatch between the model and the plant must be less then $\pi/2$.

The LMS algorithm has the advantage of simple implementation and usually the disadvantage of slow adaptation if $r(i)$ is not a white noise or a tone. Convergence coefficient μ is responsible for the speed of adaptation (the bigger μ the faster adaptation) and for fluctuations about the steady-state solution with variance (Widrow and Stearns, 1985, Haykin, 1991, Rafaely and Elliott, 1996a):

$$E\left\{\hat{W}(i) - W_{opt}\right\} \approx 2\mu E\left\{y^2(i)\right\}. \tag{92}$$

Widrow and Stearns, 1985, showed that FXLMS algorithm converges provided the convergence coefficient is bounded by the following value:

$$\mu < \frac{2}{N\sigma_r}, \tag{93}$$

where N is the length of the FIR control filter, $\hat{W}(i)$, and σ_r^2 is the variance of the filtered reference signal, $r(i)$.

As seen from the analysis presented above, the choice of value of μ has crucial influence on all features of an adaptive control system with LMS identification. It is obvious that when the level of input signal to the controller changes the convergence coefficient should be properly adapted. Therefore, a normalised version is commonly used in practice where the convergence coefficient becomes time-dependant as follows:

$$\mu \Rightarrow \mu(i) = \frac{\mu}{\mathbf{r}^T(i)\mathbf{r}(i)}. \tag{94}$$

Another modification to the LMS algorithm which increases its robustness is used in the term of the Leaky LMS (Widrow and Stearns, 1985). It has the following update equation:

$$\hat{W}(i+1) = \xi\,\hat{W}(i) + \mu\,\mathbf{r}(i)y(i), \tag{95}$$

where ξ is the leak factor and $0 << \xi \le 1$. The effect of adding the leak factor is equivalent to using the conventional steepest descent algorithm with a modified objective function:

$$L(i) = E\{y^2(i)\} + \frac{1-\xi}{\mu/2}\hat{\mathbf{w}}^T(i)\hat{\mathbf{w}}(i). \tag{96}$$

This prevent the filter coefficient from obtaining high values without a significant reduction in the mean square error, which has a stabilising effect on the adaptive process. This is equivalent to minimising the cost function given by Eq. (44) where:

$$\beta = \frac{1-\xi}{\mu/2}. \tag{97}$$

4.3.2 Fully adaptive IMC with FXLMS

In IMC structure a model of the plant is required (see Fig. 6). Its accuracy has significant influence on the reference signal, controller identification, and finally on the performance of the entire system and its stability. If no other "precautions" are made (see section 4.3.3) the model should follow the plant in terms of its changes, if they appear. Rafaely and Elliott, 1996, showed that the plant response (the model) cannot be extracted from the observable signals in the IMC control system when the only disturbance is present. Therefore, an extraneous identification signal is added to the control signal. The simplest and most obvious form of this fully adaptive system is presented on Fig. 14.

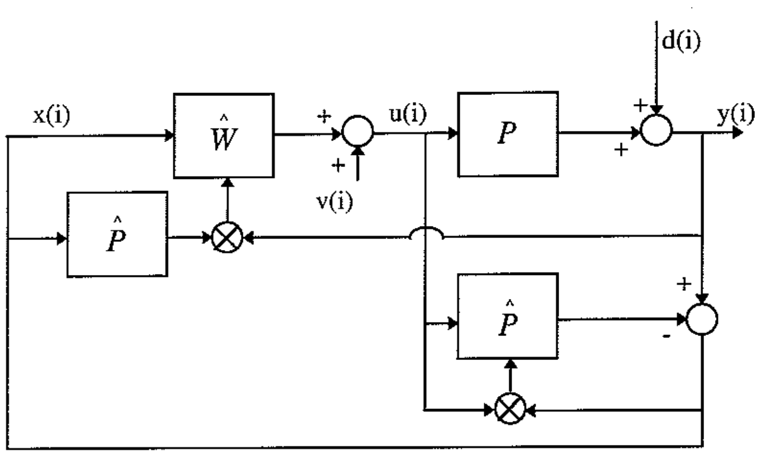


Fig. 14 Fully adaptive IMC system with injection of low-level random noise v(i). The control filer identified via Filtered-x LMS, and the model of the plant - via LMS.

However, the observed signal (equivalent to the reference signal for IMC) whose power is minimised to adopt the model is of the form:

$$x(i) = \frac{1}{1 - \hat{W}\left(P - \hat{P}\right)} d(i) + \frac{\left(P - \hat{P}\right)}{1 - \hat{W}\left(P - \hat{P}\right)} d(i). \tag{98}$$

Thus, the identification error surface:

$$L_{\hat{P}}(i) = E\left\{x^2(i)\right\} \tag{99}$$

is not quadratic and LMS might have difficulties in converging to the optimal value.
Rafaely and Elliott, 1996, suggest two other methods of identification of the model during controller adaptation.

The first one is called 'joint input / output adaptation of the plant model' and is performed in three different stages (refer to Fig. 14):
1) An LMS identification system identifies the response from $v(i)$ to $u(i)$.
2) An LMS identification system identifies the response from $v(i)$ to $y(i)$.
3) An LMS identification system extracts the model from 1) and 2).

In the second plant identification structure, proposed by Rafaely and Elliott, 1996, in the first stage the response from $v(i)$ to $x(i)$ is identified by an LMS system:

$$F_a = \frac{\left(P - \hat{P}\right)}{1 - \hat{W}\left(P - \hat{P}\right)} \tag{100}$$

and then, by $P - \hat{P}$ is extracted by another LMS identification system, as:

$$F_b = P - \hat{P} = \frac{-F_a}{1 + \hat{W} F_a}. \tag{101}$$

Finally, the plant model is updated as:

$$\hat{P}_{new} = \hat{P} + F_b. \tag{102}$$

All the fully adaptive algorithms presented above suffer problems of modelling accuracy connected to the presence of signal $d(i)$ and finite lengths of the temporary filter used. Apparently, they involve multiple identification procedure to be performed on-line. As also seen from Fig. 14, the extraneous noise injected affects the system output and reduces the performance.

4.3.3 Robust IMC - MFXLMS

In the previous section different fully adaptive IMC algorithms were presented. Although they seem to be able to track changes in the plant and adapt the controller to them, many obstacles emerge which significantly influence efficiency and stability of the entire control system.

To protect the closed-loop IMC control system against instability due to a mismatch between the plant and its model (which can be caused by modelling errors or nonstationarity of the plant), instead of on-line adaptation of the model of the plant, another approach can be used. This approach is based on H_∞ control theory (refer to section 3.5). It aims at ensuring sufficient stability margin for a certain group of mismatch between the plant and its model which is arbitrarily assumed (e.g. on the basis of former off-line identification in the open-loop), and on-line adaptation of the controller filter, $\hat{W}(i)$.

Let us now recall the approximated (the H_∞ term replaced by a weighted H_2 term) cost function, defined for IMC system with perfect model of the plant given by Eq. (52). This can be equivalently expressed in the time domain as:

$$L(i) = E\{y^2(i)\} + \beta E\{z^2(i)\}, \tag{103}$$

where an artificial signal, $z(i)$, has been introduced and it has the form:

$$z(i) = W_2 \hat{P} W v(i). \tag{104}$$

$v(i)$ in Eq. (104) is the white noise, of unit variance, uncorrelated with $d(i)$. Due to orthogonality of $y(i)$ and $z(i)$ (that can be assumed because, as will be shown further, signals $v(i)$ and $z(i)$ appear only in the identification procedure and do not really exist in the system), minimisation of the cost function given by Eq. (103) is equivalent to minimisation of:

$$L(i) = E\{y^2(i) + \beta z^2(i)\} = E\{y_\beta^2(i)\}, \tag{105}$$

where (106)

$$y_\beta(i) = y(i) + \sqrt{\beta} z(i). \tag{107}$$

Following the same steps for derivation a form of the steepest descent algorithm, as presented earlier in sections 4.2.2 and 4.3.1, we obtain:

$$\hat{W}(i+1) = \hat{W}(i) + \mu \hat{P}\left[x(i) + \sqrt{\beta}\, W_2 v(i)\right]\underbrace{\left[y(i) + \sqrt{\beta}\, W_2 \hat{P}\hat{W}(i) v(i)\right]}_{y_\beta(i)}. \tag{108}$$

The weighting coefficient β can be regarded as the variance of $v(i)$ and then Eq. (108) simplifies to:

$$\hat{W}(i+1) = \hat{W}(i) + \mu \hat{P}\left[x(i) + W_2 v(i)\right]\underbrace{\left[y(i) + W_2 \hat{P}\hat{W}(i) v(i)\right]}_{y_\beta(i)}. \tag{109}$$

This algorithm, designed for optimal performance and robust stability, is referred to as Modified Filtered-x LMS (MFXLMS) and presented on Fig. 15.

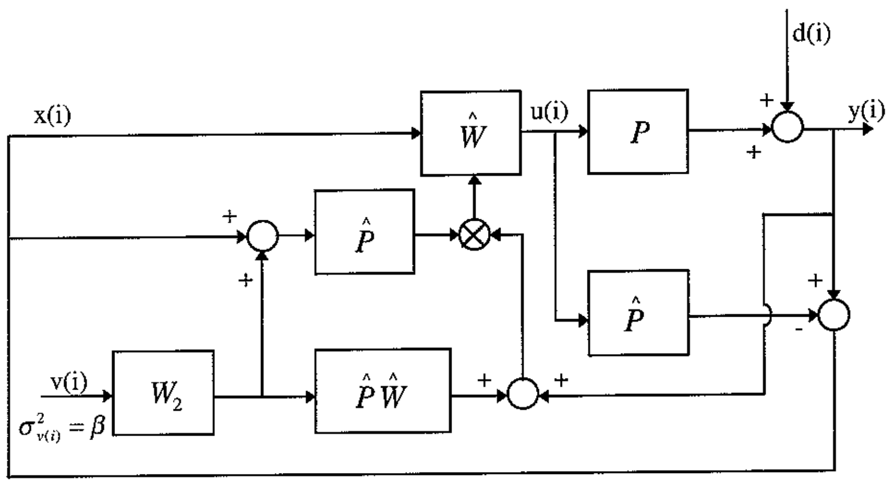


Fig. 15 Modified Filtered-x LMS for IMC - robustness included.

5 SIMULATIONS

Throughout the work many active noise control algorithms were presented in both fixed (optimal) and adaptive versions, and theoretically compared. However, as it was stressed several times, most of them were derived under some assumptions which are not always fulfilled. Apparently, during parametrization a trade-off usually appears. It concerns convergence requirements, speed of convergence, computational burden, satisfactory performance, and sufficient stability robustness. Also, during working of a control system, additional problems associated with numerical errors emerge. It is thus clear that a simulation analysis is required to give a survey of the algorithms from practical point of view. To perform the simulation test the plant and the disturbance should be chosen.

5.1 THE PLANT AND DISTURBANCE

Simulations were performed with 2 [kHz] sampling frequency. A minimum phase, second order, unit delay plant was chosen, described by Eq. (110):

$$y(i) = z^{-1}\frac{0.420 - 0.050z^{-1} - 0.160z^{-2}}{1 - 0.367z^{-1} + 0.068z^{-2}}. \tag{110}$$

Fig. 16 shows magnitude of its frequency response and zeros / poles distribution.

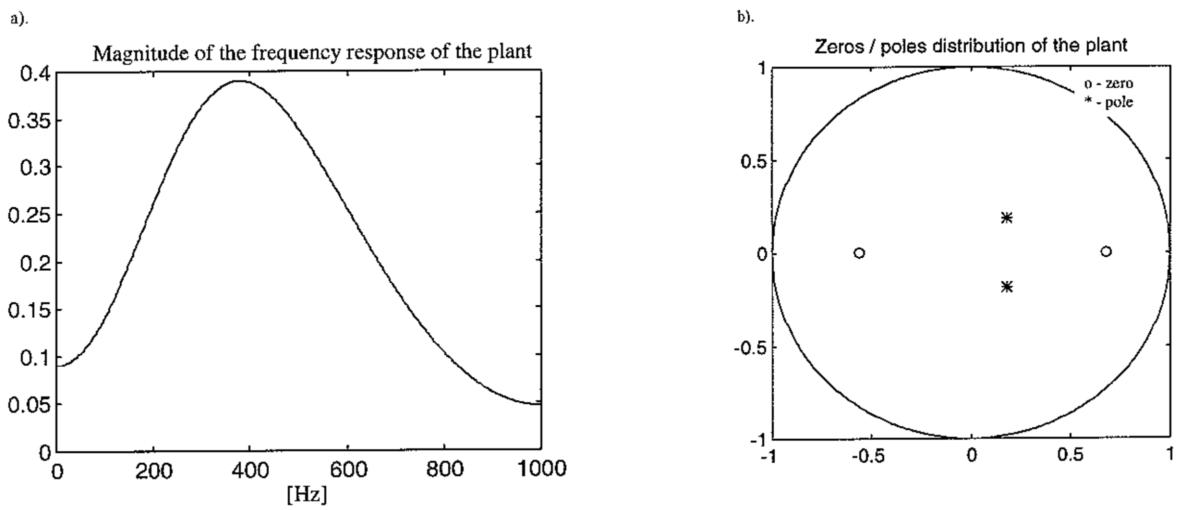


Fig. 16 The plant: k = 1; A = [1 -0.367 0.068]; B = [0.42 -0.05 -0.16]:
a). magnitude of the frequency response of the plant in linear scale;
b). zeros / poles distribution.

The disturbance shaping filter modelling the noise to be cancelled is represented by a non-minimum phase FIR filter of 20th order. Magnitude of the frequency response of the filter and its roots (zeros) distribution (before, and after spectral factorization) presents Fig. 17.

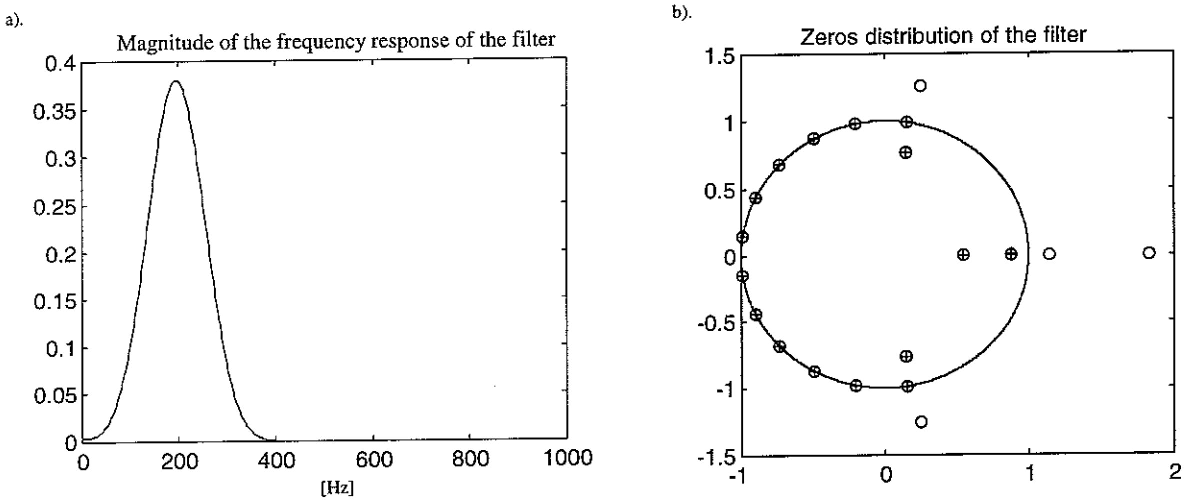


Fig. 17 The disturbance shaping filter:
a). the magnitude of the frequency response of the filter in linear scale;
b). zeros distribution: 'o' - before spectral factorisation; '⊕' - after spectral factorisation.

For further tests a plant nonstationarity will be introduced by sinusoidal change of first parameter of the plant according to the following rule (in MATLAB notation):

if time > 500, B(1) = 0.42 + 0.2*sin(2*pi*200.*time /2000); end. (111)

This affects both magnitude and phase of the frequency response. Fig. 18 shows the boundaries of zeros placement and magnitudes of frequency responses which are reached one after another at every 5 samples.

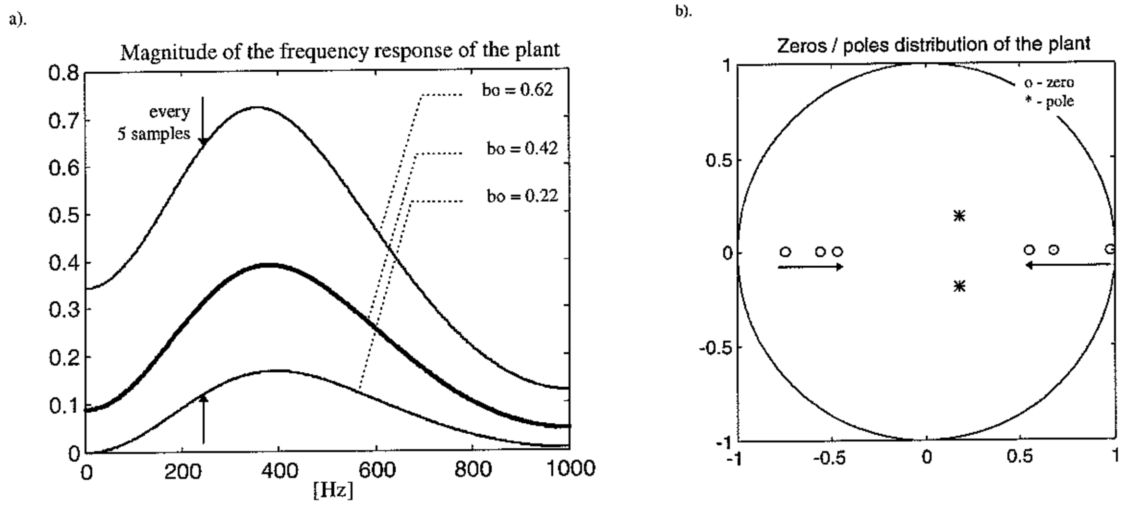


Fig. 18 The nonstationary plant:
a). the magnitude of the frequency response of the plant in linear scale;
b). zeros / poles distribution.

5.2 OPTIMAL PERFORMANCE

The best attenuation calculated according to Eq. (15) without spectral factorisation was **34.35 [dB]** and the attenuation of optimal fixed IMC filter designed as IIR filter (10) (or MVC) was

31.34 [dB] (see Fig. B.1). However, any fixed controller cannot yield such an attenuation because it is unstable. After spectral factorisation over non-minimum phase parts of polynomial C, the best optimum attenuation was calculated, according to the same formula, as **23.67 [dB]**.
Optimal fixed IMC filter designed as IIR filter produced the attenuation of **23.64 [dB]** (see Fig. B.2). The small discrepancy can come from numerical errors. Tab. B1 and Fig. B.3 show the dependence of attenuation with respect to discrete time delay of the plant. Optimal fixed IMC filter designed as an FIR (40) filter produced the performance, calculated according to Eq. (42), which depends on the number of parameters of the filter. For 550 parameters the attenuation achieved was **23.58 [dB]** (see Fig. B.6).
The dependence of the attenuation with respect to the number of parameters is illustrated in Tab. B.2 and on Fig. B.7. As it is seen, it is still possible to achieve better performance by increasing the number of parameters. This can be easily deduced on the basis of equations defining optimal controllers in which polynomial C appears in their denominators. In turn, from its roots distribution (see Fig. 17) follows that the disturbance mainly consists of random sine-waves (random phase and amplitude) because most of the roots are very close to the unit circle. Thus the discrete impulse response of the optimal feedforward IIR filter decays very slowly what illustrates Fig. B.4. However, (see Fig. B.7) the difference in attenuation for 550 and 100 parameters of the filter is less then 0.5 [dB]. Further, if we agree to loose about 3.5 [dB] we can reduce the filter length to only 4 parameters (see Tab. B.2). In practice, for real plant, it is even better to chose less parameters for many reasons:
- the more parameters the higher computational burden which can lead to a reduction of the sampling rate;
- the more parameters the more numerical errors and higher sensitivity of the whole system;
- in real applications for high number of parameters the system tries to interpolate noise of different type, like quantization noise, and measurement noise and can be far from the optimum one for different realisation;
- in adaptive systems, the more parameters the slower convergence rate (see subsections to section 5.3 for details).

5.3 ADAPTIVE CONTROLLERS

Adaptive control systems presented in the work were tested in simulations.
It should be emphasised here that while designing and parametrizing an adaptive control system aiming at cancellation of acoustic noise (of whichever type it is) a reasonable trade-off between the attenuation and the speed of convergence should be considered and taken into account. It is imposed by sensibility of applications of that kind (for some ANC applications, like an acoustic duct, the convergence time can be of order of minutes while for active headsets it should be less then one fifth of second to follow a movement of the head of a wearer).
All the adaptive systems are further analysed in terms of numbers of parameters of controllers, discrete time delay of the plant, speed of convergence, plant nonstationarities (see Eq. (111) and Fig. 18) and parametrizations assigned to each one separately. Presented attenuation results were obtained after the time of 2000 samples (1 [s] for fs = 2 [kHz]) which was found to be "enough" to allow to examine main features of the adaptive systems. By the convergence time we consider the time (in samples) after which the attenuation differs from the best possible one, under present conditions, of no more than 3 [dB].
IMC systems are also analysed in terms of a significant (in spectral features) model mismatch with respect to the plant. This mismatch was introduced by setting the first coefficient of the nominator of the model on 0.22 or 0.62 whilst for the plant it is 0.42 (see Fig. 18).

5.3.1 MVC RLS

1. The number of parameters

The controller was assumed to have the same number of parameters in numerator and denominator and furthermore the total number of parameters is only referred to. Simulations showed that this is equivalent to different numbers, in terms of performance, for more than 4 parameters.
The best performance achieved was about **21.39 [dB]** which is less then the calculated one. Starting from 14 parameters the performance is about to be saturated (**21.38 [dB]**) and this number will be considered in further investigations. It can be also deduced that increasing the number of parameters over a particular value leads to decreasing the attenuation what can be explained by numerical errors or convergence difficulties.
For details refer to Tab. B.3 and Fig. B.8.

2. Speed of convergence

The speed of convergence depends on the number of parameters of the controller and the forgetting factor in RLS identification method. For N = 14 and α = 1.0 it is about 30 samples. But for the same number of parameters and α = 0.99 it is only about 10 samples (less then the number of parameters because the speed depends only on the bigger number of parameters in numerator or denominator of the controller). It can be assessed on the basis of the trace of covariance matrix. For α < 1.0 and stationary plant and disturbance care must be taken of the conditioning of the matrix.
For N = 2 the convergence time is only 3 samples what is equivalent to 1.5 [ms] and attenuation about **9 [dB]**.

3. Effects of plant nonstationarity

The control system proved to be stable and yielded the attenuation that reaches **7.70 [dB]** for α = 1.0 and **9.26 [dB]** for α = 0.99 (see Figs. B.12 and B.13, respectively). Better results were obtained for weighting (WMVC) coefficient q = 0.05 $\neq$ 0: **8.65 [dB]** for α = 1.0 and **12.93 [dB]** for α = 0.99 (see Figs. B.14 and B.15). This testifies that the system is then more conservative and less sensitive to nonstationarities and nonlinearities. When both the plant and the disturbance shaping filter are time-invariant and linear then q $\neq$ 0 deteriorates the performance. It is also seen then that, in case of plant nonstationarities, the forgetting allows to better track the changes and improves performance.

4. Discrete time delay

The attenuation drops down with the discrete time delay in an exponential-like way. For delays more than 7 there is no sense to employ MVC - the attenuation achieved is less then 1 [dB].
For details refer to Tab. B.4 and Fig. B.8.

5.3.2 IMC IIR RLS

1. The number of parameters

The IMC IIR filter was assumed to have the same number of parameters in numerator and denominator and furthermore the total number of parameters is only referred to.
The best performance achieved was about **21.60 [dB]** which is less then the calculated one. It is also seen that starting from 14 parameters the performance is about to be saturated (**21.45 [dB]**) and this number of parameters will be considered in further investigations. It can be also deduced that increasing the number of parameters over a particular value leads to decreasing the attenuation what can be explained by numerical errors or convergence difficulties.
For details refer to Tab. B.5 and Fig. B.16.

2. Speed of convergence

Similar conclusions as for MVC (see sec. 5.3.1).

3. Effects of plant nonstationarity

The control system proved to be stable and yielded the attenuation that reaches **10.46 [dB]** for $\alpha = 0.99$ and **8.12 [dB]** for $\alpha = 1.0$ - see Fig. B.20 and B.21, respectively. Better results, **9.07 [dB]** (for $\alpha = 1.0$), were obtained for the case with control effort included in the cost function ($q = 0.05$).

4. Discrete time delay

The same conclusions as for MVC.
For details refer to Tab. B.6 and Fig. B.17.

5. Model mismatch

For B1(1) = 0.62 the attenuation achieved was **21.43 [dB]** (see Fig. B.22) and for B1(1) = 0.22: **21.34 [dB]** (see Fig. B.23) which comparing to the case with perfect model of the plant (B1(1) = 0.42), **21.60 [dB],** testifies that IMC IIR RLS control system is not sensitive to this kind of mismatch. Only the speed of convergence is slightly reduced. Worse results, **11.15 [dB]** (for B1(1) = 0.22), were obtained for $q = 0.05$.

5.3.3 IMC FIR RLS

1. The number of parameters

An FIR controller constitutes only an approximation of the IIR one. Thus, the more parameters the better attenuation can be expected. Although, after crossing a particular threshold numerical errors overcome the benefits of better approximation and the attenuation drops down. The best result obtained was about **22.33 [dB]**. However, this is for about 30 parameters what significantly reduces the speed of convergence. On the other hand, to have the opportunity to compare the behaviour of different algorithms with similar computational burden, 14 parameters of the feedforward adaptive filter were chosen which yielded the attenuation of about **21.61 [dB]**

(only 0.72 [dB] less than for 30 parameters). This choice can be also justified taking into account that in fact in IMC, the whole controller includes a model of the plant and in this case it is still of the IIR type. It is also worthy to know that that for 4 parameters only, the attenuation is about **19.89 [dB]** which in many cases can be considered sufficient due to very efficient numerical effort and convergence speed.
For details refer to Tab. B.7 and Fig. B.26.

2. Speed of convergence

The adaptive IMC FIR RLS system turned out to be relatively slow for α = 1.0 and after 2000 samples the attenuation was less than **14 [dB]** (compare Figs. B.28 and B.29). This behaviour can be explained taking into account the fact that the overall controller is of IIR type (it includes a model of the plant) and parameters in its numerator and denominator are closely correlated. Introducing a forgetting factor in RLS identification procedure dramatically speeds up the system allowing it to yield of about **8 [dB]** better attenuation in the same time (α = 0.99). However, in case of plant nonstationarity or model mismatch, the speed and attenuation results are comparable for both considered values of α.

3. Effects of plant nonstationarity

The control system proved to be stable and yielded satisfactory attenuation which reaches **9.32 [dB]** for α = 1.0 and **9.56 [dB]** for α = 0.99 (see Figs. B.30 and B.31, respectively). Similar results, **9.28 [dB],** were obtained for $q = 0.05$ (α = 1.0).

4. Discrete time delay

The same conclusions as for MVC.
For details refer to Tab. B.8 and Fig. B.27.

5. Model mismatch

For B1(1) = 0.62 the attenuation achieved was **16.47 [dB]** (Fig. B.32) and for B1(1) = 0.22: **12.07 [dB]** (Fig. B.33) which comparing to the case with perfect model of the plant (B1(1) = 0.42), **21.61 [dB],** testifies that IMC IIR RLS control system is not sensitive to this kind of mismatch. Only the speed of convergence is slightly reduced. Worse results, **10.45 [dB],** were obtained for $q = 0.05$ (B1(1) = 0.22)

5.3.4 IMC FIR MFXLMS

1. The number of parameters

Similarly to IMC FIR RLS (see section 5.3.3), the more parameters the better performance should be expected. However, numerical errors arise with the number of parameters and, after crossing a particular threshold, the attenuation drops down (see Tab. B.9 and Fig. B.36). The best attenuation, **21.44 [dB],** was obtained for 20 parameters. To have the opportunity to compare the results to those presented in previous sections the experiments were performed further for 14 parameters (**21.38 [dB]**).

2. Speed of convergence

The convergence closely depends upon the convergence coefficient (compare Figs. B.38 and B.39). However, even for the optimum one found, μ = 1.5 (which also guarantee stability), it was about 500 samples. However, for nonstationary plant or model mismatch the convergence coefficient had not so crucial importance (see Figs. B.40 and B.41).

3. Effects of plant nonstationarity

The control system proved to be stable and yielded satisfactory attenuation which reaches **13.30 [dB]** for μ = 1.5 and **11.99 [dB]** for μ = 0.8 (see Figs. B.40 and B.41). Better results were obtained when the robustness was included (β = 1e-3): **13.34 [dB]** for μ = 1.5 (Fig. B.44) .

4. Discrete time delay

The same conclusions as for MVC.
For details refer to Tab. B.10 and Fig. B.37.

5. Model mismatch

For B1(1) = 0.62 the attenuation achieved was **18.21 [dB]** (Fig. B.42) and for B1(1) = 0.22: **14.78 [dB]** (Fig. B.43) which comparing to the case with perfect model of the plant (B1(1) = 0.42), **21.38 [dB],** testifies that IMC FIR MFXLMS control system is sensitive to this kind of mismatch. The speed of convergence is slightly reduced. Better results were obtained for β = 1e-3: **14.94 [dB]** (B1(1) = 0.22).

5.3.5 IMC FIR FXLMS fully adaptive

The last control structure examined was the fully adaptive IMC system with both the model of the plant and the control filter identified on-line via Normalised LMS algorithms (control filter - with reference filtering) and injection of low-level identification signal at the input of the plant and its model (Fig. 7). Unfortunately, this structure does not assure proper identification of the model of the plant and the control filter is not identified correctly. This reflects in very poor attenuation - only **4.14 [dB]** (see graph on Fig. B.46). The convergence time is extremely long.

6 SUMMARY

6.1 Conclusions

Different feedback systems, aimed at the minimisation of energy of the undesired signal, were analysed.

Is there any sense in such a comparison?
If the model of the plant used in IMC is precise and accurately reflects the plant itself, a significant amount of information is supplied to the control system. The controller is very close to a feedforward filter. Thus better results in terms of attenuation and convergence (during the adaptation) can be expected in comparison to the case when the whole feedback has to be identified. However, when the model differs a lot from the plant, an additional large error is introduced to the identification procedure, so the results obtained are expected to be worse.

Why are both FIR and IIR IMC structures compared?
Usually an FIR filter is used for IMC. However, when properly identified via RLS on the basis of the predictive model of the plant, the cost function surface is quadratic for both FIR and IIR versions and the global minimum is found. Besides, the computational burden is exactly the same. On the other hand, it is obvious that an IIR filter should ensure better results, although for longer filters care must be taken because numerical and quantization errors are accumulated. It is extremely important to mention that the convergence time is much longer for FIR filters (at least a few times).

Why thus MVC structure is analysed?
MVC cannot tell the difference between the plant and disturbance. It treats them in the same way. Apparently, it does not require prior identification of the model of the plant except for the evaluation of its discrete time delay. That is why it is less sensitive to any nonstationarities and in that case a bit faster in convergence. When directly identified it involves much less computations than IMC.

Generally, all the adaptive systems considered in the report do not require a large number of filter parameters. Moreover, after crossing a particular threshold numerical errors overcome the benefits of better approximation of the optimal controller and the attenuation is reduced. Besides, the more parameters the lower speed of convergence.
The performance achieved by each algorithm similarly depends upon discrete time delay of the plant.
Control effort (in the Z - transform domain approach) or robustness coefficient (in the signal processing approach) introduced in the cost function being minimised, makes the systems more conservative and less sensitive to any nonstationarities or, for IMC, model mismatches.

6.2 Main contributions of this research

1. Optimal MVC and IMC algorithms have been derived in the Z - transform domain. It has been proved that for a perfect model of the plant, required by IMC, they are equivalent. Stability conditions have been also derived.

2. Provided that the plant is stable and minimum phase the only things that are necessary to precisely evaluate the best possible attenuation are: the discrete time delay of the plant and the FIR disturbance shaping filter i.e. the plant model is not required. But, if the filter is non-minimum phase, spectral factorization over it should be first performed.

3. If the plant is non-minimum phase, it is possible to precisely calculate the control effort weighting coefficient, q, and thus precisely evaluate the best possible attenuation, control signal variance and the form of the output signal.

4. For minimum phase plant, it follows from the form of the output signal that only for $k = 1$ is the output white noise. Otherwise it is an MA process and its form can be deduced prior to control, only on the basis of the disturbance shaping filter and the plant delay.

5. The effects of measurement noise have been examined and it has been shown that its presence makes the system robustly stable to plant variations defined by the multiplicative plant uncertainty bounded by the spectrum of the noise itself.

6. The performance limitations are thoroughly examined. Special attention is paid to the influence of plant delay and some solutions are suggested.

7. The predictive model of the plant has been derived which expresses polynomials of the controller (MVC or IMC) in a linear way and makes the identification surface quadratic (with only one, global minimum). It should be stressed here that MVC is identified directly - without the necessity to first identify a model of the plant and then calculate the controller. Constraints of this method are also analysed.

8. An LMS algorithm searching the global minimum (over convex identification surface) for IIR IMC filter, or MVC has been derived (with, and without control effort) which constitutes an alternative to Filtered-x LMS.

9. The proof for Modified Filtered-x LMS (robustness included for the multiplicative plant uncertainty) has been given.

REFERENCES

Burges J.C. (1981): "Active adaptive sound control in a duct: a computer simulation", Journal of Acoustical Society of America, 70, pp. 715-726.

Doyle J.C., B.A. Francis, A.R. Tannenbaum (1992): "Feedback Control Theory", Maxwell MacMillan International.

Elliott S.J., P.A. Nelson (1993): "Active Noise Control", IEEE Signal Processing Magazine, pp. 12-35.

Elliott S.J. (1994): "Active Control using Feedback", ISVR Technical Memorandum No. 732, University of Southampton, UK.

Elliott S.J., T.J. Sutton (1996): "Performance of Feedforward and Feedback Systems for Active Control", IEEE Transactions on Speech and Audio Processing, vol. 4, no. 3, pp. 214-223.

Goodall R.M., W. Forsythe (1991): "Digital Control", Mc Graw-Hill, NY.

Haykin S. (1991): "Adaptive Filter Theory", Prentice Hall, N.J.

Johansson R. (1993): "System Modeling and Identification", Prentice Hall.

Ljung L. (1977): "On positive real transfer function and convergence of some recursive schemes", IEEE Transactions on Automatic Control, AC22, pp. 539-551.

Ljung L., T. Soderstrom (1983): "Theory and Practice of Recursive Identification", The MIT Press, Cambridge, Massachusetts.

Matko D., R. Isermann, K.H. Lachmann (1992): "Adaptive Control Systems", Prentice Hall.

Morgan R.M. (1980): "An analysis of multiple correlation cancellation loops with a filter in the auxiliary path", IEEE Transactions on Acoustics, Speech and Signal Processing, 28 (4), pp. 454-467.

Nelson P.A., S.J. Elliott (1994): "Active Control of Sound", Academic Press Limited, Cambridge, UK.

Niederlinski A., J. Moscinski, Z. Ogonowski (1995): "Adaptive Control" (in Polish), PWN, Poland.

Orfanidis J. (1988): "Optimum Signal Processing. An Introduction", Macmillan Publishing Company, NY.

Papoulis A., (1984): "Signal Analysis", Mc Graw-Hill, NY.

Pawelczyk M. (1995): "Noise cancellation in active ear defender - original modifications of some known adaptive algorithms", Proceedings of Second World Automation Congress, Montpellier.

Rafaely B. (1997): "Feedback Control of Sound", Ph.D. Thesis, ISVR, University of Southampton, UK.

Rafaely B., S.J. Elliott (1996): "Internal Model Controller for Feedback Control of Sound and Vibration", ISVR Technical Memorandum No. 776, University of Southampton, UK.

Rafaely B., S.J. Elliott (1996a): "An adaptive and robust feedback controller for active control of sound and vibration", UKACC International Conference on CONTROL'96.

Stoica P., T. Soderstrom (1989): "System Identification", Prentice Hall.

Thomas D.R., P.A. Nelson (1995): "A Comparison of some Strategies for Discrete Time Feedback Control", ISVR Technical Memorandum No. 754, University of Southampton.

Widrow B., S.D. Stearns (1985): "Adaptive Signal Processing", Prentice Hall, Englewood Cliffs, NJ.

Wittenmark B., K.J. Astrom (1989): "Adaptive Control", Addison-Wesley Publishing Company, London.

Zarrop M.B., P.E. Wellstead (1991): "Self - Tuning Systems. Control and Signal Processing", John Willey & Sons, London.

APPENDIX A

EQUATIONS

A.1 MVC

Consider the plant given by an IIR filter and the acoustic disturbance given by the white noise of variance λ^2 filtered by an FIR filter (measurement noise is neglected here due to its incomparably lower level than the level of the acoustic disturbance):

$$y(i) = z^{-k}\frac{B}{A}u(i) + \underbrace{Ce(i)}_{d(i)}. \tag{A.1}$$

Shifted by k samples it takes the form:

$$y(i+k) = \frac{B}{A}u(i) + Ce(i+k). \tag{A.2}$$

Let us define the cost function as the energy at the output of the plant:

$$L(i+k) = E\{y^2(i+k)\}. \tag{A.3}$$

The goal of control is to find such a control value $u(i)$ that minimises the cost function given above.

Define the Diophantine Equation (DE):

$$C = F + z^{-k}G; \quad \dim F = k-1; \quad \dim G = \dim C - k. \tag{A.4}$$

Substituting C in Eq. (A.2) by Eq. (A.4):

$$y(i+k) = \frac{B}{A}u(i) + Fe(i+k) + Ge(i). \tag{A.5}$$

From Eq. (A.1) $e(i)$ can be expressed as:

$$e(i) = \frac{1}{C}y(i) - \frac{1}{C}z^{-k}\frac{B}{A}u(i). \tag{A.6}$$

Combining Eqs. (A.5) and (A.6), we obtain:

$$y(i+k) = \frac{B}{A}u(i) + Fe(i+k) + \frac{G}{C}y(i) - z^{-k}\frac{G}{C}\frac{B}{A}u(i), \tag{A.7}$$

or in other form:

$$y(i+k) = Fe(i+k) + \frac{G}{C}y(i) - \frac{B}{A}u(i)\left(1 - z^{-k}\frac{G}{C}\right). \tag{A.8}$$

From DE (A.4) follows that:

$$1 - z^{-k}\frac{G}{C} = \frac{F}{C} \qquad \text{(A.9)}$$

and thus Eq. (A.8) can be written as:

$$y(i+k) = F\,e(i+k) + \frac{G}{C}y(i) + \frac{B}{A}\frac{F}{C}u(i)\,, \qquad \text{(A.10)}$$

or after rearrangement the components on the RHS:

$$y(i+k) = \left(\frac{G}{C}y(i) + \frac{B}{A}\frac{F}{C}u(i)\right) + \left(F\,e(i+k)\right). \qquad \text{(A.11)}$$

Because dim F = k - 1, the two variables on the RHS of Eq. (A.11) are independent and orthogonal. The variance of the sum of two orthogonal random variables is equal to the sum of variances of these variables:

$$L(i+k) = E\left\{y^2(i+k)\right\} = E\left\{\left(\frac{G}{C}y(i) + \frac{BF}{AC}u(i)\right)^2\right\} + E\left\{(Fe(i+k))^2\right\}. \qquad \text{(A.12)}$$

From Eq. (A.12) follows that minimum of the variance of the output signal can be obtained minimising only the first component on the RHS in Eq. (A.12) because the second one is unknown in time instant i. Thus:

$$\frac{G}{C}y(i) + \frac{B}{A}\frac{F}{C}u(i) = 0\,, \qquad \text{(A.13)}$$

which defines the optimal control law as:

$$u_{opt}(i) = -\frac{A}{B}\frac{G}{F}y_{opt}(i)\,. \qquad \text{(A.14)}$$

Substituting expression given in Eq. (A.13) into Eq. (A.11) and shifting back of k samples, defines the output value:

$$y_{opt}(i) = F\,e(i)\,. \qquad \text{(A.15)}$$

Taking into account the form of the DE (A.4), this can be written as follows:

$$y_{opt}(i) = \sum_{l=0}^{k-1} c_l\, y(i-l)\,. \qquad \text{(A.16)}$$

Substituting again Eq. (A.13) into Eq. (A.12) determines the minimum cost function, which is also the minimum possible energy at the output, as:

$$\min L(i+k) = \min E\left\{y^2(i+k)\right\} = E\left\{(Fe(i+k))^2\right\}. \qquad \text{(A.17)}$$

Because $e(i)$ is the white noise of variance λ^2 and using Eq. (A.4), we have:

$$\min E\{y^2(i+k)\} = \lambda^2 \sum_{l=0}^{k-1} c_l^2 \,. \qquad \text{(A.18)}$$

Let us now examine the characteristic equation of the closed loop system (see Eqs .(A.1) and (A.14)):

$$CH = 1 + z^{-k} \frac{B}{A} \frac{AG}{BF} \,, \qquad \text{(A.19)}$$

or in other form:

$$CH = \frac{AB}{BA}\left(1 + z^{-k} \frac{G}{F}\right). \qquad \text{(A.20)}$$

Taking into account DE (A.4), we have:

$$CH = \frac{AB}{BA} \frac{C}{F}. \qquad \text{(A.21)}$$

Thus the characteristic equation can be expressed, not formally, in the following form:

$$CH = ABC \,, \qquad \text{(A.22)}$$

which imposes three constraints to have the closed loop system stable.

A.2 Adaptive MVC

Let us remind the optimal control law of MVC:

$$u_{opt}(i) = -\frac{A}{B}\frac{G}{F} y_{opt}(i), \tag{A.23}$$

which can be rewritten as follows:

$$\underbrace{BF}_{R} u_{opt}(i) + \underbrace{GA}_{S} y_{opt}(i) = 0 \quad \Rightarrow \quad R u_{opt}(i) + S y_{opt}(i) = 0, \tag{A.24}$$

where

$$R = R(z^{-1}) = \underbrace{b_0}_{r_0} + \sum_{l=1}^{\dim R} r_l z^{-l}; \qquad \dim R = \dim B + (k-1), \tag{A.25}$$

$$S = S(z^{-1}) = \sum_{l=0}^{\dim S} s_l z^{-l}; \qquad \dim S = \dim A + (\dim C - k). \tag{A.26}$$

Multiplying Eq. (A.1) by FA, we have:

$$AFy(i) = z^{-k} BFu(i) + ACFe(i). \tag{A.27}$$

Taking into account DE (A.4), Eq. (A.27) can be written as:

$$ACy(i) - z^{-k} AGy(i) - z^{-k} BFu(i) - ACFe(i) = 0, \tag{A.28}$$

or after rearrangement:

$$-BFu(i-k) - AGy(i-k) = AC\big(Fe(i) - y(i)\big). \tag{A.29}$$

Let us add to the LHS $y(i)$ and to the RHS $y(i) + Fe(i) - Fe(i)$, which does not change the equation. Then we can rewrite it to the following form:

$$y(i) - BFu(i-k) - AGy(i-k) = (AC-1)\big[Fe(i) - y(i)\big] + Fe(i). \tag{A.30}$$

The RHS can be regarded as an error and denoted as $\varepsilon(i)$.
Then using the notation introduced in Eq. (A.24), we can write this as:

$$y(i) - Ru(i-k) - Sy(i-k) = \varepsilon(i). \tag{A.31}$$

Considering the structure of polynomials R (A.25) and S (A.26) we can write this in more convenient vector form:

$$y(i) - r_0 u(i-k) - \varphi^T(i-k)\theta = \varepsilon(i), \tag{A.32}$$

where:

$$\varphi^T(i-k) = [y(i-k),\dots,y(i-k-\dim S),u(i-k-1),\dots,u(i-k-\dim R)] \quad \text{(A.33)}$$

is the data vector or in other words - the vector of regressors, and:

$$\theta^T = [s_0, s_1, \dots, s_{\dim S}, r_1, r_2, \dots, r_{\dim R}] \quad \text{(A.34)}$$

is the vector of parameters of the minimum variance controller.

In optimal control:

$$r_0\, u_{opt}(i) + \varphi^T(i)\theta = 0\,. \quad \text{(A.35)}$$

Thus the control value can be calculated from the vector form in the way as follows:

$$u_{opt}(i) = -\frac{1}{r_0}\varphi^T(i)\theta\,. \quad \text{(A.36)}$$

From Eqs. (A.32) and (A.30) follows that under optimal control (A.35):

$$y(i) = \varepsilon(i) = Fe(i) \quad \text{(A.37)}$$

because the first term of the RHS of Eq. (A.30) is nullified.
It is worthy to note that the error $\varepsilon(i)$ in Eq. (A.32) is linearly dependent on the controller parameters θ which allows to find the minimum of $\varepsilon^2(i)$. In adaptive direct minimum-variance control an estimate $\hat{\theta}$ satisfying:

$$y(i) - (\hat{r}_0)\,u(i-k) - \varphi^T(i-k)\,\hat{\theta}(i) = \varepsilon(i) \quad \text{(A.38)}$$

is searched for, which can be found as:

$$\hat{\theta}(i) = Arg \min_{\hat{\theta}(i)} E\{\varepsilon^2(i)\}\,. \quad \text{(A.39)}$$

A.3 MVC with control effort

Let us consider the same plant and Diophantine equations as it was done for MVC - see Eqs. (A.1) and (A.4), respectively.
Define now the cost function as:

$$L_q(i+k) = E\left\{y^2(i+k) + qu^2(i)\right\}, \tag{A.40}$$

which we are going to minimise. In Eq. (A.40) $q > 0$ is a weighting coefficient or in other words a penalty imposed on the control signal suppressing an excess effort.
Following the same reasoning as presented in Eqs. (A.1) through (A.12), the cost function can be now expressed as follows:

$$L_q(i+k) = E\left\{\left(\frac{G}{C}y(i) + \frac{BF}{AC}u(i)\right)^2 + qu^2(i)\right\} + E\left\{\left(Fe(i+k)\right)^2\right\}. \tag{A.41}$$

The second term on the RHS of this equation is unknown in time instant i . Thus minimisation of the cost function is equivalent to minimisation of its first term which can be done by:

$$\frac{\partial}{\partial tu(i)}\left[L_q(i+k)\right] = \frac{\partial}{\partial u(i)}\left\{\left(\frac{G}{C}y(i) + \frac{BF}{AC}u(i)\right)^2 + qu^2(i)\right\} = 0. \tag{A.42}$$

After making the partial derivation:

$$2\left(\frac{G}{C}y(i) + \frac{BF}{AC}u(i)\right)b_0 + 2qu(i) = 0, \tag{A.43}$$

the optimal control law is expressed by:

$$u_{opt}(i) = -\frac{AG}{BF + q'AC}y(i), \tag{A.44}$$

where:

$$q' = \frac{q}{b_0}. \tag{A.45}$$

Let us now substitute Eq. (A.1) for $y(i)$ into Eq. (A.44):

$$u(i) = -\frac{AG}{BF + q'AC}\left(z^{-k}\frac{B}{A}u(i) + C(i)\right). \tag{A.46}$$

Separating variables on different sides, we have:

$$A\left(BF + z^{-k}GB + q'AC\right)u(i) = -AAGCe(i). \tag{A.47}$$

Using DE (A.4), Eq. (A.47) can be rewritten to:

$$u(i) = -\frac{AAGC}{AC(B + q^{'}A)}e(i)\,. \tag{A.48}$$

Hence the characteristic equation of the closed loop system:

$$CH = AC(B + q^{'}A)\,. \tag{A.49}$$

Let us now calculate the minimum cost function at the output under optimal control. Let us substitute for $y(i)$, in Eq. (A.41), from the plant equation (A.1) and use DE (A.4). This gives:

$$\min L_q(i+k) = E\left\{\left(\frac{B}{A}u(i) + Ge(i)\right)^2 + qu^2(i)\right\} + E\{[Fe(i+k)]^2\}. \tag{A.50}$$

Substituting for $u(i)$ from Eq. (A.48) , gives:

$$\min L_q(i+k) = E\left\{\left[\left(\frac{-BG}{B+q^{'}A} + G\right)^2 + q\left(\frac{GA}{B+q^{'}A}\right)^2\right]e^2(i)\right\} + E\{[Fe(i+k)]^2\} \tag{A.51}$$

And finally, after rearranging in the first term on the RHS:

$$\min L_q(i+k) = (q^{'})^2(1+b_0^2)E\left\{\left[\left(\frac{GA}{B+q^{'}A}\right)e(i)\right]^2\right\} + E\{[Fe(i+k)]^2\}. \tag{A.52}$$

Let us check how the system output differs now from its form obtained in case of no weighting (MVC, see Eq. (A.15)):

$$\Delta(i) = y(i) - Fe(i)\,. \tag{A.53}$$

Substituting now for $y(i)$ from Eq. (A.1):

$$\Delta(i) = z^{-k}\frac{B}{A}u(i) + Ce(i) - Fe(i)\,. \tag{A.54}$$

Using Eq. (A.48) and DE (A.4):

$$\Delta(i) = -z^{-k}\frac{B}{A}\frac{AG}{B+q^{'}A}e(i) + Fe(i) + z^{-k}Ge(i) - Fe(i)\,, \tag{A.55}$$

or further:

$$\Delta(i) = -\left(\frac{BG}{B+q^{'}A} - G\right)e(i-k) = (-q^{'})\left(-\frac{AG}{B+q^{'}A}e(i-k)\right). \tag{A.56}$$

Now using again Eq. (A.48):

$$\Delta(i) = -q^{'}u(i-k)\,. \tag{A.57}$$

Thus, combining with Eq. (A.53), the output of the system satisfies the following relation:

$$y(i) + q^{'} u(i-k) = Fe(i) . \tag{A.58}$$

Let us substitute again for $u(i\text{-}k)$, (A.48):

$$y(i) = q^{'} \frac{AG}{B + q^{'} A} e(i-k) + Fe(i) . \tag{A.59}$$

Because the two random variables on the RHS of the equation are independent then the variance at the output can be expressed as follows:

$$E\{y^2(i+k)\} = E\left\{\left[q^{'} \frac{AG}{B + q^{'} A} e(i)\right]^2\right\} + E\{[Fe(i+k)]^2\} \tag{A.60}$$

or, directly from Eq. (A.58):

$$E\{y^2(i+k)\} = (q^{'})^2 E\{[u(i)]^2\} + E\{[Fe(i+k)]^2\}, \tag{A.61}$$

which shows the relation between the variance of the output of the system and the variance of the control signal.

A.4 Adaptive MVC with control effort

Let us remind the optimal MVC control law:

$$u_{opt}(i) = -\frac{AG}{BF + q^{'}AC} y_{opt}(i), \tag{A.62}$$

which can be rewritten as follows:

$$\underbrace{\left(BF + q^{'}AC\right)}_{R} u_{opt}(i) + \underbrace{GA}_{S} y_{opt}(i) = 0 \quad \Rightarrow \quad R u_{opt}(i) + S y_{opt}(i) = 0, \tag{A.63}$$

where

$$R = R(z^{-1}) = \underbrace{\left(b_0 + q^{'}\right)}_{r_0} + \sum_{l=1}^{\dim R} r_l z^{-l}; \qquad \dim R = \max(\dim B + k - 1, \dim A + \dim C), \tag{A.64}$$

$$S = S(z^{-1}) = \sum_{l=0}^{\dim S} s_l; \qquad \dim S = \dim A + (\dim C - k). \tag{A.65}$$

Multiplying Eq.(A.1) by FA, we have:

$$AFy(i) = z^{-k}BFu(i) + ACFe(i). \tag{A.26}$$

Taking into account DE (A.4), Eq. (A.66) can be written as:

$$ACy(i) - z^{-k}AGy(i) - z^{-k}BFu(i) - ACFe(i) = 0 \tag{A.67}$$

or after rearrangement:

$$-BFu(i-k) - AGy(i-k) = AC\left(Fe(i) - y(i)\right). \tag{A.68}$$

Let us add to the LHS $y(i) + q^{'}(1 - AC)u(i-k)$ and to the RHS $y(i) + q^{'}(1 - AC)u(i-k) + Fe(i) - Fe(i)$, which does not change the equation. Then we can rewrite it to the following form:

$$\gamma(i) - \left(BF + q^{'}AC\right)u(i-k) - AGy(i-k) = (AC - 1)\left[Fe(i) - \gamma(i)\right] + Fe(i), \tag{A.69}$$

where

$$\gamma(i) = y(i) + q^{'}u(i-k). \tag{A.70}$$

The RHS can be regarded as an error and denoted as $\varepsilon(i)$.
Using the notation introduced in Eq. (A.63), Eq. (A.69) can be written in terms of controller polynomials as:

$$\gamma(i) - Ru(i-k) - Sy(i-k) = \varepsilon(i). \qquad (A.71)$$

Considering the structure of polynomials R (A.64) and S (A.65), Eq. (A.71) can be expressed in a more convenient vector form:

$$\gamma(i) - r_0 u(i-k) - \varphi^T(i-k)\theta = \varepsilon(i), \qquad (A.72)$$

where:

$$\varphi^T(i-k) = \left[y(i-k),\ldots,y(i-k-\dim S), u(i-k-1),\ldots,u(i-k-\dim R)\right] \qquad (A.73)$$

is the data vector or in other words - the vector of regressors, and:

$$\theta^T = \left[s_0, s_1, \ldots, s_{\dim S}, r_1, r_2, \ldots, r_{\dim R}\right] \qquad (A.74)$$

is the vector of parameters of the controller under consideration.

In optimal control:

$$r_0\, u(i) + \varphi^T(i)\theta = 0. \qquad (A.75)$$

Thus the control value can be calculated from the vector form in the way as follows:

$$u(i) = -\frac{1}{r_0}\varphi^T(i)\theta. \qquad (A.76)$$

From Eqs. (A.72) and (A.69) follows that under optimal control (A.75):

$$\gamma(i) = \varepsilon(i) = Fe(i) \qquad (A.77)$$

because the first term of the RHS of Eq. (A.69) is nullified.

It is worthy to note that the error $\varepsilon(i)$ in Eq. (A.72) is linearly dependent on the controller parameters θ which allows to find the minimum of $\varepsilon^2(i)$. In adaptive weighted minimum-variance control an estimate $\hat{\theta}$ satisfying:

$$\gamma(i) - (\hat{b}_0)\, u(i-k) - \varphi^T(i-k)\hat{\theta}(i) = \varepsilon(i) \qquad (A.78)$$

is searched for, which can be find as:

$$\hat{\theta}(i) = Arg \min_{\hat{\theta}(i)} E\left\{\varepsilon^2(i)\right\}. \qquad (A.79)$$

A.5 IMC IIR filter

Consider the same plant, Diophantine equation and cost function forms as it was done for MVC - see Eqs. (A.1), (A.4), and (A.3), respectively:

For feedforward control with Wiener filter and measure-available acoustic disturbance control value is expressed as follows:

$$u(i) = Wd(i), \tag{A.80}$$

where $W(z^{-1})$ is the transfer function of the filter. Then, Eq. (A.1) takes the form:

$$y(i) = z^{-k}\frac{B}{A}CWe(i) + Ce(i). \tag{A.81}$$

Substituting for C from DE (A.4):

$$y(i+k) = \left(\frac{B}{A}CW + G\right)e(i) + Fe(i+k). \tag{A.82}$$

The two variables on the RHS are independent and orthogonal, that is why the variance of the output (which constitutes now the cost function) is equivalent to:

$$L(i+k) = E\left\{y^2(i+k)\right\} = E\left\{\left[\left(\frac{B}{A}CW + G\right)e(i)\right]^2\right\} + E\left\{(Fe(i+k))^2\right\}. \tag{A.83}$$

Only the first component on the RHS can be minimised because the second one is unknown in time instant i. The minimum of the cost function can be obtained for:

$$\frac{B}{A}CW + G = 0. \tag{A.84}$$

Thus the optimal IIR Wiener filter which minimises the energy of the output signal takes the following form:

$$W_{opt} = -\frac{A}{B}\frac{G}{C}. \tag{A.85}$$

Taking into account On the Eq. (A.84), Eq. (A.83) gives the form of minimum cost function:

$$\min L(i+k) = \min E\left\{y^2(i+k)\right\} = E\left\{(Fe(i))^2\right\}, \tag{A.86}$$

which for $e(i)$ being white noise of variance λ^2, and considering (A.4), is:

$$\min E\left\{y^2(i)\right\} = \lambda^2\sum_{l=0}^{k-1} c_l^2. \tag{A.87}$$

The transfer function of the whole feedforward path (see: Eqs. (A.1) and (A.85)) takes the form:

$$z^{-k}\frac{B}{A}\frac{AG}{BC}. \tag{A.89}$$

Thus the stability of the feedforward control is determined by the following equation:

$$CH = A \cdot B \cdot C. \tag{A.90}$$

which by analogy to the closed-loop system can be called, not formally, the characteristic equation of the system.

A.6 Adaptive IMC IIR filter

Let us remind the optimal feedforward filter minimising the energy at the output of the control system:

$$W_{opt} = -\frac{A}{B}\frac{G}{C} d(i), \tag{A.91}$$

which can be rewritten as follows:

$$\underbrace{BC}_{R} u_{opt}(i) + \underbrace{GA}_{S} d(i) = 0 \quad \Rightarrow \quad R u_{opt}(i) + S\, d(i) = 0, \tag{A.92}$$

where

$$R = R(z^{-1}) = r_0 + \sum_{l=1}^{\dim R} r_l z^{-l}; \qquad \dim R = \dim B + \dim C, \tag{A.93}$$

$$S = S(z^{-1}) = \sum_{l=0}^{\dim S} s_l z^{-l}; \qquad \dim S = \dim A + (\dim C - k). \tag{A.94}$$

Multiplying the plant equation (A.1) by AC, we have:

$$ACy(i) = z^{-k} BCu(i) + AC(Ce(i)). \tag{A.95}$$

Taking into account DE (A.4), Eq. (A.95) can be rewritten to:

$$ACy(i) - z^{-k} AGCe(i) - z^{-k} BCu(i) - AF(Ce(i)) = 0 \tag{A.96}$$

or after rearrangement, and substituting: $Ce(i) = d(i)$:

$$-BCu(i-k) - AGd(i-k) = AC\big(Fe(i) - y(i)\big). \tag{A.97}$$

Let us add to the LHS $y(i)$ and to the RHS $y(i) + Fe(i) - Fe(i)$, which does not change the equation. Then we can rewrite it to the following form:

$$y(i) - BCu(i-k) - AGd(i-k) = (AC-1)[Fe(i) - y(i)] + Fe(i). \tag{A.98}$$

The RHS can be regarded as an error and denoted as $\varepsilon(i)$.
Then using the notation introduced in Eq. (A.92), we can write this as:

$$y(i) - Ru(i-k) - Sy(i-k) = \varepsilon(i). \tag{A.99}$$

Considering the structure of polynomials R (A.93) and S (A.94) we can write this in more convenient vector form:

$$y(i) - r_0 u(i-k) - \varphi^T(i-k)\theta = \varepsilon(i), \tag{A.100}$$

where:

$$\varphi^T(i-k) = \left[d(i-k),\dots,d(i-k-\dim S), u(i-k-1),\dots,u(i-k-\dim R)\right] \quad \text{(A.101)}$$

is the data vector or in other words - the vector of regressors, and:

$$\theta^T = \left[s_0, s_1, \dots, s_{\dim S}, r_1, r_2, \dots, r_{\dim R}\right] \quad \text{(A.102)}$$

is the vector of parameters of optimal feedforward control filter minimising the cost function.

In optimal control:

$$r_0 u_{opt}(i) + \varphi^T(i)\theta = 0 . \quad \text{(A.103)}$$

Thus the control value can be calculated from the vector form in the way as follows:

$$u_{opt}(i) = -\frac{1}{r_0}\varphi^T(i)\theta . \quad \text{(A.104)}$$

From Eqs. (A.100) and (A.98) follows that under optimal control (A.103):

$$y(i) = \varepsilon(i) = Fe(i) \quad \text{(A.105)}$$

because the first term of the RHS of Eq. (A.98) is nullified.
It is worthy to note that the error $\varepsilon(i)$ in Eq.(A.100) is linearly dependent on the controller parameters θ which allows to find the minimum of $\varepsilon^2(i)$, which is a quadratic surface. In this control system an estimate $\hat{\theta}$ satisfying:

$$y(i) - (\hat{b}_0)u(i-k) - \varphi^T(i-k)\hat{\theta}(i) = \varepsilon(i) \quad \text{(A.106)}$$

is searched for, which can be find as:

$$\hat{\theta}(i) = Arg \min_{\hat{\theta}(i)} E\left\{\varepsilon^2(i)\right\} . \quad \text{(A.107)}$$

A.7 IMC IIR filter with control effort

Consider the same plant, Diophantine equation and cost function as it was done for MVC with control effort (see Eqs. (A.1), (A.4), (A.40), respectively).
Optimum control value minimising the cost function is expressed by the following equation (see the relevant equation derived for MVC):

$$u(i) = -\frac{GA}{B + q' A} e(i). \tag{A.108}$$

Allowable disturbance in feedforward filtering is related to the white noise, for the system under consideration, as:

$$d(i) = Ce(i). \tag{A.109}$$

Provided the disturbance shaping filter is minimum phase (in other case, spectral factorisation should be done first), $e(i)$ in Eq. (A.108) can be substituted for Eq. (A.109) leading to the final form of the IIR feedforward filter:

$$u(i) = -\frac{GA}{C(B + q' A)} d(i) = Wd(i). \tag{A.110}$$

Stability condition is given by Eq.(A.48), the minimum of the cost function - by Eq. (A.52), the relation between the output and the control signals - by Eq. (A.58), and the energy at the output - by Eq. (A.60) or (A.61).

A.8 Adaptive IMC IIR filter with control effort

Let us remind the optimal control law:

$$u_{opt}(i) = -\frac{GA}{C(B + q' A)} d(i) , \tag{A.111}$$

which can be rewritten as follows:

$$\underbrace{C(B + q' A)}_{R} u_{opt}(i) + \underbrace{GA}_{S}(i) = 0 \quad \Rightarrow \quad R u_{opt}(i) + S\, d(i) = 0 , \tag{A.112}$$

where

$$R = R(z^{-1}) = \underbrace{(b_0 + q')}_{r_0} + \sum_{l=1}^{\dim R} r_l z^{-l} ; \qquad \dim R = \dim C + \max(\dim A, \dim B) , \tag{A.113}$$

$$S = S(z^{-1}) = \sum_{l=0}^{\dim S} s_l z^{-l} ; \qquad \dim S = \dim A + (\dim C - k) . \tag{A.114}$$

By analogy to adaptive IIR filter minimising only energy of the output signal and to adaptive MVC with control effort, the final predictive model equation can be written in the following form:

$$\gamma(i) - Ru(i-k) - Sd(i-k) = \varepsilon(i) , \tag{A.115}$$

where $\gamma(i)$ is expressed by Eq. (A.70).
Considering the structure of polynomials R (A.113) and S (A.114) we can write this in a more convenient vector form:

$$\gamma(i) - b_0 u(i-k) - \varphi^T(i-k)\theta = \varepsilon(i) , \tag{A.116}$$

where:

$$\varphi^T(i-k) = [y(i-k), \ldots, y(i-k-\dim S), d(i-k-1), \ldots, d(i-k-\dim R)] \tag{A.117}$$

is the data vector or in other words - the vector of regressors, and:

$$\theta^T = [s_0, s_1, \ldots, s_{\dim S}, r_1, r_2, \ldots, r_{\dim R}] \tag{A.118}$$

is the vector of parameters of the control filter.
In optimal control:

$$r_0 u_{opt}(i) + \varphi^T(i)\theta = 0 . \tag{A.119}$$

Thus the control value can be calculated from the vector form in the way as follows:

$$u_{opt}(i) = -\frac{1}{r_0}\varphi^T(i)\theta\,. \tag{A.120}$$

From Eqs. (A.116) and (A.119) follows that under optimal control (A.110):

$$\gamma(i) = \varepsilon(i) = Fe(i)\,. \tag{A.121}$$

It is worthy to note that the error $\varepsilon(i)$ in Eq. (A.116) is linearly dependent on the controller parameters θ which allows to find the minimum of $\varepsilon^2(i)$, which is a quadratic surface. In adaptive version of this control algorithm an estimate $\hat{\theta}$ satisfying:

$$\gamma(i) - \hat{r}_0\, u(i-k) - \varphi^T(i-k)\hat{\theta}(i) = \varepsilon(i) \tag{A.122}$$

is searched for, which can be find as:

$$\hat{\theta}(i) = Arg\min_{\hat{\theta}(i)} E\{\varepsilon^2(i)\}\,. \tag{A.123}$$

APPENDIX B

SIMULATION RESULTS

B.1 Optimal (fixed) controllers

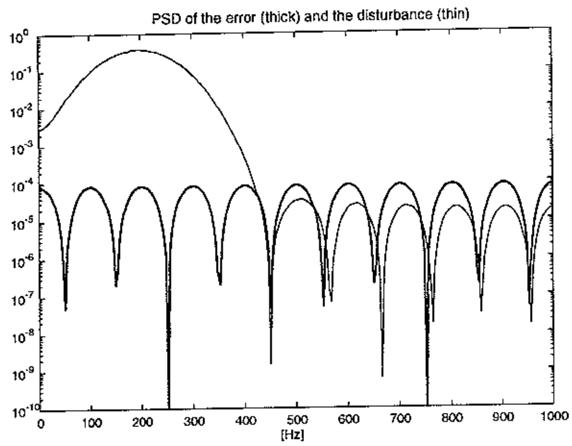


Fig. B.1 Optimum IMC IIR designed for non-minimum phase C. $J = 31.34$ [dB]. System unstable.

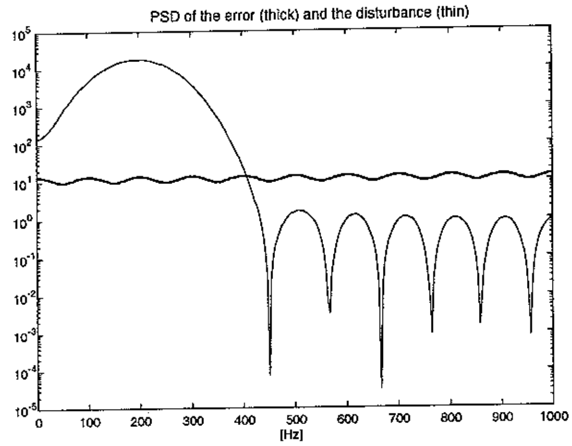


Fig. B.2 Optimum IMC IIR designed for C after spectral factorisation. $J = 23.64$ [dB]. System stable.

k	1	2	3	4	5
J [dB]	23.64	14.00	8.41	5.51	4.65
k	6	7	8	9	10
J [dB]	4.64	3.50	1.67	0.64	0.43

Tab. B.1 Attenuation vs. discrete time delay of the plant for optimal IMC IIR.

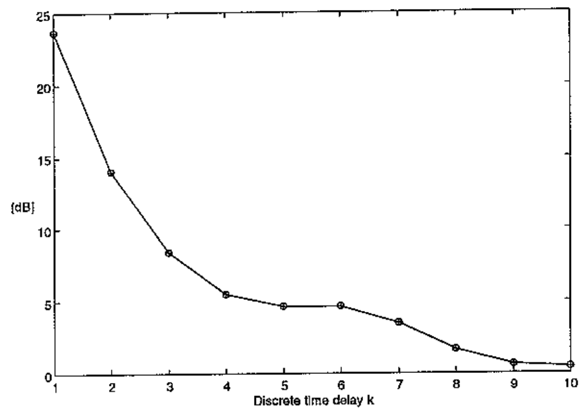


Fig. B.3 Attenuation vs. discrete time delay of the plant for optimal IMC IIR

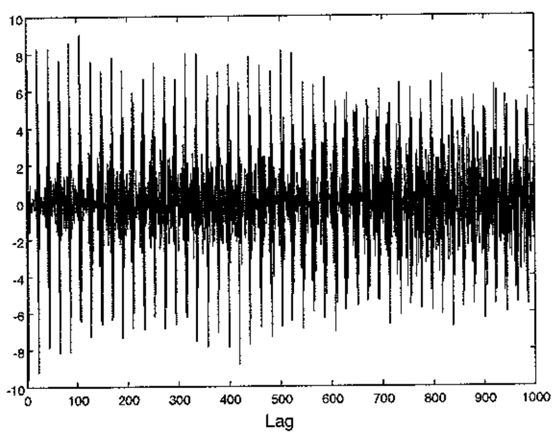


Fig. B.4 Discrete impulse response of optimal IIR feedforward filter.

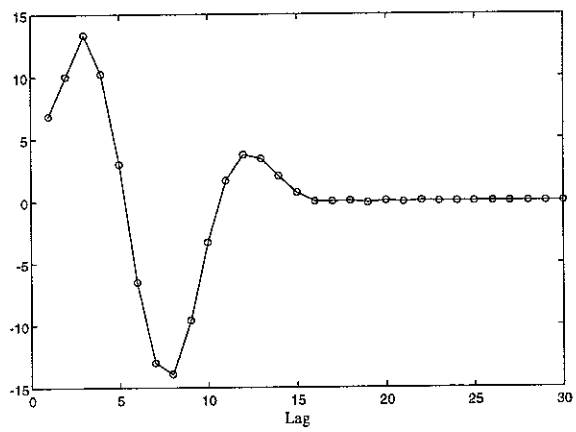


Fig. B.5 Discrete impulse response of optimal feedback controller.

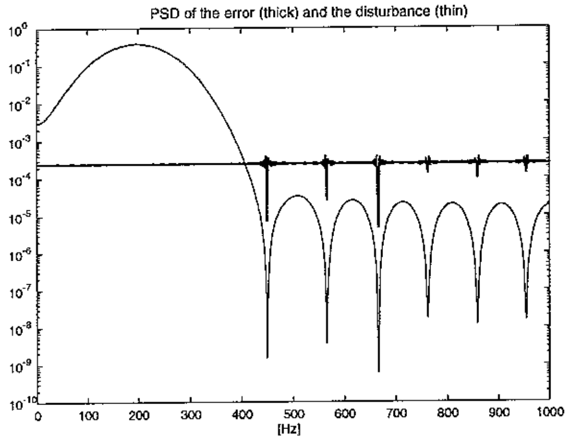


Fig. B.6 Attenuation result for optimal IMC FIR (Wiener filter). $J = 23.58$ [dB].

N	2	4	10	12
J [dB]	10.86	19.97	21.54	21.65
N	14	16	20	30
J [dB]	21.69	21.72	21.77	22.56
N	30	50	100	150
J [dB]	22.56	22.89	23.21	23.35
N	200	250	300	350
J [dB]	23.42	23.47	23.50	23.52
N	400	450	500	550
J [dB]	23.54	23.56	23.57	23.58

Tab. B.2 Attenuation vs. number of parameters of optimal IMC FIR (Wiener filter).

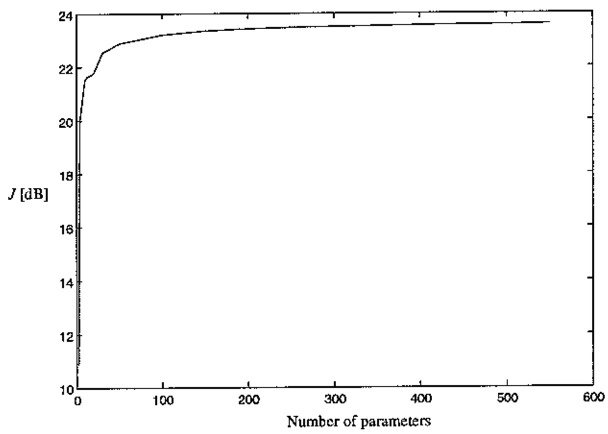


Fig. B.7 Attenuation vs. number of parameters of optimal IMC FIR (Wiener filter).

B.2 MVC RLS

N	2	4	6	8	10
J [dB]	8.90	10.67	16.79	19.03	19.66
N	12	14	16	20	30
J [dB]	20.01	21.38	21.38	21.39	21.10

Tab. B.3 Attenuation vs. number of parameters of MVC for $\alpha = 0.0$ and $q = 0.0$.

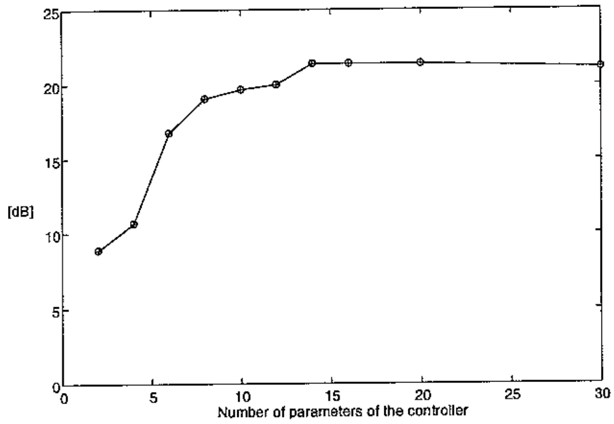


Fig. B.8 Attenuation vs. number of parameters of MVC for $\alpha = 0.0$ and $q = 0.0$.

k	1	2	3	4	5
J [dB]	21.38	11.71	6.67	5.39	4.23
k	6	7	8	9	10
J [dB]	3.00	2.73	0.87	0.42	0.33

Tab. B.4 Attenuation vs. discrete time delay of the plant for MVC ($\alpha = 0.0$ and $q = 0.0$, N = 14).

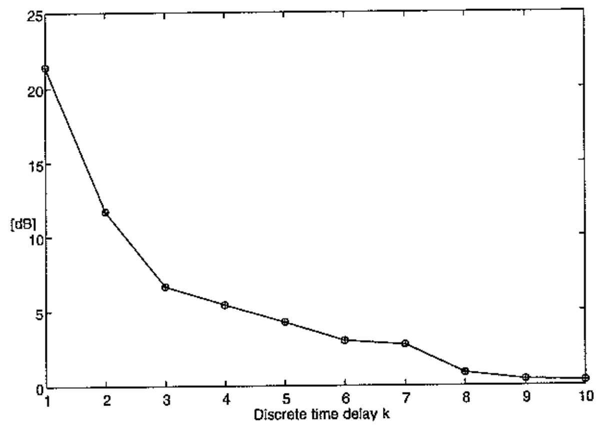


Fig. B.9 Attenuation vs. discrete time delay of the plant for MVC ($\alpha = 0.0$ and $q = 0.0$, N = 14).

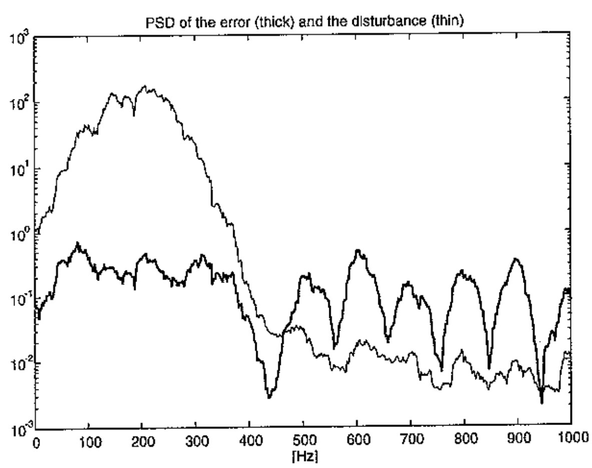


Fig. B.10 Attenuation result for MVC (α = 1.0, q = 0.0, N = 14). J = 20.86 [dB].

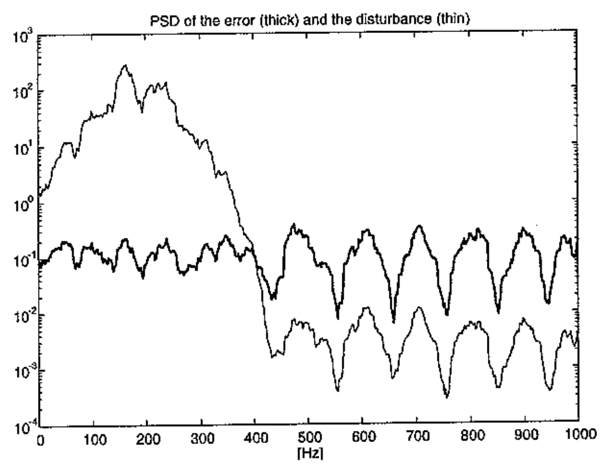


Fig. B.11 Attenuation result for MVC (α = 0.99, q = 0.0, N = 14). J = 21.39 [dB].

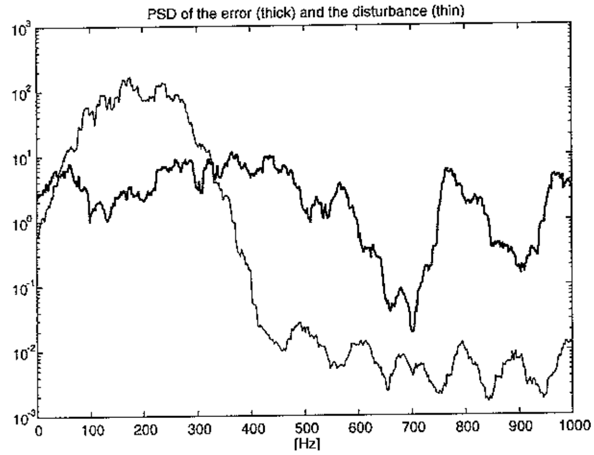


Fig. B.12 Attenuation results for MVC (α = 1.0, q = 0.0, N = 14) and nonstationary plant. J = 7.70 [dB].

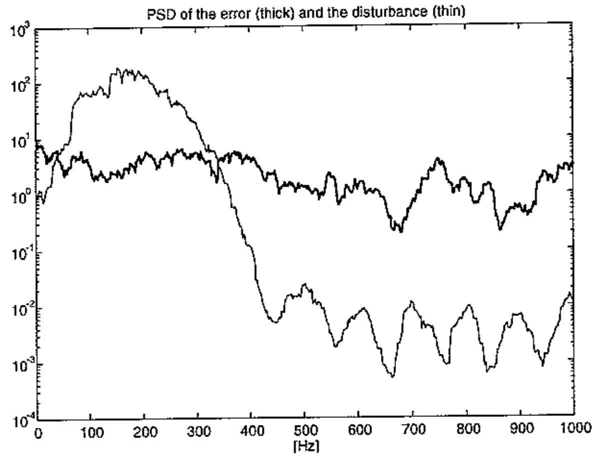


Fig. B.13 Attenuation results for MVC (α = 0.99, q = 0.0, N = 14) and nonstationary plant. J = 9.26 [dB].

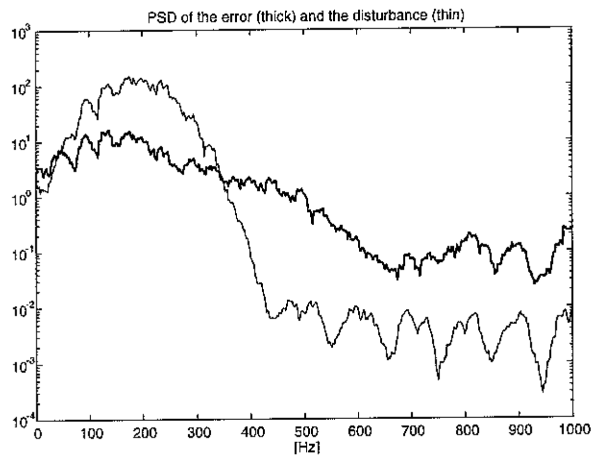


Fig. B.14 Attenuation result for MVC (α = 1.0, q = 0.05, N = 14) and nonstationary plant. J = 8.65 [dB].

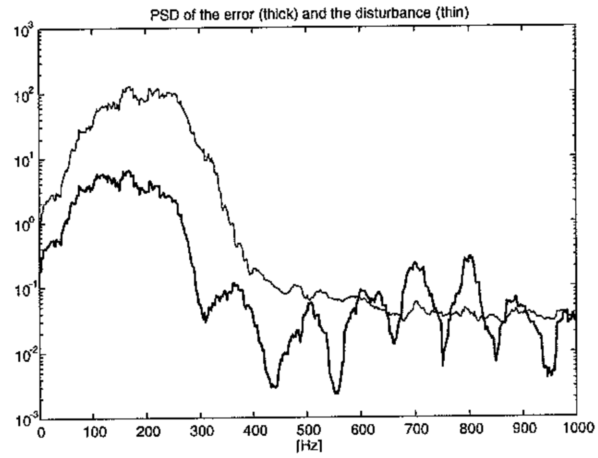


Fig. B.15 Attenuation result for MVC (α = 1.0, q = 0.05, N = 14). J = 12.93 [dB].

B.3 IMC IIR RLS

N	2	4	6	8	10
J [dB]	10.70	12.85	18.29	19.27	20.24
N	12	14	16	20	30
J [dB]	20.59	21.45	21.47	21.60	20.57

Tab. B.5 Attenuation vs. number of parameters of IMC IIR RLS (α = 1.0).

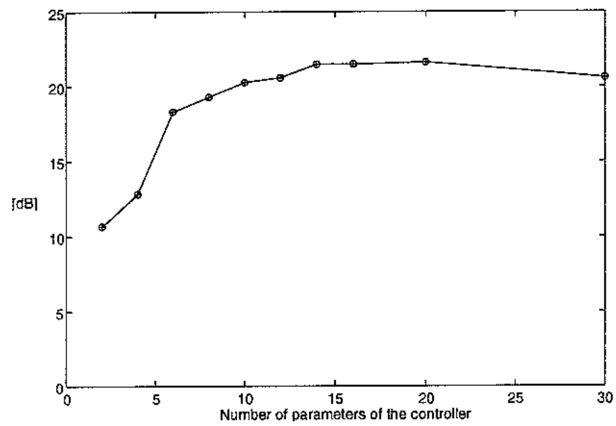


Fig. B.16 Attenuation vs. number of parameters of IMC IIR RLS (α = 1.0).

k	1	2	3	4	5
J [dB]	21.45	11.84	6.72	4.51	3.94
k	6	7	8	9	10
J [dB]	3.90	2.74	0.86	-0	-0

Tab. B.6 Attenuation vs. discrete time delay of the plant for IMC IIR RLS (α = 1.0, N = 14).

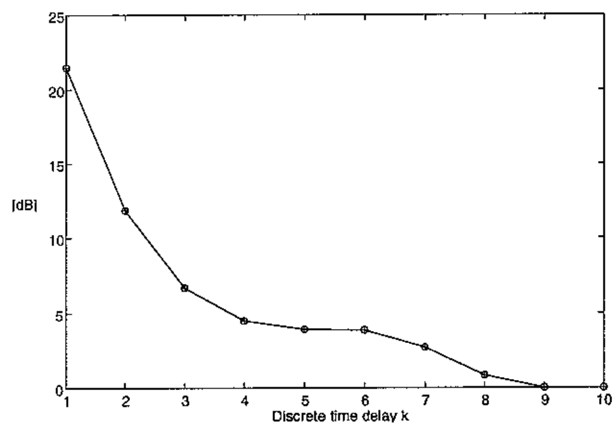


Fig. B.17 Attenuation vs. discrete time delay of the plant for IMC IIR RLS (α = 1.0, N = 14).

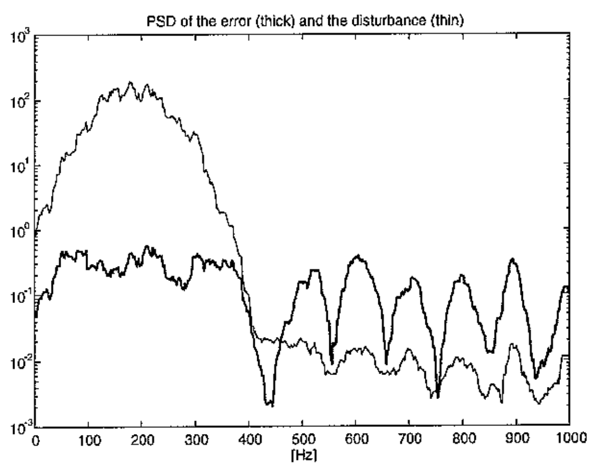


Fig. B.18 Attenuation result for IMC IIR RLS (α = 1.0, N = 14). J = 20.99 [dB].

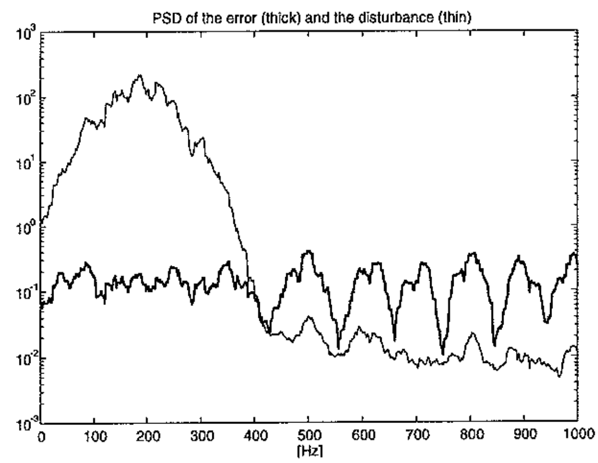


Fig. B.19 Attenuation result for IMC IIR RLS (α = 0.99, N = 14). J = 21.63 [dB].

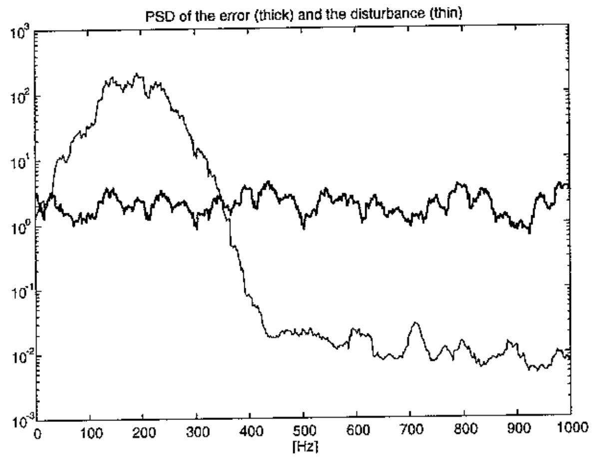


Fig. B.20 Attenuation result for IMC IIR RLS (α = 0.99, N = 14) and nonstationary plant. J = 10.46 [dB].

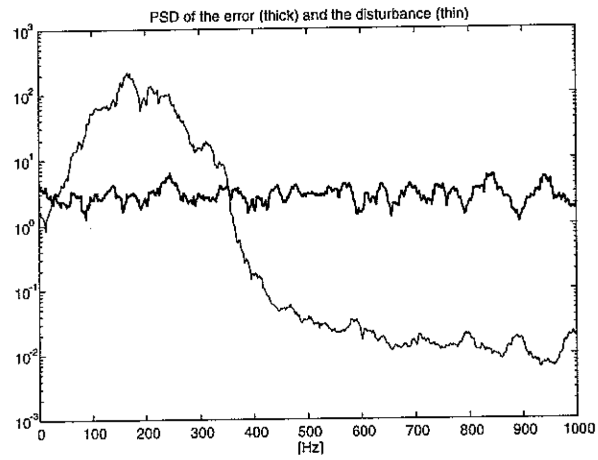


Fig. B.21 Attenuation result for IMC IIR RLS (α = 1.0, N = 14) and nonstationary plant. J = 8.12 [dB].

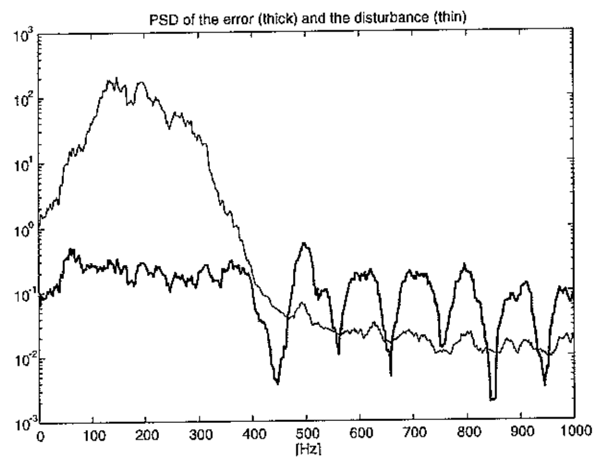


Fig. B.22 Attenuation result for IMC IIR RLS (α = 1.0, N = 14) and model mismatch: B1(1) = 0.62. J = 21.43 [dB].

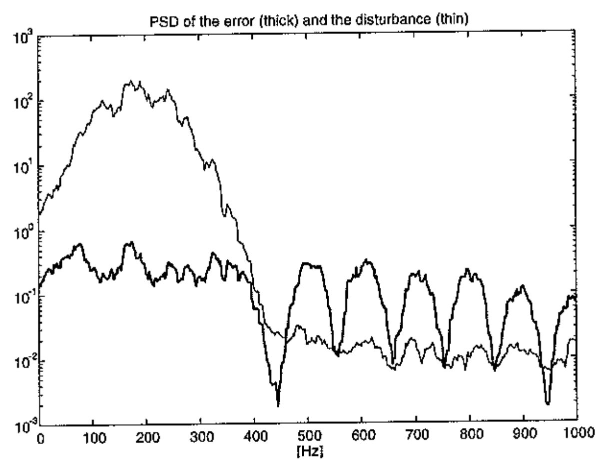


Fig. B.23 Attenuation result for IMC IIR RLS (α = 1.0, N = 14) and model mismatch: B1(1) = 0.22. J = 21.34 [dB].

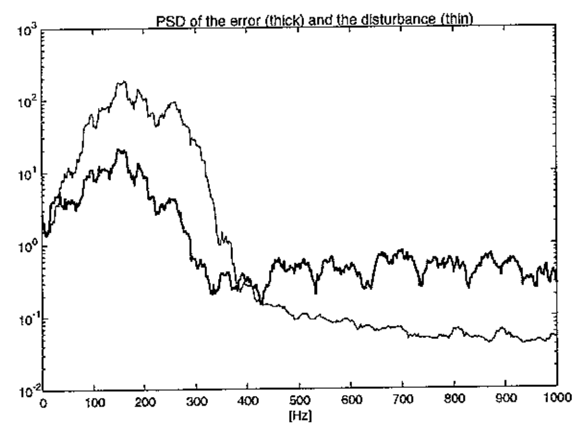


Fig. B.24 Attenuation result for IMC IIR RLS (α = 1.0, q = 0.05, N = 14) and nonstationary plant. J = 9.07 [dB].

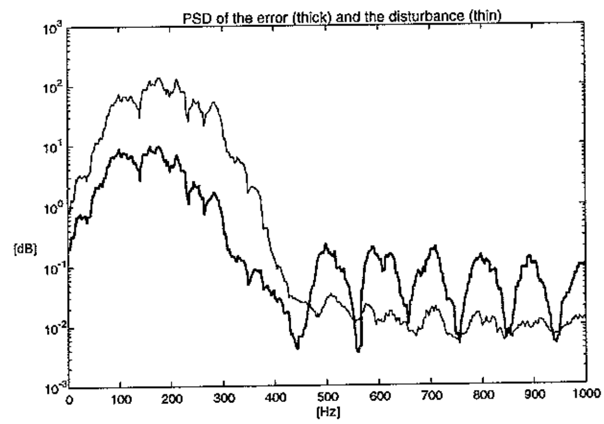


Fig. B.25 Attenuation result for IMC IIR RLS (α = 1.0, q = 0.05, N = 14) and model mismatch: B1(1) = 0.22. J = 11.15 [dB].

B.4 IMC FIR RLS

N	2	4	10	12	14
J [dB]	10.85	19.69	21.38	21.48	21.61
N	16	20	30	50	100
J [dB]	21.70	21.95	22.33	22.21	22.11

Tab. B.7 Attenuation vs. number of parameters of IMC FIR RLS (α = 0.99).

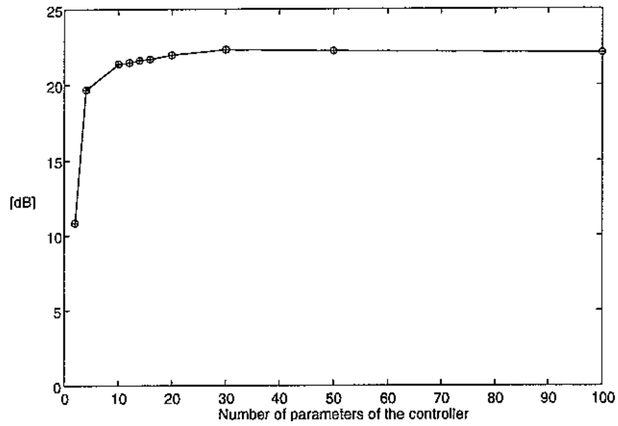
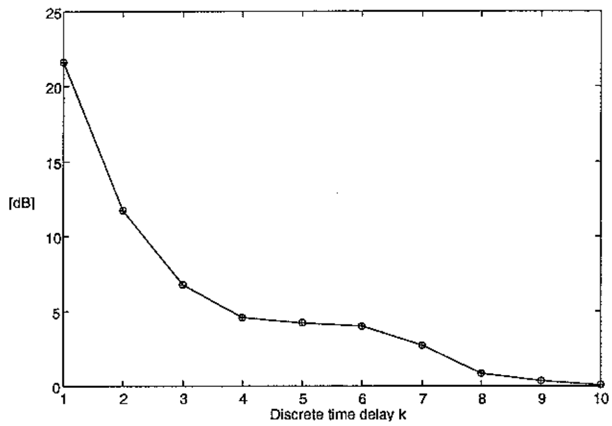


Fig. B.26 Attenuation vs. number of parameters of IMC FIR RLS (α = 0.99).

k	1	2	3	4	5
J [dB]	21.61	11.76	6.82	4.62	4.27
k	6	7	8	9	10
J [dB]	4.01	2.75	0.86	0.35	0.11

Fig. B.8 Attenuation vs. discrete time delay of the plant for IMC FIR RLS (α = 0.99, N = 14).



Tab. B.27 Attenuation vs. discrete time delay of the plant for IMC FIR RLS (α = 0.99, N = 14).

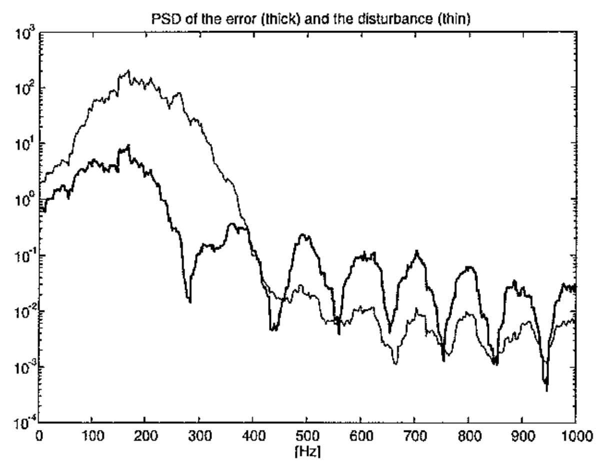


Fig. B.28 Attenuation result for IMC FIR RLS (α = 1.0, N = 14). J = 13.64 [dB]. Slow convergence.

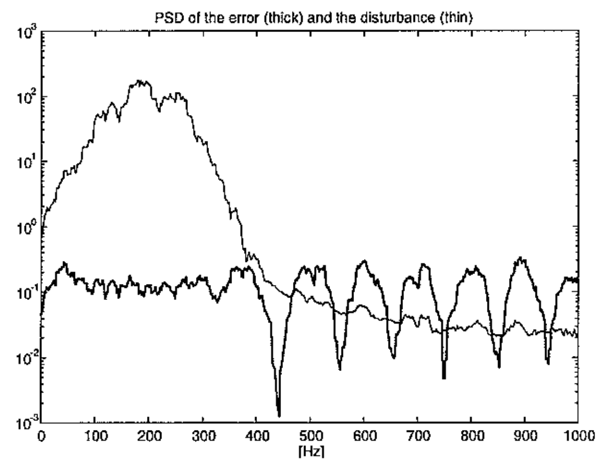


Fig. B.29 Attenuation result for IMC IIR RLS (α = 0.99, N = 14) J = 21.61 [dB]. Fast convergence.

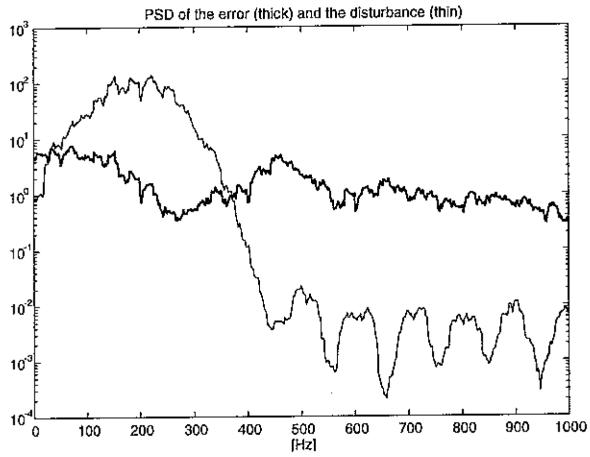


Fig. B.30 Attenuation result for IMC FIR RLS (α = 1.0, N = 14) and nonstationary plant. J = 9.32 [dB].

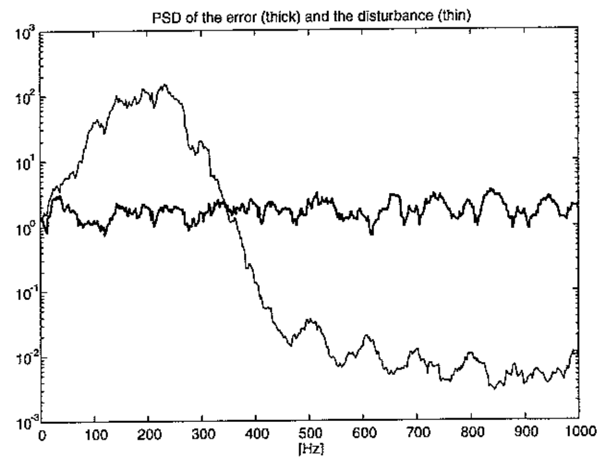


Fig. B.31 Attenuation result for IMC FIR RLS (α = 0.99, N = 14) and nonstationary plant. J = 9.56 [dB].

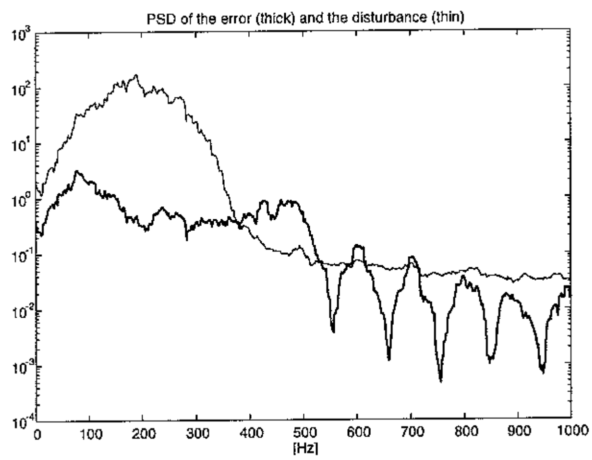


Fig. B.32 Attenuation result for IMC FIR RLS (α = 1.0, N = 14) and model mismatch: B(1) = 0.62. J = 16.47 [dB].

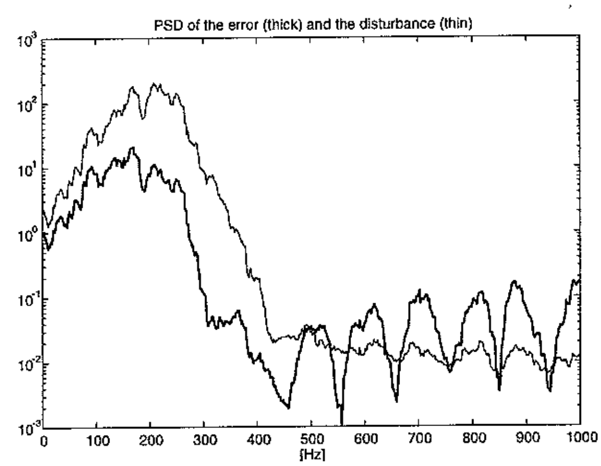


Fig. B.33 Attenuation result for IMC FIR RLS (α = 1.0, N = 14) and model mismatch: B(1) = 0.22. J = 12.07 [dB]

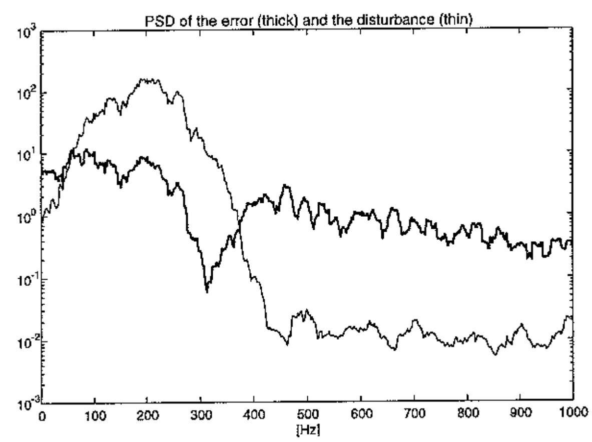


Fig. B.34 Attenuation result for IMC FIR RLS (α = 1.0, q = 0.05, N = 14) and nonstationary plant. J = 9.28 [dB].

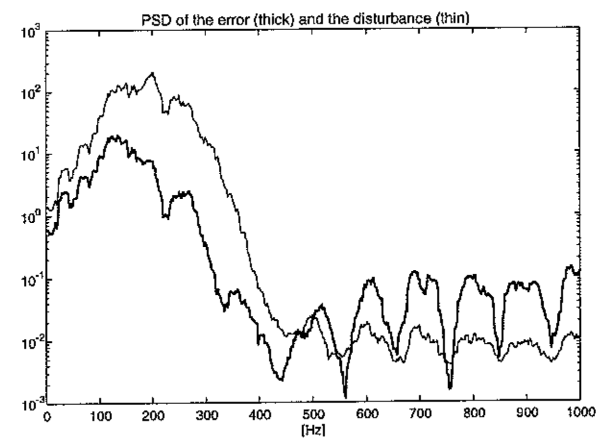


Fig. B.35 Attenuation result for IMC FIR RLS (α = 1.0, q = 0.05, N = 14) and model mismatch: B(1) = 0.22. J = 10.45 [dB]

B.5 IMC FIR MFXLMS

N	2	4	10	12	14
J [dB]	16.40	18.29	19.34	20.82	21.38
N	16	20	30	50	100
J [dB]	21.42	21.44	21.28	21.05	20.64

Tab. B.9 Attenuation vs. number of parameters of IMC FIR MFXLMS.

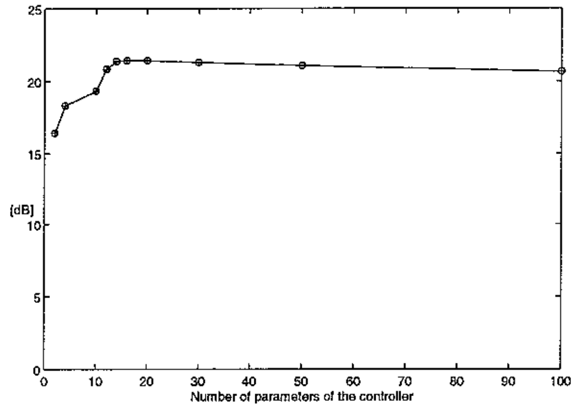


Fig. B.36 Attenuation vs. number of parameters of IMC FIR RLS (α = 0.99).

k	1	2	3	4	5
J [dB]	21.38	11.43	6.48	4.53	4.21
k	6	7	8	9	10
J [dB]	3.98	2.73	0.81	0.30	0.05

Fig. B.10 Attenuation vs. discrete time delay of the plant for IMC FIR RLS (α = 0.99, N = 14).

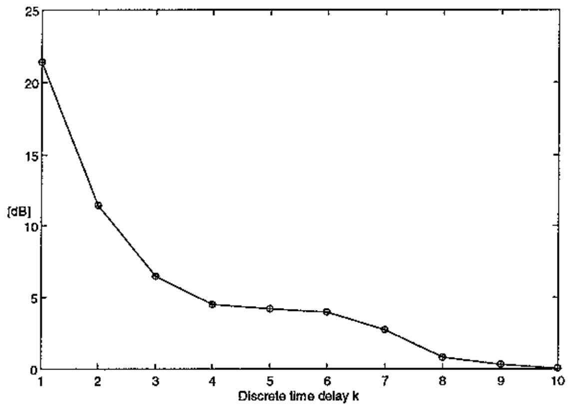


Fig. B.24 Attenuation vs. discrete time delay of the plant for IMC FIR RLS (α = 0.99, N = 14).

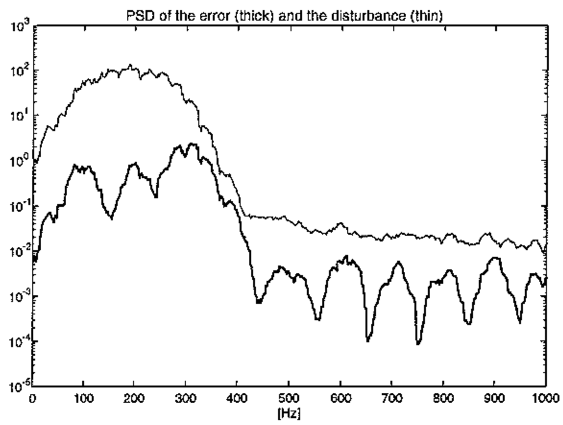


Fig. B.38 Attenuation result for IMC FIR MFXLMS, (μ = 1.5, β = 0.0 N = 14). J = 21.38 [dB].

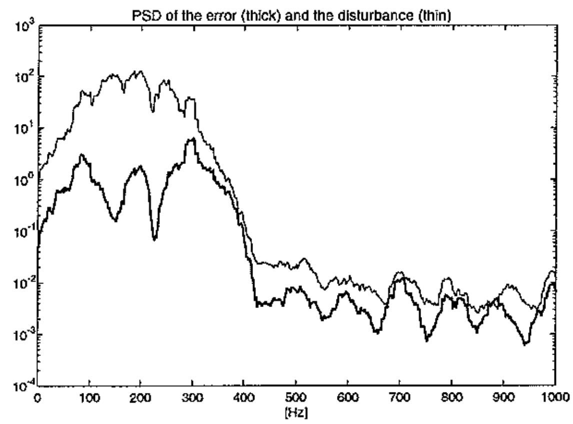


Fig. B.39 Attenuation result for IMC FIR MFXLMS, (μ = 0.8, β = 0.0 N = 14). J = 16.04 [dB].

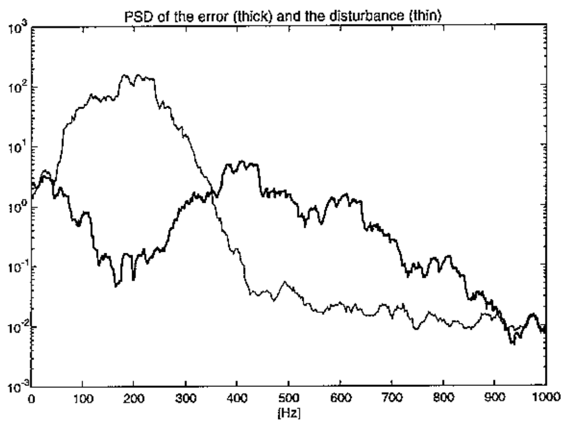


Fig. B.40 Attenuation result for IMC FIR MFXLMS (μ = 1.5, N = 14) and nonstationary plant. J = 13.30 [dB].

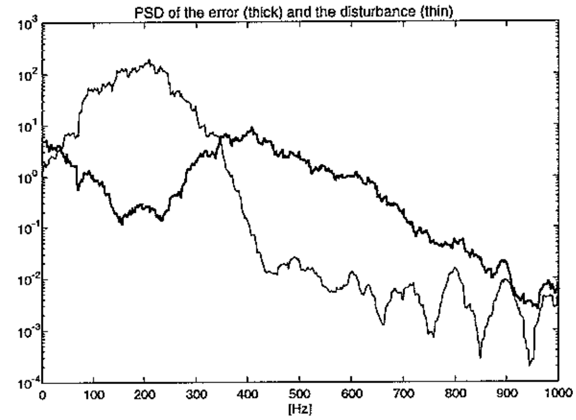


Fig. B.41 Attenuation result for IMC FIR MFXLMS (μ = 0.8, N = 14) and nonstationary plant. J = 11.99 [dB].

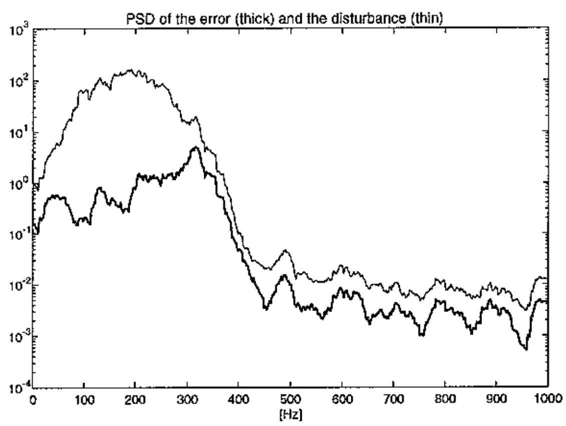


Fig. B.42 Attenuation result for IMC FIR MFXLMS (μ = 1.5, N = 14) and model mismatch: B(1) = 0.62. J = 18.21 [dB].

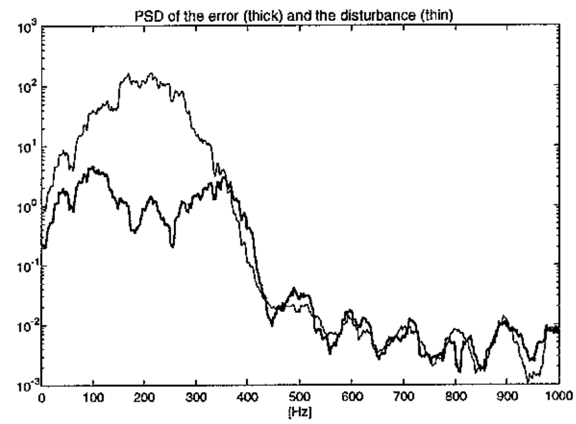


Fig. B.43 Attenuation result for IMC FIR MFXLMS (μ = 1.5, N = 14) and model mismatch: B(1) = 0.22. J = 14.78 [dB].

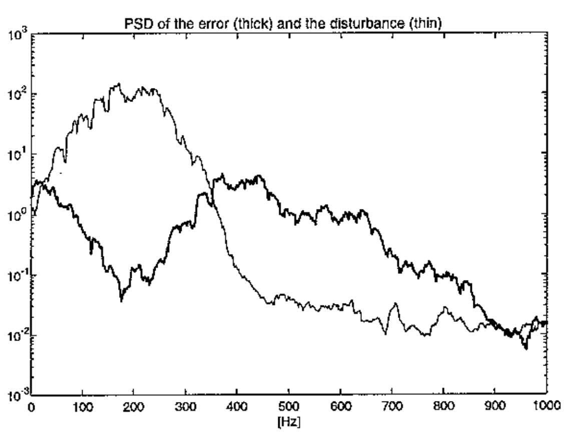


Fig. B.44 Attenuation result for IMC FIR RLS (μ = 1.5, β = 1e-3, N = 14) and nonstationary plant. J = 13.34 [dB].

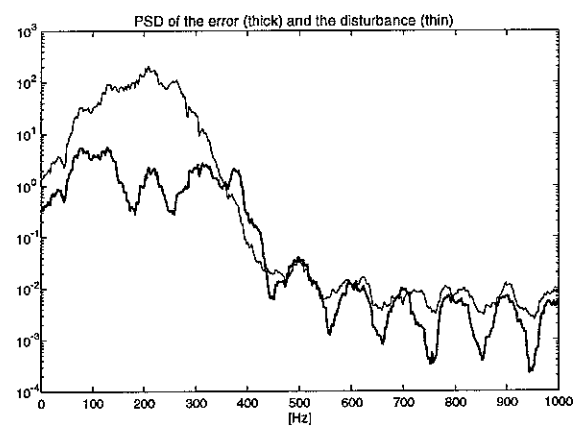


Fig. B.45 Attenuation result for IMC FIR MFXLMS (μ = 1.5, β = 1e-3, N = 14) and model mismatch: B(1) = 0.22. J = 14.94 [dB].

B.6 IMC FIR FXLMS fully adaptive

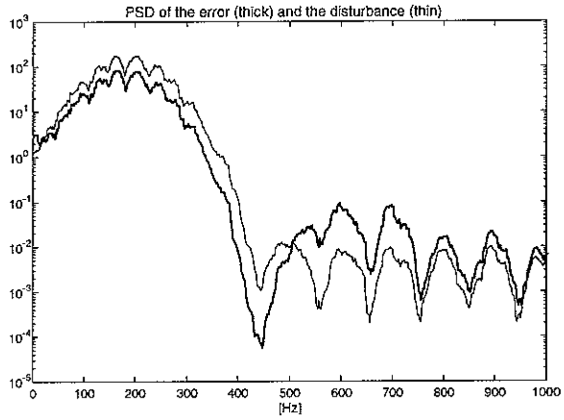


Fig. B.46 Attenuation result for IMC FIR FXLMS fully adaptive (N = 14, $\dim \hat{P}$ = 10, SNR = 1e+6). $J = 4.14$ [dB].