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Boundary Element Methods for Elastodynamic Problems in Ground Vibration in Two Dimensions Part III: Quadratic Elements

A.T. Peplow and M. Petyt

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UNIVERSITY OF SOUTHAMPTON

INSTITUTE OF SOUND AND VIBRATION RESEARCH

DYNAMICS GROUP

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Part III: Quadratic Elements

by

A.T. Peplow and M. Petyt

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Dr M J Brennan
Group Chairman

Boundary Element Methods for Elastodynamic Problems in Ground Vibration in Two-Dimensions Part III: Quadratic Elements

Andrew Peplow
Maurice Petyt

Dynamics Group

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Abstract

It is shown how elastodynamic boundary integral equations (EBIE) formulated for bounded domains are solved numerically. A detailed analysis using quadratic elements is presented. For domains with infinite boundaries, the boundary is simply truncated at a pre-determined value. Results for vertical and horizontal responses due to a constant surface load over a finite width are also presented.

Contents

1 Introduction

This report shows how elastodynamic boundary integral equations (EBIE) formulated for bounded domains are solved numerically. The form of the singular integrals is found by investigating the low- frequency behaviour of the Green's functions. A detailed analysis using piecewise quadratic elements is presented. Using a simple example of ground vibration represented by a half–space, it is shown that quadratic elements are more efficient than constant elements.

2 Boundary Integrals for Elastodynamics

A system of fixed rectangular cartesian co-ordinates is used to present the theory. The summation convention is used, whereby a repeated sub-index implies a summation. Quantities with one or two sub–indices denote components of a vector or of a second-order tensor, respectively. Vectors are written in bold where appropriate.

The co-ordinate axes are denoted by x_j, where $j = 1, 2$. The displacement vector at a point $\mathrm{x} = (x_1, x_2)$ is denoted by $\mathbf{u}(\mathrm{x})$ and the stress vector by $\mathbf{t}(\mathrm{x})$.

In [1] it was shown that, for a bounded domain Ω with boundary Γ, the following boundary integral equation may be formulated

$$c_{lk}(\mathrm{x})u_k(\mathrm{x}) \; + \; \int_\Gamma T_{lk}(\mathrm{x}|\mathrm{y})\, u_k(\mathrm{y})\, d\Gamma(\mathrm{y}) \; = \; \int_\Gamma U_{lk}(\mathrm{x}|\mathrm{y})\, t_k(\mathrm{y})\, d\Gamma(\mathrm{y}), \quad \mathrm{x} \in \overline{\Omega}, \qquad (1)$$

where the integrals are Cauchy Principal Value integrals and where c_{lk} is given by

$$c_{lk}(\mathrm{x}) \; = \; \begin{cases} \delta_{lk}, & \text{if} \quad \mathrm{x} \in \Omega, \\ \frac{1}{2}\delta_{lk}, & \text{if} \quad \mathrm{x} \in \Gamma, \\ 0, & \text{if} \quad \mathrm{x} \in \mathrm{R}^2 \backslash \Gamma, \end{cases} \qquad (2)$$

$U_{lk}(\mathrm{x}|\mathrm{y})$ and $T_{lk}(\mathrm{x}|\mathrm{y})$ are displacements and tractions for the fundamental solution (see [1]).

This boundary integral equation permits solving the general boundary value problem of time-harmonic elastodynamics. If displacements are known over Γ, (1) produces an integral equation of the first-kind. If tractions are known over the boundary, an integral equation of the second-kind is obtained. Finally, a combination of both types of boundary conditions results in a mixed integral equation.

3 A numerical method for solving the BIE

A simple method for solving (1) numerically will now be described. The first step is to divide Γ into N divisions or elements $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ to give an approximation $\tilde{\Gamma}$. In the nexy section it is shown how each element is analysed using a quadratic approximation.

3.1 Quadratic elements

Consider a curved piece of boundary and define three nodal points on the prescribed section, two nodes at the ends and one node along the element, not necessarily in the middle. Define also a homogeneous coordinate ξ which varies between $\xi = -1$ and $\xi = +1$ along the section of the boundary. The traction and displacement variables are allowed to vary along the element in terms of their three nodal values and three *interpolation functions* N_1, N_2, N_3 where the interpolation functions are quadratic polynomials in ξ. Each function takes the value 1 at the associated node and 0 at the others.

$$N_1 = \frac{1}{2}\xi(\xi - 1); \quad N_2 = (1 - \xi)(\xi + 1); \quad N_3 = \frac{1}{2}\xi(\xi + 1). \tag{3}$$

The BIE (1) can be written as

$$\begin{aligned} c_{lk}u_k(\mathbf{x}) \quad + \quad & \sum_{n=1}^{N} \left\{ \int_{\Gamma_n} T_{lk}(\mathbf{x}|\mathbf{y})\, u_k(\mathbf{y})\, d\Gamma(\mathbf{y}) \right\} \; - \\ & \sum_{n=1}^{3\times NE} \left\{ \int_{\Gamma_n} U_{lk}(\mathbf{x}|\mathbf{y})\, t_k(\mathbf{y})\, d\Gamma(\mathbf{y}) \right\} \; = \; 0 \quad \mathbf{x} \in \tilde{\Gamma} \end{aligned} \tag{4}$$

where NE is the number of elements and N is the number of nodes. (For *closed* domains $N = 2 \times NE$).

Note that we have assumed that displacement is continuous over the boundary, *i.e.*, vertical and horizontal displacement at the left–hand node, say, of one element is equal to vertical and horizontal displacements at the right–hand node of the adjoining element. However, the traction variable may be discontinuous across elements, if adjoining elements do not share the same unit–normal vector, resulting in a non–square system of equations. The solution to this problem is the subject of a forthcoming report. An illustrative example follows:

Example

Consider a square domain enclosed by a boundary discretised into four quadratic elements, one element per-side. Thus $NE = 4$ and $N = 8$. For each *element* there are three values of vertical and horizontal traction to be determined (*i.e.* $3 \times 4 = 12$ in total). In addition vertical and horizontal displacements must also be determined at each node, ($N = 8$ in total).

To proceed further let h_n denote the length of Γ_n for $n = 1, 2, \ldots, N$ and let

$$h := \max_{1 \le n \le N} h_n. \tag{5}$$

Then, provided h is small enough, $u_k(\mathrm{y})$ and $t_k(\mathrm{y})$, $k = 1, 2$ vary quadratically over each element Γ_n. Thus, where y_{n-1}, y_n, y_{n+1}, denotes the *left hand-side node*, *mid-point* and *right hand-side node* of Γ_n : $u_k^1(\mathrm{y}) = u_k(\mathrm{y}_{n-1}), u_k^2(\mathrm{y}) = u_k(\mathrm{y}_n), u_k^3(\mathrm{y}) = u_k(\mathrm{y}_{n+1})$ and similarly for the tractions, then

$$\begin{aligned}
\int_{\Gamma_n} T_{lk}(\mathrm{x}|\mathrm{y}) u_k(\mathrm{y}) d\Gamma(\mathrm{y}) &\approx \sum_{i=1}^{3} \int_{\Gamma_n} N_i(\mathrm{y}) T_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) \;\; u_k^i(\mathrm{y}_n) \\
\int_{\Gamma_n} U_{lk}(\mathrm{x}|\mathrm{y})\, t_k(\mathrm{y})\, d\Gamma(\mathrm{y}) &\approx \sum_{i=1}^{3} \int_{\Gamma_n} N_i(\mathrm{y}) U_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) \;\; t_k^i(\mathrm{y}_n).
\end{aligned} \tag{6}$$

The approximation (6) is the basis of the numerical method, but in fact a further approximation has to be made. The integrals

$$\int_{\Gamma_n} N_i(\mathrm{y}) T_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) \text{ and } \int_{\Gamma_n} N_i(\mathrm{y}) U_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) \tag{7}$$

cannot be calculated exactly and so are replaced by approximations

$$H^{lk}(\mathrm{x}, \Gamma_n) \quad \text{and} \quad G^{lk}(\mathrm{x}, \Gamma_n) \tag{8}$$

respectively. We will discuss how $H^{lk}(\mathrm{x}, \Gamma_n)$ and $G^{lk}(\mathrm{x}, \Gamma_n)$ are calculated later. Thus, the integrals in (4) have the approximations

$$\begin{aligned}
\int_{\Gamma_n} \left\{ T_{lk}(\mathrm{x}|\mathrm{y})\, u_k(\mathrm{y}) \; - \; \int_{\Gamma_n} U_{lk}(\mathrm{x}|\mathrm{y})\, t_k(\mathrm{y}) \right\} \, d\Gamma(\mathrm{y}) \approx \\
H^{lk}(\mathrm{x}, \Gamma_n) u_k(\mathrm{y}_n) \; - \; G^{lk}(\mathrm{x}, \Gamma_n) t_k(\mathrm{y}_n)
\end{aligned} \tag{9}$$

From (4) and (9) we deduce that

$$c_{lk}(\mathbf{x})u_k(\mathbf{x}) \approx \sum_{n=1}^{3\times NE} G^{lk}(\mathbf{x},\Gamma_n)t_k(\mathbf{y}_n) - \sum_{n=1}^{N} H^{lk}(\mathbf{x},\Gamma_n)u_k(\mathbf{y}_n), \quad \mathbf{x} \in \overline{\Omega} \tag{10}$$

provided h is small enough.

3.2 Collocation

Note that in particular the approximation (10) holds when $\mathbf{x} = \mathbf{y}_m$, for $m = 1, 2, .., N$. That is,

$$c_{lk}u_k(\mathbf{y}_m) \approx \sum_{n=1}^{3\times NE} G^{lk}(\mathbf{y}_m,\Gamma_n)t_k(\mathbf{y}_n) - \sum_{n=1}^{N} H^{lk}(\mathbf{y}_m,\Gamma_n)u_k(\mathbf{y}_n),\ m = 1, 2, \ldots, N. \tag{11}$$

Note that, for $m = 1, 2, \ldots, N$ $c_{lk} = \frac{1}{2}\delta_{lk}$.

The equations (11) are a set of linear equations which are approximately satisfied by the $2N$ unknowns, $u_k(\mathbf{y}_m)$ and $3NE$ unknowns $t_k(\mathbf{y}_m)$ $k = 1, 2$. Since $u_k(\mathbf{y}_m)$ can be written as

$$u_k(\mathbf{y}_m) = \sum_{n=1}^{N} \delta_{mn}\, u_k(\mathbf{y}_n), \tag{12}$$

the approximations (11) can be written as

$$\sum_{n=1}^{N} H^{lk}_{mn}u_k(\mathbf{y}_n) \approx \sum_{n=1}^{3NE} G^{lk}_{mn}t_k(\mathbf{y}_n), \quad \text{for } m = 1, 2, \ldots, N. \tag{13}$$

where

$$\begin{aligned} H^{lk}_{mn} &= \frac{1}{2}\delta_{mn} + H^{lk}(\mathbf{y}_m,\Gamma_n), \quad m, n = 1, 2, \ldots, N. \\ G^{lk}_{mn} &= G^{lk}(\mathbf{y}_m,\Gamma_n), \quad n = 1, 2, \ldots, N, m = 1, 2, \ldots, 3NE \end{aligned} \tag{14}$$

Note that equation (10) says that, once we have determined $u_k(\mathbf{y}_n)$ for $n = 1, 2, \ldots, N$ and $t_k(\mathbf{y}_n)$ for $n = 1, 2, \ldots, 3NE$ we can calculate $u_l(\mathbf{x})$ approximately for any value of $\mathbf{x} \in \overline{\Omega}$. An obvious approximate procedure is to :

(i) replace the $\approx$ by $=$ in (13), replace the unknown tractions or displacements via the boundary conditions and solve these linear equations to obtain approximate values for $u_k(\mathbf{y}_n)$ and $t_k(\mathbf{y}_n)$.

(ii) replace the $\approx$ in (10), and use this equation to calculate an approximation to $u_l(\mathbf{x})$ for any $\mathbf{x} \in \overline{\Omega}$

Thus what we are proposing, is to calculate an approximation $u_l^N(\mathbf{x})$ to $u_l(\mathbf{x})$, this approximation being defined by

$$c_{lk}(\mathbf{x})u_k^N(\mathbf{x}) = \sum_{n=1}^{3NE} G^{lk}(\mathbf{x},\Gamma_n)t_k^N(\mathbf{y}_n) - \sum_{n=1}^{N} H^{lk}(\mathbf{x},\Gamma_n)u_k^N(\mathbf{y}_n), \quad \mathbf{x} \in \overline{\Omega}, \qquad (15)$$

which implies that

$$\sum_{n=1}^{N} H_{mn}^{lk} u_k^N(\mathbf{y}_n) \approx \sum_{n=1}^{3NE} G_{mn}^{lk} t_k^N(\mathbf{y}_n), \quad \text{for } m = 1,2,\ldots,N. \qquad (16)$$

Suppose that the tractions t_k, on the boundary, are known. Hence, the computational method is to :

(a) Calculate the matrices $[H_{mn}^{lk}]$, $[G_{mn}^{lk}]$ and the right hand-side of (16), and then solve the equations (16) to determine $u_l^N(\mathbf{y}_n)$ for $n = 1,2,\ldots,N$.

(b) Using the values calculated in (16) and equations (15) calculate $u_l^N(\mathbf{x})$ at any given receiver point $\mathbf{x}$.

(c) From the discussion in section 3.1 note that the matrix H is a $2N \times 2N$ fully populated complex–valued matrix and G is a $2N \times 6NE$ fully populated complex–valued matrix. Also the vectors $\mathbf{u}$, of length $2N$, and $\mathbf{t}$, of length $6NE$, respectively represent displacement and traction solutions on the boundary.

In most practical situations the stress, is given as a boundary condition (given force or stress free condition) and the given geometry is *smooth.* Thus, we are permitted to argue that the given tractions at adjoining nodes are equal, yielding a well- posed problem ($2N$ equations with $2N$ unknowns).

4 Evaluation of the integrals

The evaluation of the integrals contained in G^{lk} and H^{lk} must be done numerically to a high degree of accuracy and analytically at the singularity.

The integrals along Γ_n need to be transformed to the homogeneous coordinates system ξ. Hence the integrals become

$$\int_{\Gamma_n} \phi_i(\mathrm{y}) T_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) = \int_{-1}^{+1} \phi_i(\xi) T_{lk}(\mathrm{x}|\xi)\ |J|\ d\xi \tag{17}$$

$$\int_{\Gamma_n} \phi_i(\mathrm{y}) U_{lk}(\mathrm{x}|\mathrm{y}) d\Gamma(\mathrm{y}) = \int_{-1}^{+1} \phi_i(\xi) U_{lk}(\mathrm{x}|\xi)\ |J|\ d\xi \tag{18}$$

where the Jacobian $|J|$ is given by

$$\begin{aligned} |J| = \quad & \{\{(x_1^3 - 2x_1^2 + x_1^1)\xi + \frac{1}{2}(x_1^3 - x_1^1)\}^2 + \\ & \{(x_2^3 - 2x_2^2 + x_2^1)\xi + \frac{1}{2}(x_2^3 - x_2^1)\}^2\}^{\frac{1}{2}} \end{aligned} \tag{19}$$

The expressions for the fundamental solution displacements U_{lk} and tractions T_{lk} were derived in [1]. The displacements are given by

$$U_{lk} \ = \ \frac{1}{2\pi\mu}\left(\psi\delta_{lk} \ - \ \phi\frac{\partial r}{\partial x_l}\frac{\partial r}{x_k}\right). \tag{20}$$

where

$$\psi(r) \ = \ K_0(k_2 r) \ + \ \frac{1}{k_2 r}\left(K_1(k_2 r) - \left(\frac{c_1}{c_2}\right) K_1(k_1 r)\right) \tag{21}$$

$$\phi(r) \ = \ K_2(k_2 r) - \left(\frac{c_1}{c_2}\right)^2 K_2(k_1 r) \tag{22}$$

and $r^2 = x^2 + y^2$. K_i are the modified Bessel functions of the second kind of order i.

The k component of the traction on a surface whose unit external normal is n, when a unit load is applied in the l direction is given by

$$\begin{aligned} T_{lk} \ = \quad & \frac{1}{2\pi}\left\{\frac{\partial\psi}{\partial r} \ - \ \frac{\phi}{r}\right\}\left[\delta_{lk}\frac{\partial r}{\partial n} \ + \ \frac{\partial r}{\partial x_k}n_l\right] \\ & -\frac{2\phi}{r}\left[n_k\frac{\partial r}{\partial x_l} - 2\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial n}\right] \ - \ 2\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_k}\frac{\partial r}{\partial n} \end{aligned} \tag{23}$$

$$+ \ \left[\left(\frac{c_1}{c_2}\right)^2 - 2\right]\left\{\frac{\partial\psi}{\partial r} \ - \ \frac{\partial\phi}{\partial r} \ - \ \frac{\phi}{r}\right\}\frac{\partial r}{\partial x_l}n_k \tag{24}$$

To take into account the possibility of having two different values of any traction component at the nodes connecting two consecutive elements (required for corner points), the nodal tractions are arranged in a $6 \times NE$ array. The nodal displacements are arranged in a $2 \times N$ array with two components per node.

Notice that only two unknowns per node is acceptable. This takes place naturally in any smooth boundary point of a *well-posed problem.* However at true corner points, the unknown for a given coordinate will be the displacement, the traction *before* the node, the traction *after* the node, or the unique traction *before* and *after* the node. The so-called corner problem i.e., two different unknown tractions along one nodal coodinate, is treated in a forthcoming report. In any case, such a situation is somehow unrealistic, if one knows the displacements along the two directions going towards the corner, the displacement derivatives and hence the strain and stress tensors can be obtained from linear theory of the kinematic equations and constitutive relations.

4.1 Non–singular integrals

The evaluation of the matrices $[H]$ and $[G]$ that relate a collocation point $\mathbf{x}$ outside an integration element Γ_n is based on a ten–point Gaussian quadrature rule [2]. However, the entries of the elements are defined as :

$$G = \int_{-1}^{+1} \phi_i(\xi) U_{lk}(\mathrm{x}|\xi) \; |J| \; d\xi \tag{25}$$

$$H^{\omega} = \int_{-1}^{+1} \phi_i(\xi) \{T_{lk}(\mathrm{x}|\xi) - T^0_{lk}(\mathrm{x}|\xi)\} \; |J| \; d\xi \tag{26}$$

$$H^0 = \int_{-1}^{+1} \phi_i(\xi) T^0_{lk}(\mathrm{x}|\xi) \; |J| \; d\xi \tag{27}$$

where T^0_{lk} is the static fundamental solution traction tensor. Details of how the fundamental solutions are evaluated using a low– frequency approximnation is given in reference [3].

For the case where the collocation point is within the integration element then a special routine is implemented for the $[G]$ matrix only.

4.2 Weakly–singular integrals

The matrix entries for $[G]$, which correspond to an element when the collocation point is one of the three in the element, are considered as *weakly–singular* integrals as the kernel has a logarithmic behaviour which require special attention. The integrals are split into two parts: one without a singularity, which is integrated by standard Gaussian integration formulae, and the other which contains a logarithmic singularity, is evaluated using a special integration technique.

1. Collocation point at node (1).

   The singularity of U_{lk} is determined is of the type (see reference [1]).

   $$\lim_{r\to 0} U_{lk} = \frac{1}{8\pi\mu}\delta_{lk}\frac{(3-4\nu)}{(1-\nu)}\ln r. \tag{28}$$

   Using this result one can write the function

   $$\frac{\psi}{(3-4\nu)}4(1-\nu) + \ln(1+\xi)/2 \tag{29}$$

   which is non-singular and may be integrated using standard Gaussian quadrature.

   The singular part is integrated after a change of variables $\eta = (1+\xi)/2$. Using a special integration formula of the type [2]

   $$I = \int_0^1 \ln(\eta) f(\eta) d\eta \approx \sum_{i=1}^{n} w_i f(\eta_i)$$

   where the shape functions and Jacobian are contained within the function $f(\eta)$.

2. Collocation point at node (2).

   In order to integrate the two singularities that appear on both sides of the nodes, the integral is divides into two parts,

   $$G = \int_{-1}^{+1} N_i(\xi) U_{lk}(\mathrm{x}|\xi)\,|J|\,d\xi \tag{30}$$

   $$= \int_{-1}^{0} N_i(\xi) U_{lk}(\mathrm{x}|\xi)\,|J|\,d\xi + \tag{31}$$

   $$\int_{0}^{+1} N_i(\xi) U_{lk}(\mathrm{x}|\xi)\,|J|\,d\xi \tag{32}$$

   and the method described for the previous case is repeated.

5 Boundary integral representation for unbounded domains

Details of how the method can be extended to unbounded domains are given in reference [1]. (Note that for open domains $N = 2NE + 1$).

6 Analysis of the horizontal and vertical responses on the surface of a half-space

In order to use a theoretical model for ground vibration it is necessary to obtain parameters which are valid for the ground. The Young's moduli, Poisson's ratio and densities of the soil at a number of sites have been measured using p-wave and s-wave seismic surveys. At the site used in [4] it was found that the ground can be modelled as a half-space with the following properties : shear and compression wave speeds in the half-space, 335 and 854 m/s, density for the half-space was found to be 1759 kg/m^3. Damping has been included in the model as a loss factor proportional to frequency (0.013 at 32 Hz), [4].

6.1 Comparison of quadratic BEM model with a model using constant elements

The boundary element model using quadratic elements is compared (see Figure 1) with a similar model using constant elements. Nodes used in the quadratic element model numbered $N_Q = 109$ and nodes in the constant element model numbered $N_C = 349$. Although it is not shown here, the result for the quadratic element approximation agrees with a semi-analytical solution [4] with less degrees of freedom than the constant element approximation. In general, it is conjectured, that the error for the quadratic element model is $O(N_Q^{-5})$ compared to $O(N_Q^{-2})$ for the constant element model.

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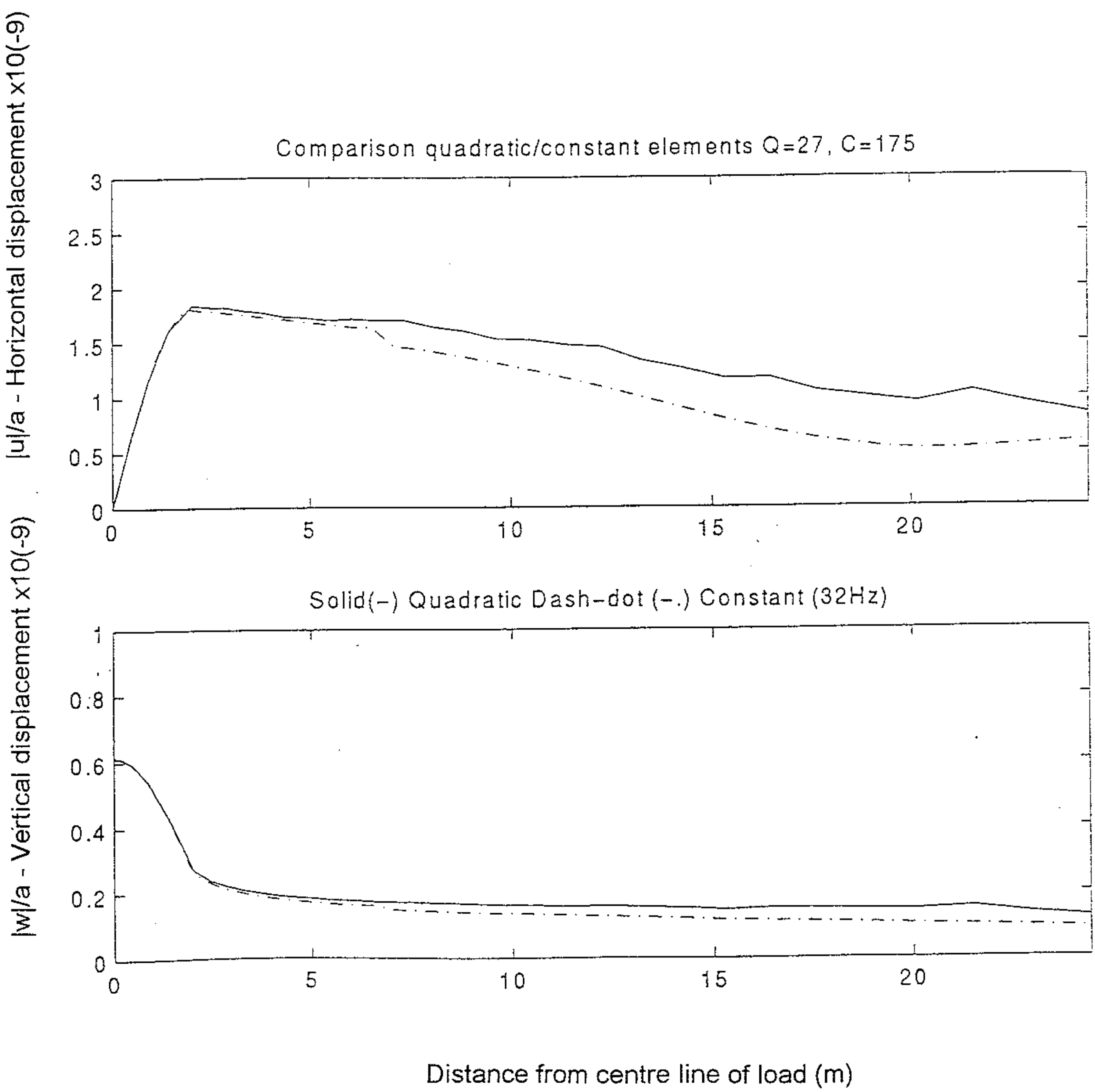

Figure 1 : Vertical and horizontal displacements due to constant load (32 Hz) of width, 2a=1.5m, over the surface of an elastic half-space. Solid line (-) represents quadratic element solution and dashed-dot line (-.) represents constant element solution.