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Discrete Time LQG Feedback Control of Vibrations

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INSTITUTE OF SOUND AND VIBRATION RESEARCH

FLUID DYNAMICS AND ACOUSTICS GROUP

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by

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Approved: Group Chairman, P. A. Nelson
Professor of Acoustics

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ABSTRACT

The aim of this work was to investigate the use of the Linear Quadratic Guassian (LQG) technique for the design of feedback controllers in sound and vibration control problems. A polynomial approach to the solution of the LQG problem is examined. The solution of 'Diophantine equations' is found to be a major part of this approach and these have been examined extensively. Computer simulations comparing the performance of the LQG controller to that of the, more common, filtered-x LMS controller were then performed with simulated plant and disturbance dynamics. In the case where a second order plant and disturbance were simulated, the performance of the LQG controller was found to compare very favourably with that of the LMS controller. A plant was also chosen which the LMS controller found difficult to invert. It was expected that the LQG method would work under these conditions and this was found to be the case. The results of these simulations are presented.

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1. Introduction.

Feedback control systems may have to be used rather than feedforward control systems when no detection of a disturbance signal is possible prior to its arrival at the ouput of the plant to be considered. Additional reference signals, which may be difficult to obtain in practice, are required for feedforward control [1]. Thus, in applications where no reference signal is available and the disturbance is broadband (as opposed to periodic), feedforward control cannot be used and feedback control methods must be employed instead.

There are various methods of designing feedback contollers which have previously been employed successfully. These methods include the frequency domain (Nyquist) method and the conventional sampled-time domain approach. Another method using Internal Model Control (IMC) has been developed recently in which the feedback problem can be reformulated into a standard feedforward problem which can be solved using the filtered-x LMS (Least Mean Square) method. For a summary of the analysis and design of some of these feedback control systems, the reader is referred to Elliott [1] and Nelson and Elliott [2].

At the ISVR, some preliminary investigations have been made into possible alternative methods of feedback control and, more specifically, into a polynomial approach to the solution of the feedback control problem which falls within the framework of the Linear, Quadratic, Guassian (LQG) design technique for discrete controllers [3, 4]. The controller can be designed using the LQG technique under the following conditions

- A cost function is minimised
- The disturbance can be represented by a filtered white noise sequence
- Closed loop stability is required

The LQG technique has been widely employed previously for several different applications (see, for example, [5]). However, this method had not previously been applied to the active control of sound and vibrations until recently when Thomas and Nelson conducted some initial investigations [3, 4, 6]. The aim of the work undertaken during this four month study and described in this report has been to further these initial investigations into the use of LQG techniques for feedback control.

This report is divided into three sections. In section 2, the polynomial solution to the LQG method is introduced. It is shown that the solution of 'Diophantine equations' is a fundamental task which must be undertaken in order to obtain the optimal filter coefficients. These equations and their solutions are discussed in detail. In section 3, the computer implementation of the LQG technique is discussed. The LQG technique is then compared to the filtered-x LMS method through the use of computer simulations. At the outset it had been suggested that, when the transfer function of the plant to be controlled had zeros close to the unit circle in the z-plane, the LMS method may not work as well as the LQG method. This is examined in more detail. Finally, section 4 summarises the results of the work carried out in the project and suggests some directions for future work.

2. Theory.

2.1. Background to LQG methods.

The basic LQG problem arises from the need to determine a closed-loop controller which minimises a cost function composed of the sum of the squared outputs of a system plus a weighted control effort. In the case of broadband signals this is equivalent to the minimisation of the sum of the system output variances plus the weighted sum of the system input variances. This optimum must be achieved whilst minimising the effects of process and measurement noise, and subject to the constraint of closed-loop stability [5].

The LQG problem has been extensively dealt with in the literature on control theory and has been applied in several practical applications but its application to the active control of sound and vibrations is relatively recent. There are thus several methods commonly used for the design of LQG control systems. The most fully developed of these use state-space representations of the plant and disturbance dynamics and are solved in the time domain using the 'separation principle' outlined by Thomas and Nelson [3]. The main disadvantage of this particular method is that delays in the plant dynamics are not represented by state-space models. However, significant delays may occur in a 'practical' plant. Another commonly used method is that of Youla and Kûcera in which the solution is found in the frequency domain and spectral factorisation and parameterisation of stabilising controllers methods are used [4]. A polynomial approach has also been applied to the solution of the LQG problem [6]. This arises through the discrete time formulation of the Wiener-Hopf equation and has the benefit of taking into account delays in plant dynamics. The resulting controllers are found to be stable when causality of time domain operators is taken into account. Nelson and Thomas have further shown that the polynomial method of solution could in principle be extended to deal with problems in which the control effect sought is not simply at the directly measurable output of the plant to be controlled but when the output to be controlled is the far field acoustic radiation as monitored by a number of error sensors [6].

Some initial investigations into the use of LQG feedback control in sound and vibration problems have been made recently by Nelson and Thomas in an attempt to find the 'most appropriate method for the design of discrete time feedback controllers for use in broadband active control' [4]. The purpose of this work has been to investigate the polynomial approach to the LQG problem more thoroughly and compare it to the filtered-x LMS method of control, especially in applications where controllers obtained through the latter design method may not work well.

2.2 LQG feedback control problems.

The general block diagram of the feedback control problem is shown in figure 1. In this figure, $W_p(q^{-1})$ represents the 'plant' (physical system) to be controlled and $W_d(q^{-1})$ the 'disturbance' i.e. the signal produced at a given point by the primary field[1]. The disturbance sequence $d(t)$ is assumed to be generated by driving W_d with a white noise sequence $w(t)$. If the delay operators q^{-1} in the polynomials representing the plant and disturbance dynamics in figure 1 are replaced by z^{-1} (the unit sample delay, thus z^{-k} implies a delay of k samples), then the plant and disturbance have the polynomial representations

$$\begin{aligned} W_p(z^{-1}) &= \frac{z^{-k}B_p(z^{-1})}{A_p(z^{-1})} \\ W_d(z^{-1}) &= \frac{B_d(z^{-1})}{A_d(z^{-1})} \end{aligned} \tag{1}$$

where B_p and A_p are assumed to be relatively prime meaning that they do not have any common roots (and similarly for the disturbance polynomials).

It is also clear from figure 1 that

$$y(t) = W_p(q^{-1})u(t) + W_d(q^{-1})w(t) \tag{2}$$

which, after substituion of the z-transform polynomial representation in (1), can be written

$$A_p(z^{-1})A_d(z^{-1})y(t) = z^{-k}B_p(z^{-1})A_d(z^{-1})u(t) + B_d(z^{-1})A_p(z^{-1})w(t) \tag{3}$$

which is more commonly expressed in the ARMAX (Auto Regressive Moving Average with eXogonous variable) form

$$A(z^{-1})y(t) = z^{-k}B(z^{-1})u(t) + C(z^{-1})w(t) \tag{4}$$

where

$$\begin{aligned} A(z^{-1}) &= A_p(z^{-1})A_d(z^{-1}) \\ B(z^{-1}) &= B_p(z^{-1})A_d(z^{-1}) \end{aligned} \tag{5 a, b}$$

and

---

[1] t denotes a discrete time index and q^{-1} is the delay operator

$$C(z^{-1}) = A_d(z^{-1})A_p(z^{-1}) \tag{5 c}$$

This ARMAX representation is commonly used in System Identification (SI) methods [7] which would be required in a practical application using feedback controllers.

Recalling that the cost function to be minimised penalises the sum of squared outputs and weighted control inputs, and restricting ourselves to the single input, single output (SISO) case, we choose for a cost function

$$J = E\left[y^2(t) + bu^2(t)\right] \tag{6}$$

where β is the weighting factor on the control input and $y(t)$ and $w(t)$ represent the system input and control output respectively which are assumed to be stationary random sequences.

The optimal compensator is then assumed to take the form

$$G_0(z^{-1}) = \frac{H(z^{-1})}{1 + H(z^{-1})W_p(z^{-1})} \tag{7}$$

as shown in figure 2. For a discussion of this choice for the form of controller, the reader is referred to Elliott [1].

The compensator can be implemented in practice by identifying a model for the plant $W_p(z^{-1})$ with SI methods and using this within the model shown in figure 2. This controller design is called 'Internal Model Control' (IMC) and has been discussed in some detail [1]. One important point which arises from this particular controller implementation is that the feedback problem can be redrawn as the feedforward problem shown in figure 3 which allows easy comparison with the filtered-x LMS method which has the same block diagram.

In the method which follows, the controller is not actually implemented as shown in figure 2. Instead, the 'controller parameterisation' shown in figure 3 is used and the filter, $H(z^{-1})$, which minimises the cost function (10) is used to determine the optimal compensator. This method ensures that the closed-loop system is always stable provided that the plant $W_p(z^{-1})$ is stable.

The filter $H(z^{-1})$ is found by the solution of a discrete-time Wiener-Hopf equation. This solution is described in detail in appendices 1-3 of Nelson and Thomas [6].

$$H_0(z^{-1}) = \frac{-Q_w A(z^{-1})A(z^{-1})}{D_c(z^{-1})D_f(z^{-1})}\left\{\frac{z^k B(z)C(z)C(z^{-1})}{D_c(z)D_f(z)A(z^{-1})}\right\}_+ \tag{8}$$

The result is expressed in terms of the polynomials $A(z^{-1})$, $B(z^{-1})$ and $C(z^{-1})$ in the ARMAX model, as defined above, and the 'spectral factors' $D_c(z^{-1})$ and $D_f(z^{-1})$ (see below). $Q_w=E[w^2(t)]$ is the variance of the white noise signal $w(t)$ and the notation $\{\ \}_+$ implies that the causal part of the polynomial function of z should be taken.

2.3. Calculating the optimal compensator.

The controller which arises from the solution of the LQG problem is given in equation (8). The derivation for this optimal controller can be broken down into 4 distinct parts which are discussed in greater detail below.

1) System Identification in order to give the ARMAX representation of the system.
2) Spectral factorisation.
3) Solution of Diohantine equations.
4) Calculation of the optimal filter.

2.3.1. SI and ARMAX representation.

In a practical application, SI would be the first step in the calculation of the controller given by equation (8). The plant and disturbance dynamics could be obtained through several well established methods which are discussed in detail for example, by Söderström and Stöica [7].

The ARMAX representation is then a straightforward matter of converting equations of the form of (1) to the form of equation (5).

2.3.2. Spectral factorisation.

Spectral factorisation usually refers to the process of splitting a rational power spectral density into a causal, stable part and an anti-causal stable part such that

$$S_{xx}(z^{-1}) = X(z^{-1})X(z) \tag{9}$$

where $X(z^{-1})$ is causal and stable and its inverse $1/X(z^{-1})$ is also causal and stable. $X(z^{-1})$ also has the property of being unique to within a costant multiplier and exists because a power spectrum is a real, even, non-negative function of frequency.

In these calculations, the spectral factors are defined by

$$\begin{aligned} D_c(z)D_c(z^{-1}) &= B(z)B(z^{-1}) + bA(z)A(z^{-1}) \\ D_f(z)D_f(z^{-1}) &= Q_w C(z)C(z^{-1}) + Q_v A(z)A(z^{-1}) \end{aligned} \tag{10}$$

where $Q_v=E[v^2(t)]$ is the variance of the (measurement) noise signal $v(t)$. The right hand side of the equations (10) are similarly real, even and non-negative polynomials in z^{-1}.

2.3.3. Diophantine equations and calculation of the optimal controller.

The solution of Diophantine equations has been the major task in this study. Their relevance in the LQG technique is to find the casual part of the term in brackets in equation (8) that is given by;

$$\left\{ \frac{z^k B(z)C(z)C(z^{-1})}{D_c(z)D_f(z)A(z^{-1})} \right\}_+ \tag{11}$$

The use of the Diophantine equation to obtain the causal part of this term (11) can be shown by substitution of the spectral factors (10) which, after rearranging becomes

$$\left\{ \frac{z^k B(z)C(z)C(z^{-1})}{D_c(z)D_f(z)A(z^{-1})} \right\}_+ = \frac{1}{Q_w} \left\{ \frac{z^k B(z)D_f(z^{-1})}{D_c(z)A(z^{-1})} - \frac{Q_w B(z)A(z)}{D_c(z)D_f(z)} \right\}_+ \tag{12}$$

The second term here is entirely non-casual and, by using the Diophantine equation,

$$D_c(z)G(z^{-1}) + z^g F(z^{-1})A(z^{-1}) = z^k B(z)D_f(z^{-1}) \tag{13}$$

we find

$$\frac{z^k B(z)D_f(z^{-1})}{D_c(z)A(z^{-1})} = \frac{G(z^{-1})}{A(z^{-1})} + \frac{z^g F(z^{-1})}{D_c(z)} \tag{14}$$

When g and the order of $F(z^{-1})$ are then chosen appropriately, the second term again becomes non-causal leaving only the term $G(z^{-1})/A(z^{-1})$. i.e.

$$\left\{\frac{z^k B(z)C(z)C(z^{-1})}{D_c(z)D_f(z)A(z^{-1})}\right\}_+ = \frac{1}{Q_w}\frac{G(z^{-1})}{A(z^{-1})} \tag{15}$$

When this result, for the causal part of (11) is substituted into the equation for the optimal compensator, it is found that

$$H_0(z^{-1}) = \frac{A(z^{-1})G(z^{-1})}{D_c(z^{-1})D_f(z^{-1})} \tag{16}$$

and the feedback controller

$$G_0(z^{-1}) = \frac{A(z^{-1})G(z^{-1})}{z^{-k}B(z^{-1})G(z^{-1}) - D_c(z^{-1})D_f(z^{-1})} \tag{17}$$

2.4. Solutions of the Diophantine equations.

Having obtained the spectral factors $D_c(z^{-1})$ and $D_f(z^{-1})$, the only step required to obtain the optimal compensator is to obtain the polynomial $G(z^{-1})$ which arises through the solution of Diophantine equations. These polynomial equations, for the LQG problem, are found to take the form of (13) where $D_c(z)$, $A(z^{-1})$, $B(z)$ and $D_f(z^{-1})$ are defined above and represent transfer functions in terms of z-transforms, k is a known integer corresponding to a pure delay of k samples in the plant, $G(z^{-1})$ and $F(z^{-1})$ are unknown (required) polynomials and g is an unknown integer.

The solution required is the 'minimal degree solution' corresponding to the smallest number of terms in the polynomials $G(z^{-1})$ and $F(z^{-1})$ for which a unique solution can be found.

We now define the orders of the polynomials $D_c(z)$, $D_f(z^{-1})$, $A(z^{-1})$, $B(z)$, $G(z^{-1})$ and $F(z^{-1})$ by n_{dc}, n_{df}, n_a, n_b, n_g and n_f respectively. For example,

$$A(z^{-1}) = (a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_{n_a} z^{-n_a}) \tag{18}$$

When written out in full, equation (13) thus becomes

$$\begin{aligned}&\left(d_{c0} + d_{c1}z^1 + d_{c2}z^2 + \ldots + d_{cn_{dc}} z^{n_{dc}}\right)\left(g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots + g_{n_g} z^{-n_g}\right)\\ &+ z^g\left(f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots + f_{n_f} z^{-n_f}\right)\left(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{n_a} z^{-n_a}\right)\\ &= z^k\left(b_0 + b_1 z^1 + b_2 z^2 + \ldots + b_{n_b} z^{n_b}\right)\left(d_{f0} + d_{f1} z^{-1} + d_{f2} z^{-2} + \ldots + d_{fn_{dc}} z^{-n_{df}}\right)\end{aligned} \tag{19}$$

which can be rewritten as

$$z^{n_{dc}}\left(d_{cn_{dc}}+\ldots+d_{c2}z^{(2-n_{dc})}+d_{c1}z^{(1-n_{dc})}+d_{c0}z^{-n_{dc}}\right)\left(g_0+g_1z^{-1}+g_2z^{-2}+\ldots+g_{n_g}z^{-n_g}\right)$$
$$+z^{g}\left(f_0+f_1z^{-1}+f_2z^{-2}+\ldots+f_{n_f}z^{-n_f}\right)\left(a_0+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n_a}z^{-n_a}\right) \tag{20}$$
$$=z^{(n_b+k)}\left(b_{n_b}+\ldots+b_2z^{(2-n_b)}+b_1z^{(1-n_b)}+b_0z^{-n_b}\right)\left(d_{f0}+d_{f1}z^{-1}+d_{f2}z^{-2}+\ldots+d_{fn_{dc}}z^{-n_{df}}\right)$$

where the product of each of the bracketed terms now gives a polynomial in z^{-1}. It is then clear that, in order to solve the equations, g should be chosen to be the maximum of either n_{dc} or n_b+k and the solution of the Diophantine equations then depends on whether $g=n_{dc}$ or $g=n_b+k$. Although the methods for obtaining the solutions in both cases are similar, they will be dealt with seperately for the sake of clarity. The final case in which $n_{dc}=n_b+k$ $(=g)$ will not be discussed as its solution is a simplified version of case (i) or (ii) below i.e. the solution is obtained through direct matrix inversion methods.

i) $n_{dc}>n_b+k$ ($g=n_{dc}$).

When g is set equal to n_{dc} equation (20) can be rewritten

$$\left(d_{cn_{dc}}+\ldots+d_{c2}z^{(2-n_{dc})}+d_{c1}z^{(1-n_{dc})}+d_{c0}z^{-n_{dc}}\right)\left(g_0+g_1z^{-1}+g_2z^{-2}+\ldots+g_{n_g}z^{-n_g}\right)$$
$$+\left(f_0+f_1z^{-1}+f_2z^{-2}+\ldots+f_{n_f}z^{-n_f}\right)\left(a_0+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n_a}z^{-n_a}\right)$$
$$=z^{(n_b+k-g)}\left(b_{n_b}+\ldots+b_2z^{(2-n_b)}+b_1z^{(1-n_b)}+b_0z^{-n_b}\right)\left(d_{f0}+d_{f1}z^{-1}+d_{f2}z^{-2}+\ldots+d_{fn_{dc}}z^{-n_{df}}\right) \tag{21}$$

On the right hand side of this equation are two polynomials with coefficients which are known. The product of these polynomials can be obtained through their convolution or, equivalently, by the matrix product

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ e_{n_b+n_{df}} \end{bmatrix} = \begin{bmatrix} b_{n_n} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & b_{n_n} & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_{n_n} \\ b_1 & b_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_0 & b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 \\ 0 & b_0 & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & b_0 \end{bmatrix} \begin{bmatrix} d_{f0} \\ d_{f1} \\ d_{f2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{fn_{df}} \end{bmatrix} \tag{22}$$

It then follows that equation (21) can be rewritten

$$
\left[\begin{array}{ccccc|ccccc}
d_{cn_{dc}} & 0 & . & . & 0 & a_0 & 0 & . & . & 0 \\
 & d_{cn_{dc}} & . & . & 0 & a_1 & a_0 & . & . & 0 \\
 & & . & . & 0 & a_2 & a_1 & . & . & a_0 \\
d_{c2} & & . & . & d_{cn_{dc}} & . & a_2 & . & . & a_1 \\
d_{c1} & d_{c2} & . & . & & & & & & a_0 \\
d_{c0} & d_{c1} & . & . & d_{c2} & a_{na} & & & & \\
0 & d_{c0} & . & . & d_{c1} & & a_{na} & & & \\
0 & 0 & . & . & d_{c0} & & & & & a_{na}
\end{array}\right]
\begin{bmatrix} g_0 \\ g_1 \\ \\ g_{n_g} \\ f_0 \\ f_1 \\ \\ f_{n_f} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ e_0 \\ e_1 \\ e_2 \\ \\ \\ e_{nb+ndf} \end{bmatrix}
\tag{23}
$$

where the number of zeros preceeding the vector on the right hand side is given by $g\text{-}(n_b\text{+}k)$ and in this example is chosen to be 2. The length of the right hand side vector is then $n_b\text{+}n_{df}\text{+}(g\text{-}n_b\text{-}k)\text{+}1$.

The orders of the polynomials $G(z^{-1})$ and $F(z^{-1})$ are chosen to be smallest possible giving a unique solution to (13) and so, n_g and n_f are, if possible, chosen such that

$$
\left[n_g + 1\right] + \left[n_f + 1\right] = \left[g - k + n_{df} + 1\right] \tag{24}
$$

It is found that when $(n_a\text{-}1)\text{>}(n_{df}\text{-}k)$, no unique solution can be found when the right hand side vector has length

$$
g - k + n_{df} + 1 \tag{25}
$$

equation (24) cannot be satisfied and trailing zeros must be added. To see why, note that the number of rows in the submatrices to the left and right of the line dividing the matrix in (23) must be less than or equal to the length of the vector on the right of the equation. i.e.

$$
\mathrm{n_f} + \mathrm{n_a} \le \mathrm{n_{df}} - \mathrm{k} + \mathrm{g} \tag{26}
$$

and

$$
\mathrm{n_g} + \mathrm{n_{dc}} \le \mathrm{n_{df}} - \mathrm{k} + \mathrm{g} \tag{27}
$$

or, in this case, as $g\text{=}n_{dc}$,

$$
n_g \le n_{df} - k \tag{28}
$$

Now, let

$$
x = n_a - 1 \tag{29}
$$

and so, from (26) above,

$$n_f + x \le n_{df} - k + g - 1 \tag{30}$$

For the minimum value of x (as $x>n_{df}-k$),

$$x = n_{df} - k + 1 \tag{31}$$

and (30) then becomes

$$n_f \le g - 2 \tag{32}$$

which also holds for any other possible value of x.

Combining the inequalities (28) and (32) then shows that

$$n_g + n_f + 2 < g + n_{df} - k + 1 \tag{33}$$

and so the equation

$$n_g + n_f + 2 = g + n_{df} - k + 1 \tag{34}$$

which is the desired solution, cannot be satisfied. Trailing zeros thus have to be added to the vector on the right hand side of (23); for each trailing zero in this right hand vector, there must be at least two, non-zero, elements in each of the corresponding rows of the matrix on the left and so, the condition for a unique minimal degree solution is given by

$$n_g + g + 1 = n_f + n_a + 1 = n_g + n_f + 2 \tag{35}$$

It is then clear that orders of the unknown polynomials are chosen to be

$$\begin{aligned} n_g &= n_a - 1 \\ n_f &= g - 1 \end{aligned} \tag{36}$$

So far, only the case in which $(n_a-1)>(n_{df}-k)$ has been examined and trailing zeros must be added. In the case where no trailing zeros are required, either $(n_g+g)=(n_{df}-k+g)$ or $(n_f+n_a)=(n_{df}-k+g)$ depending on which gives minimal order for $G(z^{-1})$ and $F(z^{-1})$. Thus three possible cases arise:

a) $g>n_a$ so $(n_g+g)=(n_{df}-k+g)$ gives the minimal order. In this case,

$$n_g = n_{df} - k \tag{37}$$

and, as the matrix in (23) must be square and (34) is satisfied, we substitute (37) into (34) to obtain

$$n_f = g - 1 \tag{38}$$

b) $g<n_a$ so $(n_f+n_a)=(n_{df}-k+g)$ gives minimal order. Now,

$$n_f = n_{df} - k + g - n_a \tag{39}$$

and, following the same argument as above (substitute (39) into (34))

$$n_g = n_a - 1 \tag{40}$$

c) $g=n_a$ and so $(n_g+g)=(n_f+n_a)=(n_{df}-k+g)$. The orders of n_g and n_f in this case are given by equations (37) and (39) respectively.

The main difficulty in solving Diophantine equations is in choosing the orders g, n_f and n_g. Once these have been chosen, examination of (23) shows that solutions follow simply through matrix inversion methods or, equivalently, by the solution of simultaneous equations.

<u>*ii)* $n_b+k>n_{dc}$ *i.e.* $g=n_b+k$</u>

In this case, (20) is rewritten

$$\begin{aligned}
&z^{(n_{dc}-n_b-k)}\left(d_{cn_{dc}}+\ldots+d_{c2}z^{(2-n_{dc})}+d_{c1}z^{(1-n_{dc})}+d_{c0}z^{-n_{dc}}\right)\left(g_0+g_1z^{-1}+g_2z^{-2}+\ldots+g_{n_g}z^{-n_g}\right)\\
&+\left(f_0+f_1z^{-1}+f_2z^{-2}+\ldots+f_{n_f}z^{-n_f}\right)\left(a_0+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n_a}z^{-n_a}\right)\\
&=\left(b_{n_b}+\ldots+b_2z^{(2-n_b)}+b_1z^{(1-n_b)}+b_0z^{-n_b}\right)\left(d_{f0}+d_{f1}z^{-1}+d_{f2}z^{-2}+\ldots+d_{fn_{dc}}z^{-n_{df}}\right)
\end{aligned} \tag{41}$$

The method of solution is similar to that outlined above. The vector on the right hand side of the equation can again be found through convolution or by the matrix product given by equation (22). However, this time, $g-n_{dc}$ zeros are added to the top of the left hand side of the matrix in (23) instead of to the vector on the right hand side. As a result, the length of the vector on the right hand side is given by

$$n_b + n_{df} + 1 \tag{42}$$

and so if possible we choose n_f and n_g such that

$$n_g + n_f + 2 = n_b + n_{df} + 1 \tag{43}$$

As above, when $(n_a\text{-}1)>(n_{df}\text{-}k)$, trailing zeros must be added to the right hand side vector. The number of rows in the submatrices to the left and right of the line dividing the matrix in (23) must be less than or equal to the length of the vector on the right hand side and so

$$\begin{aligned} n_f + n_a &\leq n_b + n_{df} \\ n_g + g &\leq n_b + n_{df} \end{aligned} \tag{44}$$

(the second equation arising as the the size of the left hand part of the matrix is n_g+n_{dc}+g-n_{dc}). Now substituting (29) into the first of equations (44),

$$n_f + x \leq n_b + n_{df} - 1 \tag{45}$$

and using the argument that the minimum value of x is given by (31),

$$n_f \leq n_b + k - 2 \qquad (n_b + k = g) \tag{46}$$

which holds for any value of x. Combining this inequality with the second of equations (44) further leads to

$$n_g + n_f + 2 < n_b + n_{df} + 1 \tag{47}$$

From which we see that the desired solution,

$$n_g + n_f + 2 = n_b + n_{df} + 1 \tag{48}$$

cannot be satisfied.

On examination, it is found that the orders n_f and n_g required for minimal degree solutions are the same as those required in section (i) above.

3. COMPUTER SIMULATIONS.

To demonstrate the use of the LQG feedback control technique, computer simulations were carried out using MATLAB for a given plant and disturbance in each case. For each given plant and disturbance, the filtered-x LMS controller and its performance were compared with those of the LQG method. The basic signal processing problem is shown in figure 3 where *w(t)* represents a white noise sequence which is fed through the filter W_d in order to generate the disturbance sequence *d(t)*. The disturbance sequence is then fed through the filter W_p representing the plant dynamics and for either type of control, the aim was to minimise the error sequence *y(t)*. To achieve this, the signal from the optimal compensator should be as close to *d(t)* as possible. This can be achieved by making the compensator *H* an exact inversion of $W_{p.}$

Two different simulations were performed as described below.

It should be noted that in real practical applications, the first step would in designing a suitable feedback controller using any IMC method would be to obtain a model of the plant (and disturbance). Established SI methods are available to do this and the MATLAB SI toolbox contains the necessary functions [7, 8]. Some work was carried out on SI methods but was not incorporated into the simulations described below.

3.1. Polynomial input.

A note should be made regarding the method of representing polynomials using MATLAB as the signal processing and control systems toolboxes use different conventions for storing polynomials [9, 10]. In the signal processing toolbox, polynomials are represented in ascending powers of z^{-1} such that

$$W_p\left(z^{-1}\right)=\frac{2z^{-3}+z^{-1}-3}{z^{-3}-z^{-2}+1} \tag{49}$$

is represented

$$Wpnum=[-3\ 1\ 0\ 2]\ , \qquad Wpden=[1\ 0\ -1\ 1] \tag{50}$$

This convention follows the derivations above. In the control systems toolbox, polynomials are represented in descending powers of *z*. To store a polynomial such as (49) it must first be turned into its counterpart with positive powers of z such that

$$W_p\left(z^{-1}\right)=\frac{2+z^2-3z^3}{1-z+z^3} \tag{51}$$

This polynomial can then be stored as in (50) above.

In chapter 2 it was shown that polynomials with positive powers of z were assumed to be anti-causal. The solution for representing anti-casual poynomials in MATLAB is to 'flip' the polynomials. Thus,

$$W_p(z)=\frac{2z^3+z^1-3}{z^3-z^2+1} \tag{52}$$

Is represented

$$Wpnumi=[2\ 0\ 1\ \ \text{-}3]\ ,\qquad Wpdeni=[1\ \ \text{-}1\ 0\ 1] \tag{53}$$

3.2. Description of computer simulations.

At the outset, it had been suggested that the LMS method should work well in controlling plants with poles close to the unit circle but, it was expected that this would not be the case in controlling plants with zeros close to the unit circle (in the z-plane). This can be explained by looking again at figure 3. The aim of the controller is to invert the plant and therefore, also its poles and zeros. A plant with zeros close to the unit circle will then give rise to a controller with poles in these same positions and a recursive, infinite impulse response. As the LMS method gives rise to recursive, finite impulse response (FIR) filters, it cannot approximate the required controller. In contrast, the LQG method results in a recursive, infinite impulse repsonse (IIR) filter and an approximation to the required controller can be achieved with relatively few filter coefficients.

MATLAB functions have been created to carry out the simulations described below. After creating the plant and disturbance transfer functions by inputting the poles and zeros, giving rise to a polynomial representation, the LMS algorithm is called with a given number of filter coefficients and convergence coefficient. The filtered-x LMS method has a simple algorithm and has been examined closely [11]. The program then solves the Diophantine equation and uses the results to obtain the LQG controller.

Recall the steps involved in the solution of the Diophantine equation;

1) *Identification of plant and disturbance dynamics.* Assume this has already been achieved (using the MATLAB SI toolbox for instance).

2) *Obtain ARMAX representation* through convolution of the relevant plant and disturbance numerators and denominators.

3) *Spectral Factorisation* through convolution, also taking care how causal and non-causal polynomials are represented (see 3.1.).

4) *Solution of Diophantine equations* using the equations derived in 2.4. followed by matrix inversion.

5) *Calculation of the optimal filter* using convolution of polynomials.

Two simulations were carried out and were designed to find out under which circumstances LQG controllers might perform better than LMS based controllers:

1) *Plant and disturbance designed to have two zeros at the origin of the z-plane and two poles close to the unit circle* (figures 4 and 5).

Representing simple second order systems. The inverse filter should have zeros close to the unit circle which can be easily 'recreated' by either the LMS (FIR) or LQG (IIR) controllers. It was expected that in this case, the LMS method should work as well as the LQG method.

2) *Plant and disturbance designed to have poles close to origin and zeros close to the unit circle* (figures 6 and 7).

The advantages of the LQG method were expected to be seen in this problem where the LMS method has difficulty inverting the plant due to its FIR nature.

3.3. Results of computer simulations.

3.3.1. Plant with poles close to the unit circle.

The plant and disturbance are shown in figure 3 and their pole-zero maps in figure 4. The transfer function of a second order system can be written [2]

$$W_*(z^{-1}) = \frac{1}{1 - 2r\cos(\omega_0 T)z^{-1} + r^2 z^{-2}} \tag{54}$$

where r is related to the damping of the system, and ω_0T to the resonant frequency. In this case, a lightly damped plant and disturbance were chosen with

$$\begin{aligned} r_p &= 0.98, \qquad \omega_0 T_p = \pi/4 \\ r_d &= 0.96, \qquad \omega_0 T_d = \pi/5 \end{aligned} \tag{55}$$

giving rise to the polynomial representations

$$\begin{aligned} W_{pnumid} &= [1\ 0\ 0], \qquad W_{pdenid} = [1\ \ -1.3859\ 0.9604] \\ W_{dnumid} &= [1\ 0\ 0], \qquad W_{ddenid} = [1\ \ -1.5533\ 0.9216] \end{aligned} \tag{56}$$

The frequency response and impulse response of the plant, LMS compensator and LQG compensator are shown in figures 8 and 9. The LMS filter used three coefficients and it is clear from figure 8 that both methods provide a close approximation to the inverse of the plant. Proof that the LMS coefficients have converged to their final values is given by figure 10. However, when four (or more) filter coefficients were used, the LMS controller did not converge (figure 11) but remained stable and still gave satisfactory results. This is an example of an underdetermined set of equations with an infinite number of solutions; there are more filter coefficients than there are degrees of freedom. Note also that the LMS method works slightly better at high frequencies than the LQG method and that the the impulse response of the LQG contoller remains slightly fluctuating showing its IIR nature. The results obtained confirm the theory at the outset of the project, that the LMS and LQG controllers should have similar performances for these plant dynamics.

3.3.2. Plant with zeros close to the unit circle.

Further simulations were performed in order to contrast simulation 3.3.1. and show an application where the LQG controller clearly outperforms the LMS-based control system with a limited number of coefficients.

The new plant and disturbance are shown in figure 6 and their pole-zero maps in figure 7. This time, the plant and disturbance were chosen to be 'inverse' second-order systems with the polynomial representations

$$W_{pnumid} = [1\ \ -1.8\ 0.98], \qquad W_{pdenid} = [1\ \ -0.01\ 0.01]$$
$$W_{dnumid} = [1\ \ -1.3\ 0.7], \qquad W_{ddenid} = [1\ \ -0.01\ 0.1] \tag{57}$$

Figures 12 and 13 show the frequency responses and impulse responses of the compensators. This time, the LMS filter has six coefficients. It is clear that, in this application, the LQG method can still invert the plant and the LMS method cannot, as expected. The IIR nature of the LQG controller is seen very clearly in graph 13. However, it may be that the LMS controller has not had time to converge sufficiently or that a larger number of coefficients are required.

Finally, graphs 14 and 15 show the results of the same simulation where the LMS controller has more coefficients. It is again evident that the LMS method cannot give rise to a good controller for these plant dynamics. Implementing LMS controllers with a large number of filter coefficients is a time-consuming, computationally intensive method and the LQG method is more attractive under these conditions.

4) Conclusions and suggestions for future work.

A polynomial approach to the design of LQG feedback controllers has been examined. This approach may have advantages over state-space solutions as it takes into account delays in the plant dynamics. The solution of Diophantine equations have been examined in some detail and an algorithm designed for their solutions has been implemented in MATLAB. These form a fundamental part of the calculation of the filter coefficients for the optimal controller.

Computer simulations were carried out in order to give an indication of possible applications for LQG feedback controllers. In particular, it was anticipated that for certain plant dynamics, the LQG method might work considerably better than other IMC methods, particularly the filtered-x LMS method. For example, when the plant transfer function has zeros close to the unit circle which cannot be inverted easily using the LMS method, due to its FIR nature. Simulations were performed comparing the LMS and LQG methods, in the first case on a second order system where the performance of the LQG controller compared favourably with that of the LMS controller. In the second case, where a plant was chosen with zeros close to the unit circle, the LQG controller was indeed found to perform considerably better than the LMS method due to its IIR nature. However, it is possible that more LMS filter coefficients were required or that the LMS controller needed longer to converge.

The programs developed are now available to perform similar simulations for other plant and disturbance dynamics.

The LQG method of designing feedback contollers is clearly an attractive one and should be investigated further. There are a number of directions in which work could be continued:

1) The computer simulations could be extended to deal with more complex systems and used to compare the performance of different controller design methods further.

2) On an experimental side, a test rig has been built which is also expected to behave as a two-degree of freedom system (i.e. as simulation 1, above). This rig could be used to see how computer simulations and experimental results compare and also if the method works well in practical applications. Some investigations into system identification methods, an essential part of any practical system, have also been investigated.

3) It has been suggested that, of the several cases of Diophantine equations discussed in chapter 2.2., only some will occur in practice [12]. The MATLAB files which are, at present, relatively slow (but can solve any type of Diophantine equation) could be simplified accordingly.

4) The theory of the LQG method has been widely extended to deal with vibrations from flexible surfaces and multi-channel feedback control [6]. the latter requiring the solution of a second set of Diophantine equations. It would be fairly straightforward to implement computer simulations of these.

In summary, LQG feedback controllers hold some promise for use in the active control of sound and vibration.

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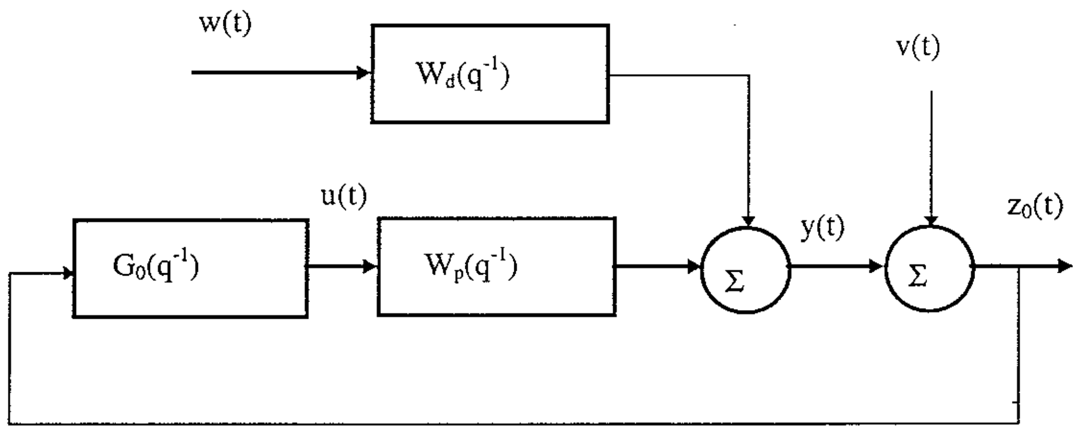

Figure 1: Block diagram of a feedback control system.

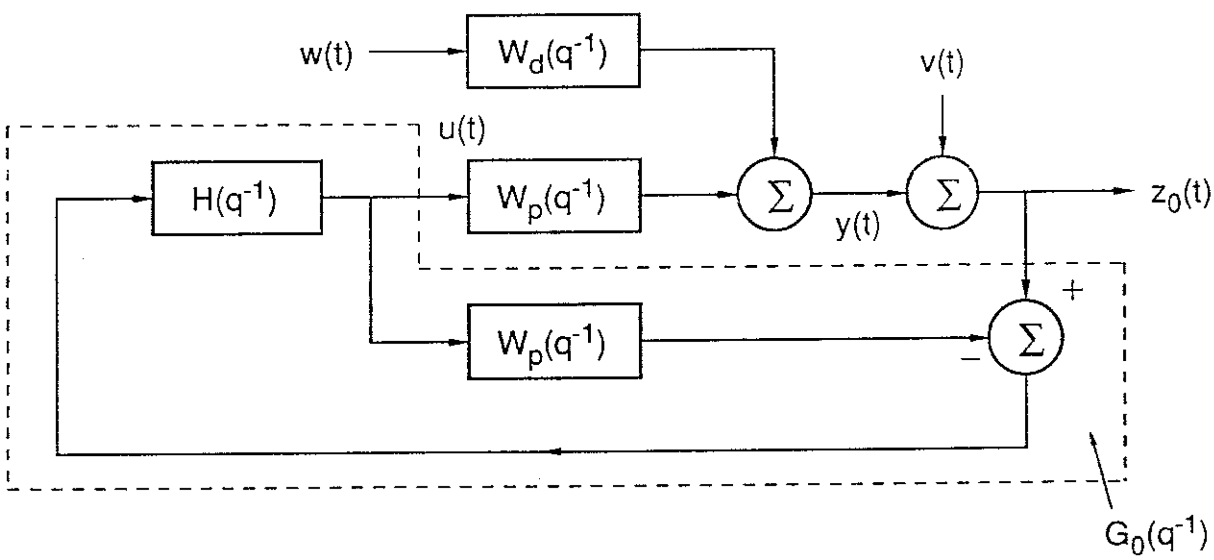

Figure 2: Form of optimal compensator used for IMC or
LQG feedback control.

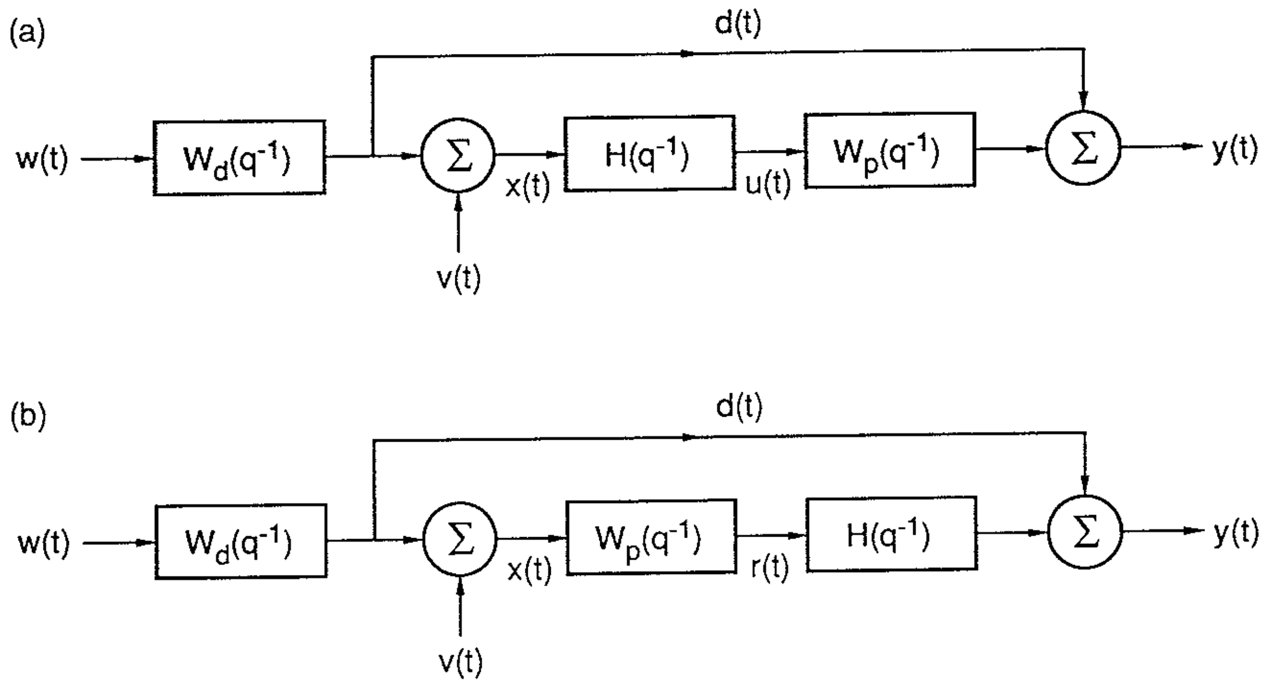

Figure 3: (a) Feedforward representation of figure 2.
(b) Redrawn to show definition of filtered reference signal r(t).

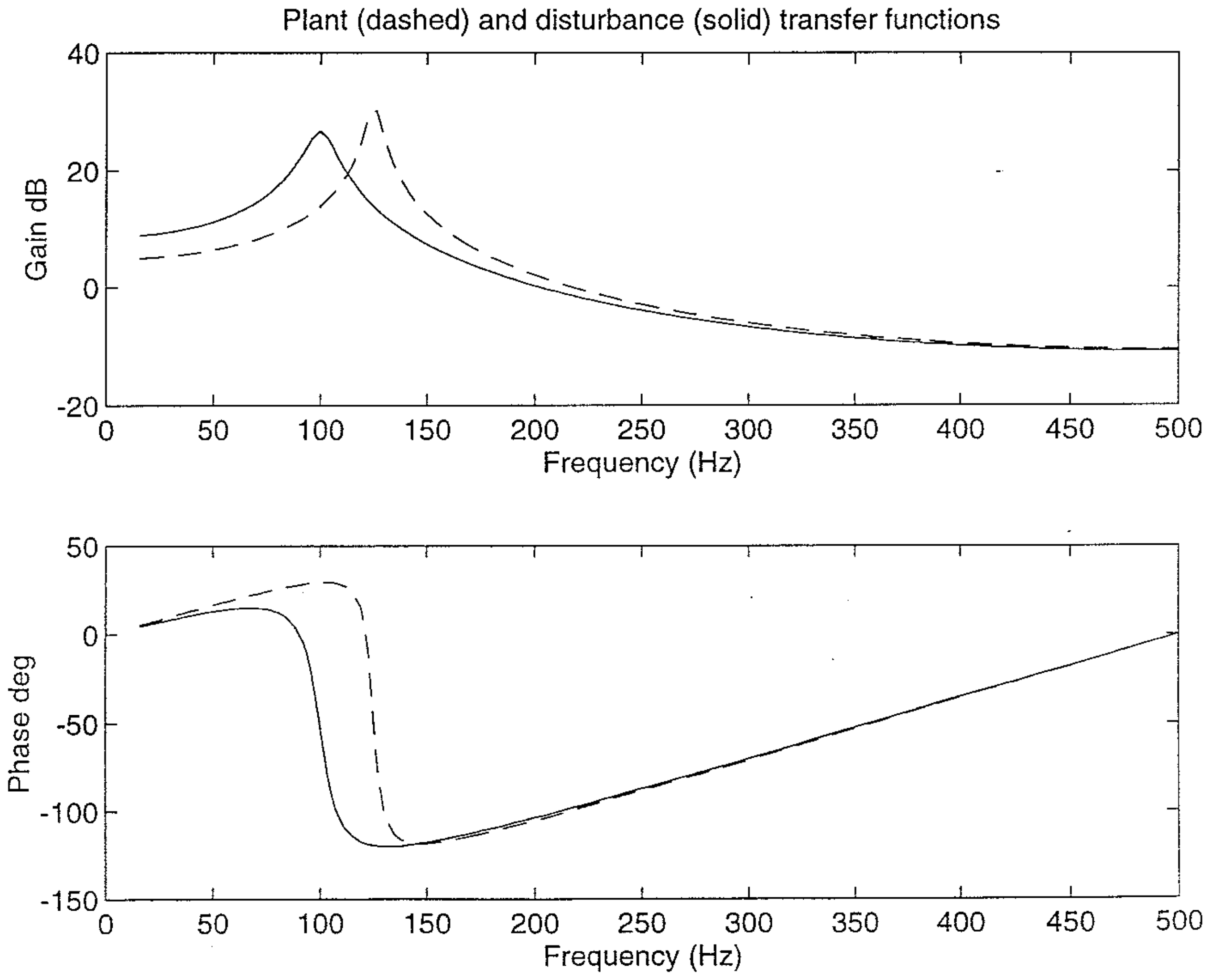

Figure 4: Transfer function of plant and disturbance dynamics with two zeros close to origin and two poles close to unit circle.

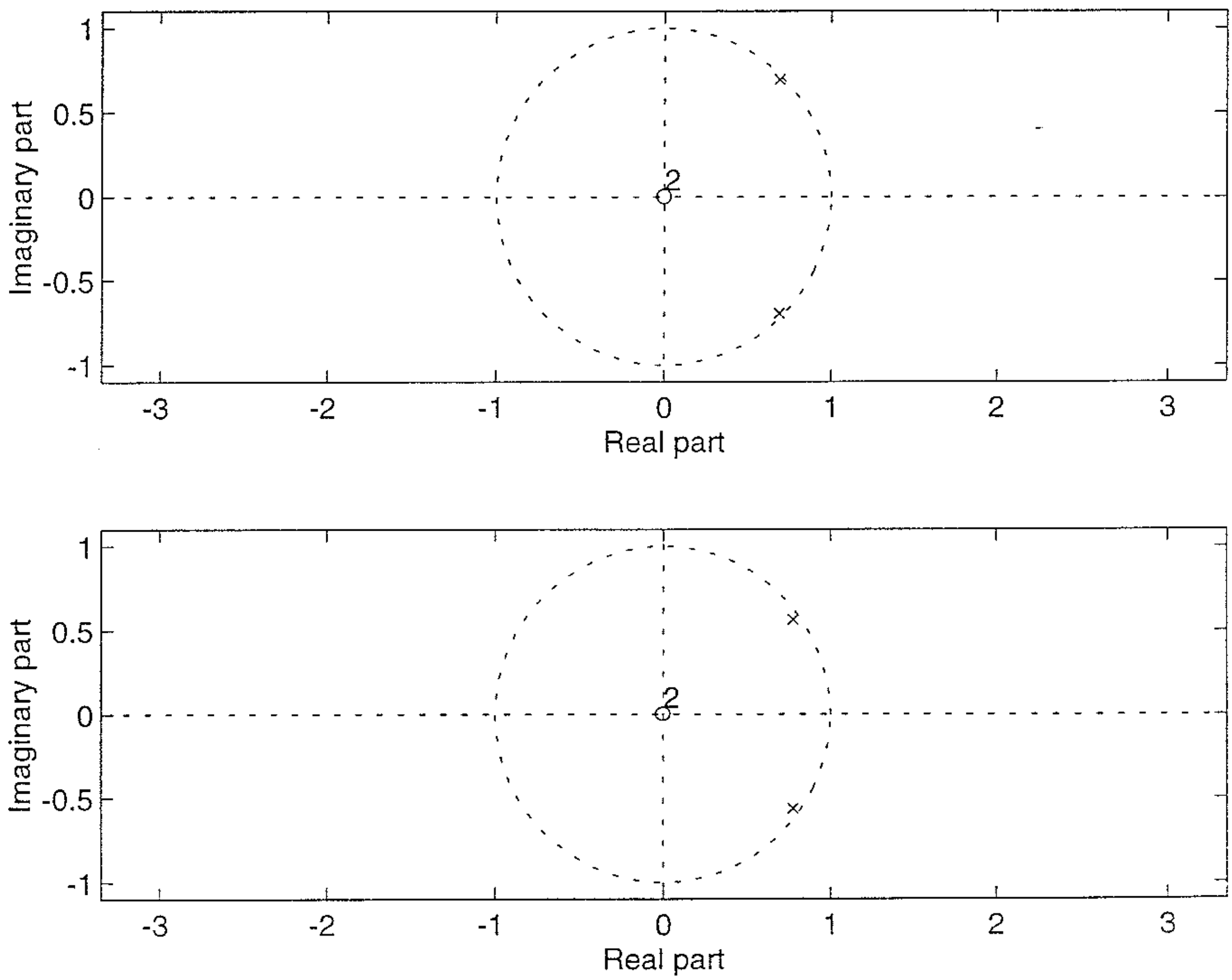

Figure 5: Pole-zero map of plant (top) and disturbance (bottom) with two zeros close to origin and two poles close to unit circle.

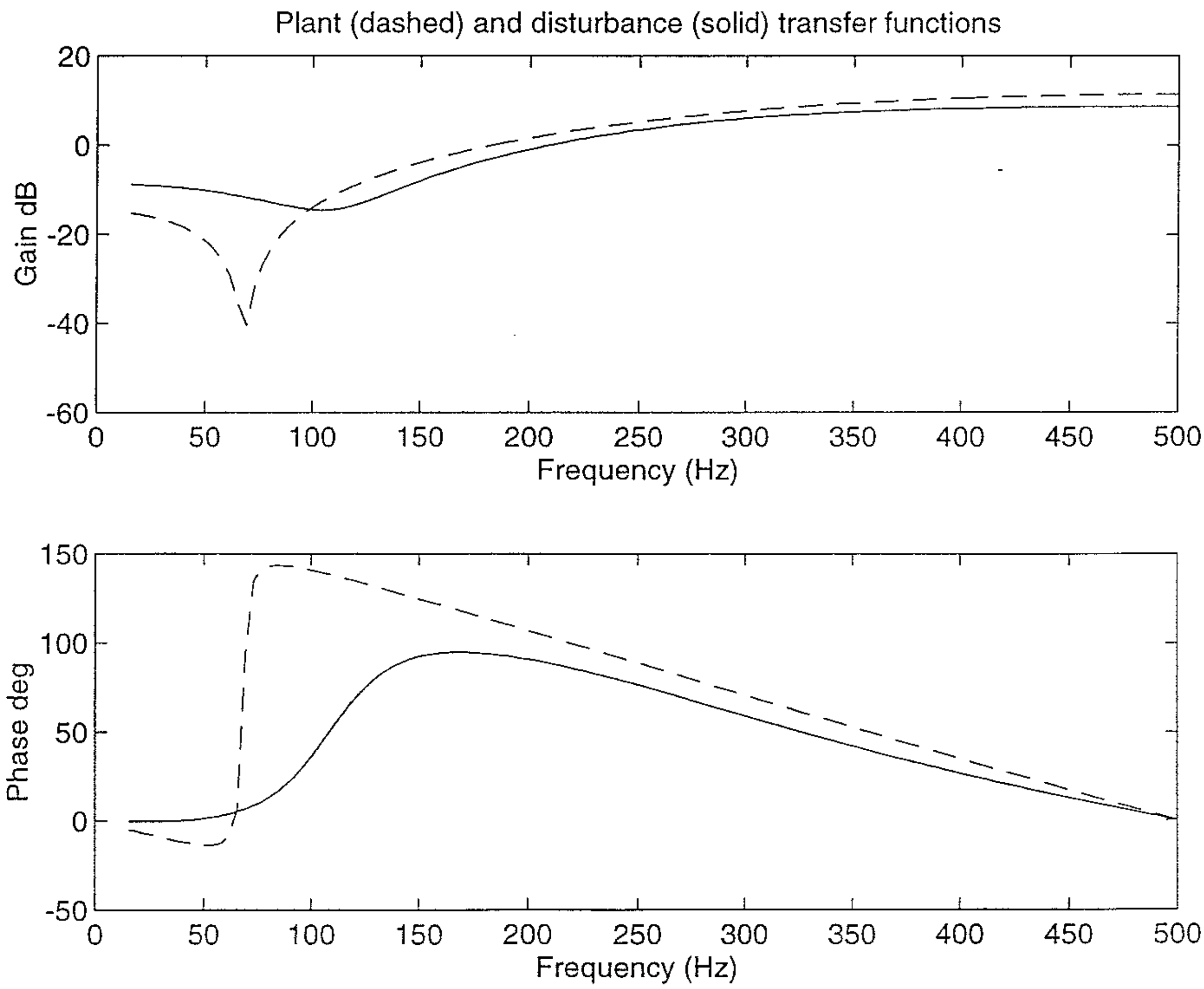

Figure 6: Transfer function of plant and disturbance dynamics with two poles close to origin and two zeros close to unit circle.

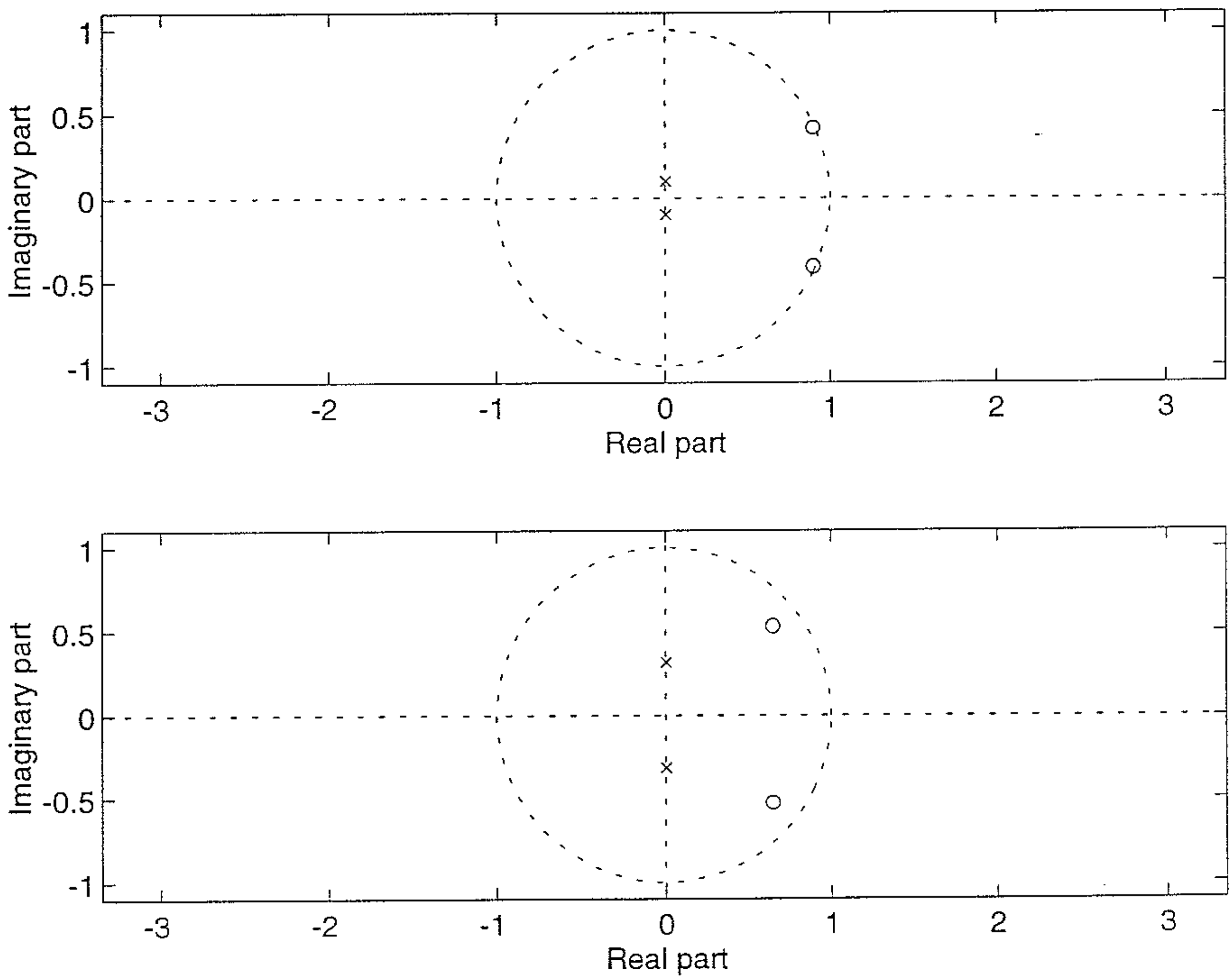

Figure 7: Pole-zero map of plant (top) and disturbance (bottom) with two poles close to origin and two zeros close to unit circle.

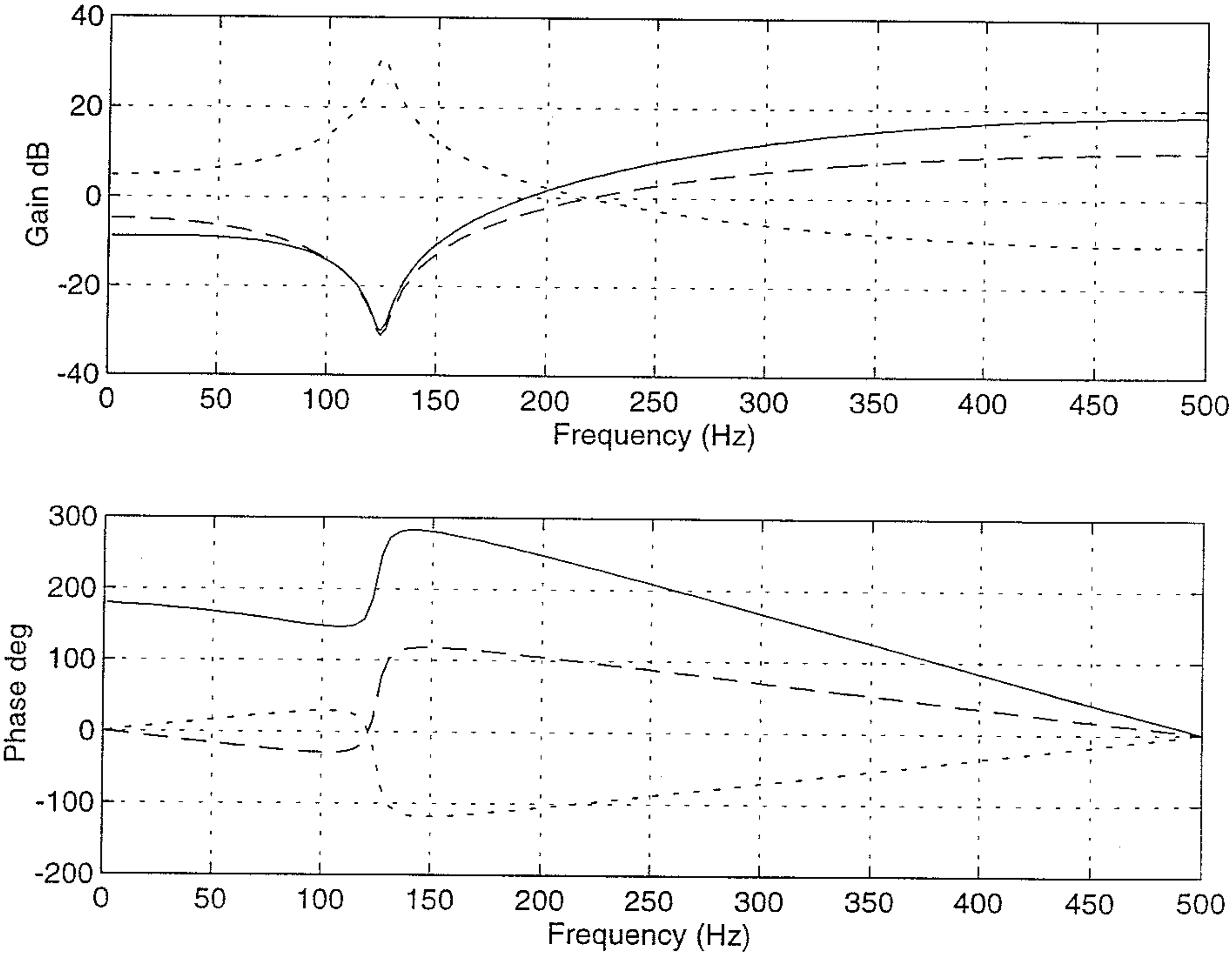

Figure 8: Frequency response of plant (dotted), LMS controller (dashed) and LQG controller (solid) for simulation 1. (LMS filter has 3 coefficients).

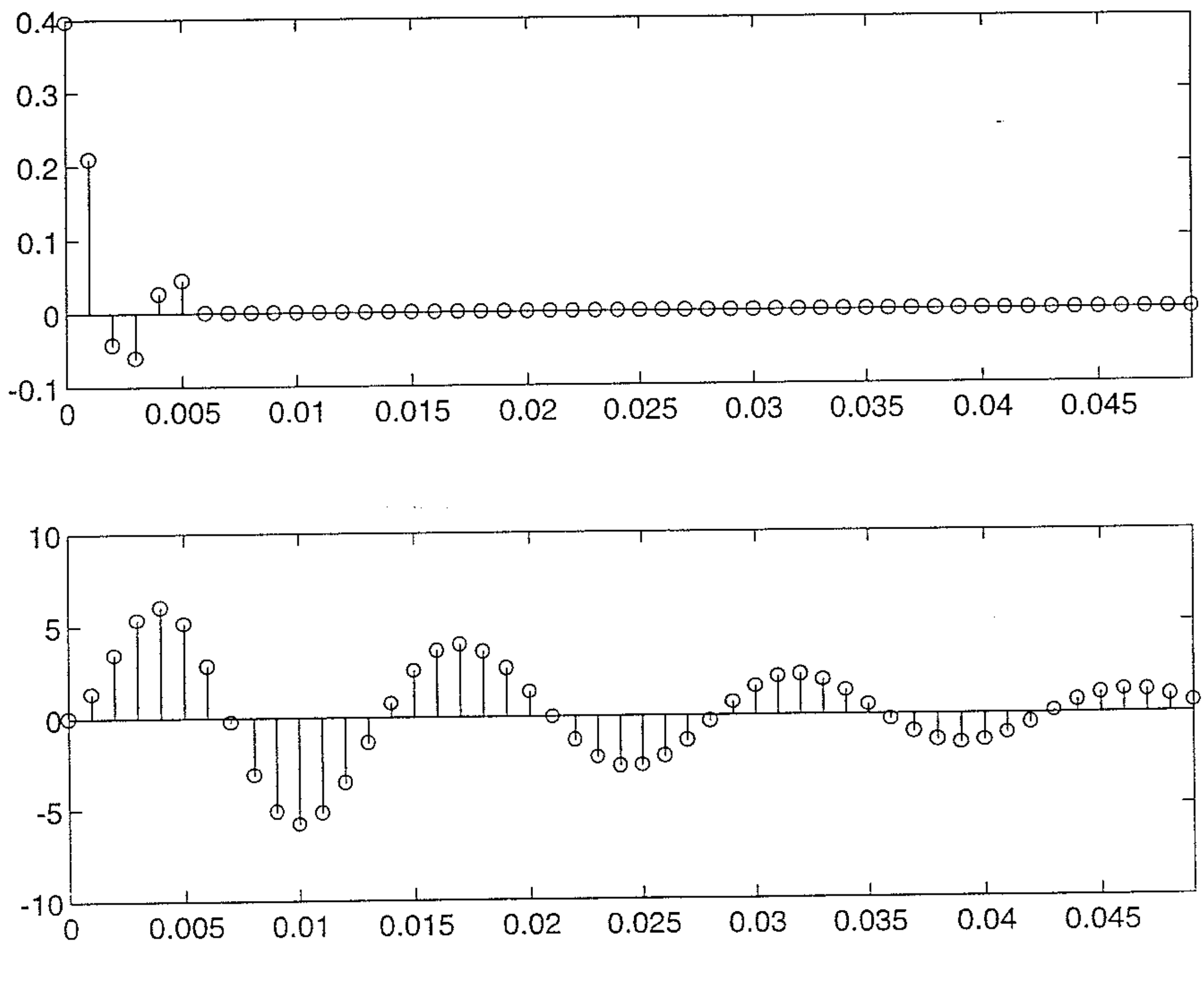

Figure 13: a) Impulse response of LMS controller (6 coefficients) for simulation 2.
b) Impulse response of LQG controller for simulation 2.

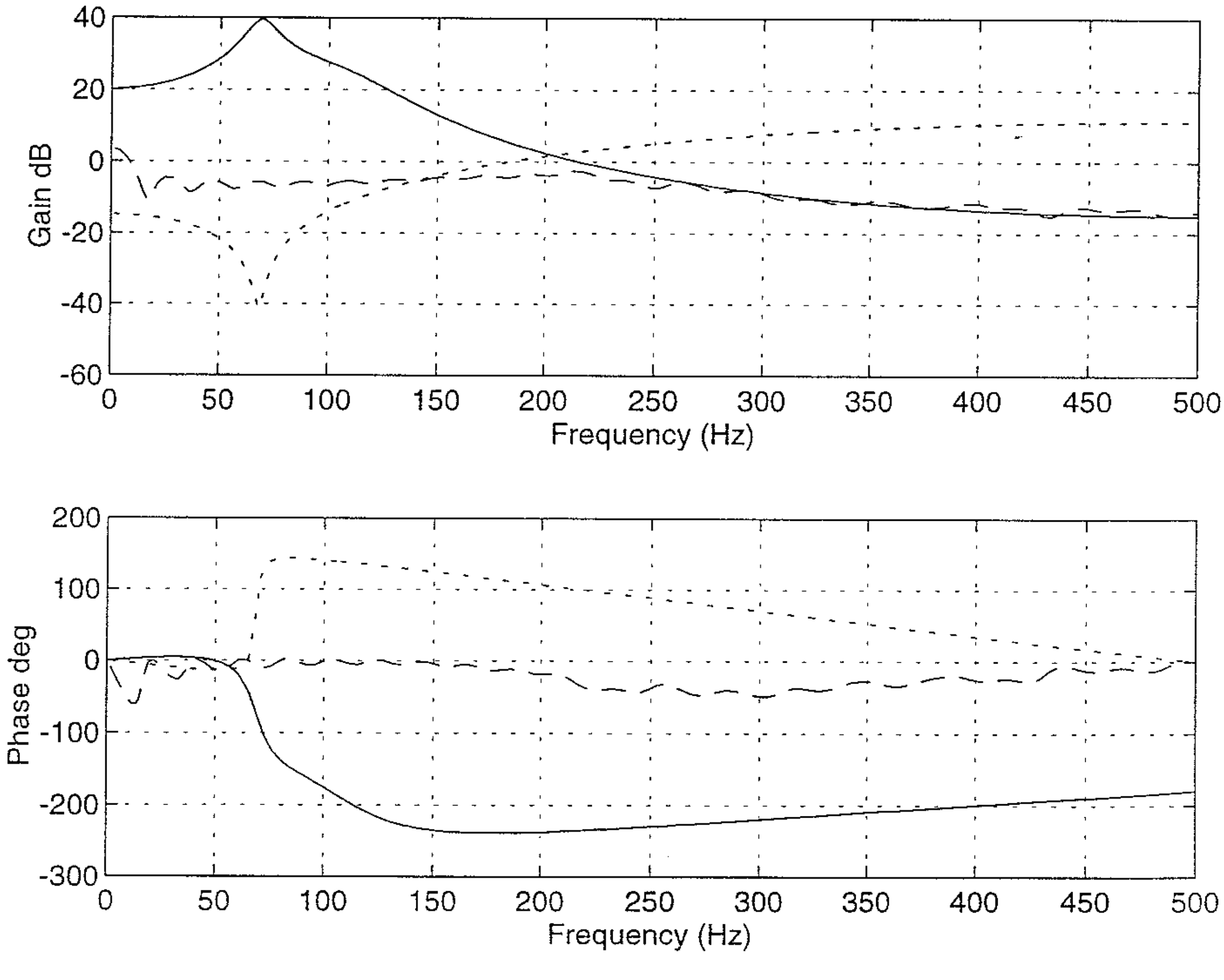

Figure 14: Frequency response of plant (dotted), LMS controller (dashed) and LQG controller (solid) for simulation 2. (LMS filter has 49 coefficients).

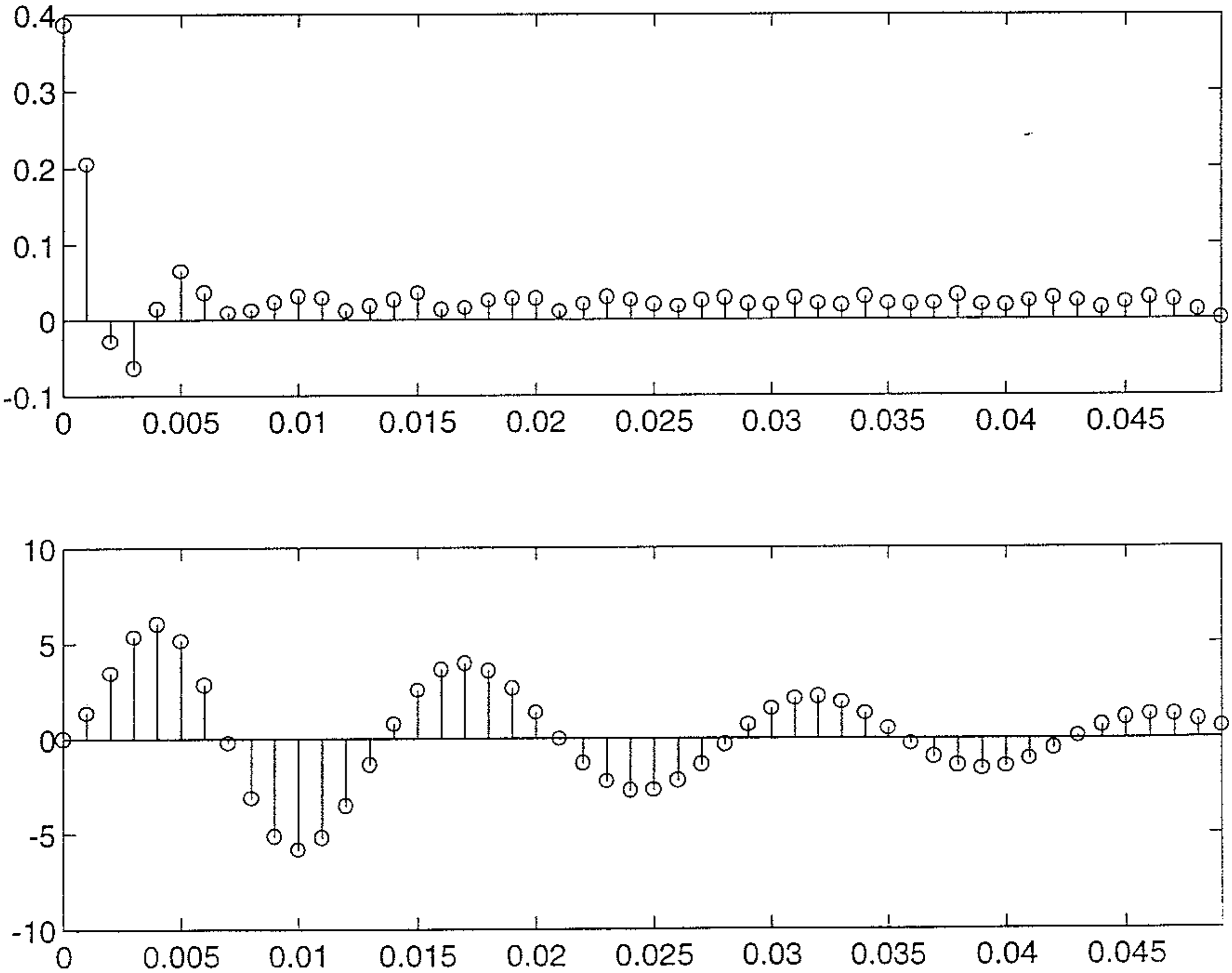

Figure 15: a) Impulse response of LMS controller (49 coefficients) for simulation 2.
b) Impulse response of LQG controller for simulation 2.