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# Analysis of Bubble Scattering Data using Higher Order Statistics

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This paper describes methods based on Higher Order Statistics for the analysis of scattering data. Specifically it demonstrates that any non-linearity in the scatterer can be partially characterised, giving information about the nature of the non-linearity involved. This is designed for use in identifying bubbles. By projecting a Gaussian random noise signal and capturing the scattered signal, estimates of the non-linear character of a bubble are made.

## 1. Introduction

Scattering from single bubbles has long been studied [1]. Such scattering is highly resonant, the location of the resonance betraying the bubble size. Further, it is non-linear, a fact which is useful for identification purposes. Hence, by acoustically exciting a medium not only can the presence of bubbles be detected but also size information is available. The form which this acoustic excitation takes varies. Systems might assume that a bubble is a linear scatterer and therefore, rely on its resonant nature, searching for resonant peaks. Such systems must be multi-frequency in nature in order to identify peaks correctly. A single frequency measurement is ambiguous since two bubbles of different sizes can have the same acoustic back scattering cross-section. These resonance-based systems are open to errors caused by the presence of other linear scatterers being identified as resonant bubbles.

To mitigate this problem one can exploit the non-linear nature of the bubble scatterer. Bubbles scatter harmonics and ultraharmonics, or when excited by a pair of tones, say at frequencies $f_1$ and $f_2$, they generate components at combination frequencies, *e.g.* at $f_1 \pm f_2$. These effects can only arise as the result of non-linearities. Such non-linearities may arise through effects other than a bubble, *e.g.* transduction, propagation and turbulence effects. The aim of this paper is to demonstrate how more information about the form of any non-linearity can be obtained, and exploited to add confidence in the bubble identification problem. To achieve this a random (Gaussian) signal is projected and the scattered data analysed using Higher Order Spectra (HOS). This paper will examine the use of the bispectrum, the lowest order HOS, in both its auto- and cross- formulations.

## 2. Higher Order Spectra

The conventional (second order) spectrum of a signal is defined as the Fourier transform, denoted $\mathcal{F}\{\ \}$, of the correlation function :

$$S_{xx}(f) = \mathcal{F}\{ R_{xx}(\tau) \} \qquad R_{xx}(\tau) = E[\, x(t)\, x(t-\tau)\, ]$$

$R_{xx}(\tau)$ is also termed the second order moment function. The natural extension to the third order moment function is provided by the function :

$$R_{xxx}(\tau_1,\tau_2) = E[\, x(t)\, x(t-\tau_1)\, x(t-\tau_2)\, ] \qquad (1)$$

whose double Fourier transform, $B_{xxx}(f_1,f_2)$ is dubbed the bispectrum. These observations can be generalised to arbitrary order, with one additional modification : for higher orders one uses the cumulant, rather than moment, functions which are invariant in the presence of additive Gaussian noise. In the first and second order cases then the cumulant and moment functions are identical[1]. Analysis in this paper is restricted to consideration of third order statistics, *i.e.* the bispectrum is exploited.

The recent surge of interest in HOS is driven by the property that all HOS are, asymptotically, unaffected by the presence of additive, stationary, Gaussian noise. Commonly, in studying HOS, it is implicitly assumed that the signals of interest are non-Gaussian in nature. In this application the source of this non-Gaussianity is directly related to the non-linearity in the bubble dynamics. Studying the traditional, second order, spectrum reveals no information about the Gaussianity, or otherwise, of a signal. In this application second order statistics yield no information about the linearity of any processes involved.

Bispectra can be estimated using techniques analogous to those used for conventional spectral estimation. Here a segment averaging (or direct) approach is employed, the bispectral counterpart of Welch's method [2]. There are many issues involved in the estimation of bispectra, only a few of which are highlighted here. The first problem arises because of the fact that a bispectral estimate is the average of product of three random variables (rather than two as in the second order case). This means that typically bispectral estimates are more variable than their spectral counterparts. A rule of thumb for segment averaging [3], is that one should average across as many (unique) segments as there are points in one's FFT, which implies that if one uses a 128 point FFT, then it necessary to capture a 16,384 ($128^2$) point time history.

A second problem arises when estimating a bispectrum because the variability of the bispectral estimate depends on the spectral composition of the signal. Specifically a bispectral estimate at the frequency pair $(f_1,f_2)$ has a variability proportional to $S_{xx}(f_1)S_{xx}(f_2)S_{xx}(f_1+f_2)$, so that whilst the theoretical bispectrum has no dependence on the second order statistics of a signal, in practice *estimates* of

[1] We continue to use the term "moment function" solely because it is familiar to a wider audience.

the bispectrum do have such a dependence. Such variability is often misleading. There are several techniques for removing these effects. Here only the skewness function, $s^2(f_1,f_2)$, is used, which is defined as :

$$s^2(f_1,f_2)=\frac{|B_{xxx}(f_1,f_2)|^2}{S_{xx}(f_1)S_{xx}(f_2)S_{xx}(f_1+f_2)} \tag{2}$$

## 3. Volterra Series

The input-output relationship for many weakly, non-linear systems can be expressed as a so-called Volterra series. This can be thought of as a generalisation of the classical, linear, convolution relationship. The Volterra series expansion can be written [4] :

$$y(t)=\int h_1(\tau)x(t-\tau)d\tau+\iint h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1\,d\tau_2+\cdots$$

The non-linear system is characterised by the set of functions $h_p(\tau_1,\ldots,\tau_p)$, called the kernel functions. A complete representation of a system via a Volterra series is normally impractical since one needs to construct an infinite set of kernel functions whose dimensions are also increasing. In practice the Volterra series is artificially truncated at some order to render the problem practical. This is only reasonable if the Volterra series is convergent, conditions for which can be found in [5].

Here characterisation of the non-linear scattering behaviour of a single bubble will be attempted by identifying the elements of a second order Volterra model, *i.e.* a Volterra series with only the first two terms retained. This will, necessarily, be an incomplete representation.

The kernel functions can be estimated by exploiting HOS. Given measurements of the input and output time series the Volterra kernel functions can be estimated (assuming the series is truncated at second order and that the input, x(t), is Gaussian), via :

$$\hat{H}_1(f)=\frac{\hat{S}_{xy}(f)}{\hat{S}_{xx}(f)} \quad \text{and} \quad \hat{H}_2(f_1,f_2)=\frac{\hat{B}_{xxz}(f_1,f_2)}{2\hat{S}_{xx}(f_1)\hat{S}_{xx}(f_2)} \tag{3}$$

where ^ is used to denote estimates of quantities and $H_1(f)$ and $H_2(f_1,f_2)$ are the Fourier transforms of $h_1(\tau)$ and $h_2(\tau_1,\tau_2)$, the latter being a two dimensional Fourier transform. The cross-bispectrum, $B_{xxz}(f_1,f_2)$, is defined as the (double) Fourier transform of $E[\,x(t)\,x(t-\tau_1)\,z(t-\tau_2)\,]$. The signal z(t) is defined as :

$$z(t) = y(t) - \hat{h}_1(t)*x(t)$$

where $*$ represents linear convolution. The point of using z(t) in (3) is that it leads to an unbiased estimate of $H(f_1,f_2)$ with a minimised variance [6]. Note also that one is able to use a conventional estimator of the linear transfer function, without it being affected by the second order term. This independence arises because of the orthogonality of second and third order products of Gaussian signals.

## 4. Method

Measurements were made in a 1.8m × 1.2m × 1.2m vibration isolated, reinforced plastic, tank, at a depth of 0.15m. The acoustic field was generated using an underwater loudspeaker (Gearing and Watson UW60). The received signal was measured using a B&K 8103 hydrophone. When needed a bubble was tethered to a wire. The loudspeaker was driven by Gaussian white noise which was band-pass filtered between 1.2kHz and 7kHz. The received signals were then acquired onto a p.c. (after low pass filtering at 9kHz) using a sampling rate of 20kHz. Measurements were taken both in the presence and absence of a bubble.

## 5. Results

Figure 1 shows the spectrum of the scattered signal, $y_s(t)$, as a solid line, the dooted line shows this spectrum in the absence of a bubble. This signal is constructed by using data obtained in the absence of a bubble to estimate the direct (linear) path from the loud speaker to the receiver, $h_d(t)$. Once a bubble has been introduced to the system, the output, y(t), is processed by forming

$$y_s(t) = y(t) - h_d(t) * x(t)$$

where x(t) is the driving signal, in this case Gaussian noise. The reason for working with this scattered signal is that the Signal to Noise Ratio is greatly improved. There is a very large peak in the spectrum at about 2.6kHz, indicative of a bubble size of about 1.1 mm. The resolution of the spectral estimate is around 20Hz equivalent to a 2.5% error in the estimate of bubble radius.

The results shown in Figure 1 allow one to be confident that a resonant scatterer was in the field of view of the system and to estimate a bubble radius assuming that the scatterer was a resonant bubble. However other scatterers could yield indistinguishable results. Weight to the argument that the scatterer in question was a bubble is found by searching for the presence of non-linear effects. This is achieved be calculation of the skewness function (2), the results of which are shown in Figure 2. There is a clear peak in this function at the bifrequency (2.6,2.6)kHz. Such a peak is indicative of coupling between the frequencies 2.6kHz and 5.2kHz in the scattered signal, characteristic of a non-linearity. Notice that to obtain statistical significance and to keep the computational load reasonable a large frequency resolution is used in the estimation of the skewness function. This means that whilst, in principle, solely the skewness function could be used to size bubbles, it is best to use a combination of both the spectrum and skewness functions.

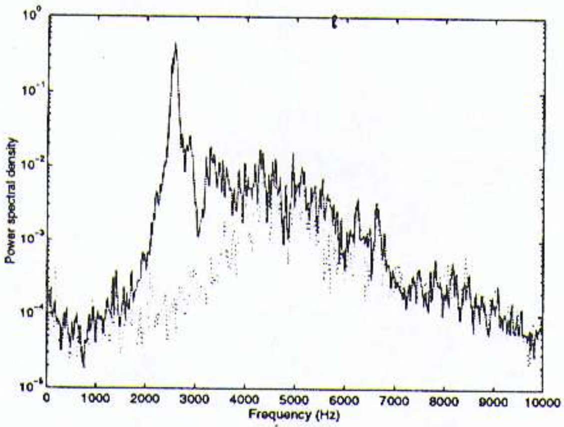


**Figure 1** *Spectrum of Scattered Signal from a Bubble*

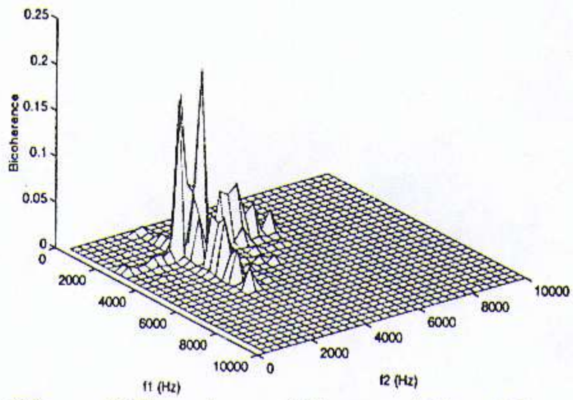


**Figure 2** *Bispectrum of Scattered Signal from Bubble*

To obtain more information about the nature of the non-linearity involved in the scattering phenomenon one can estimate the associated Volterra kernels, as discussed in Section 3. These kernels will depend on the non-linearity under consideration. Examinantion of the estimated kernels leads to the ablity to distinguish between non-linearities due to bubbles and other sources, *e.g.* transducer effects. The results of applying the second order transfer function estimator in (3) is shown in Figure 3. Note in this case, where the input signal is Gaussian, the second order transfer function resembles the skewness function; a comparison of (2) and (3) reveals that the two functions share many common terms. Once again the resolution in this estimator is poorer than that of the spectral estimate, as shown in Figure 1, so it yields a less precise estimate of bubble radius. However examination of such transfer functions can only mis-classify bubbles in extraordinary circumstances; specifically confusion can only arise if the scatterer is resonant and has a similar second order Volterra kernel.

The second order transfer function can also be inverse Fourier transformed to yield an estimate of the second order impulse response, $\hat{h}(\tau_1,\tau_2)$, which is shown in Figure 4. This response has its major contributions along the line $\tau_1=\tau_2$, usually referred to as the main diagonal. This observation allows one to conclude that a simpler model of the scattering from a bubble may be appropriate. Specifically one might consider a model of the form :

$$y(t)=\int h_1(\tau)x(t-\tau)d\tau+\int h_2(\tau)x(t-\tau)^2 d\tau+\cdots$$

where $h_2(\tau)=h_2(\tau,\tau)$.

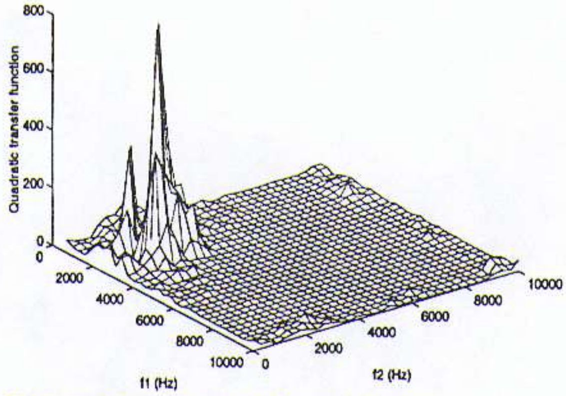


**Figure 3** *Estimate of Second Order Transfer Function*

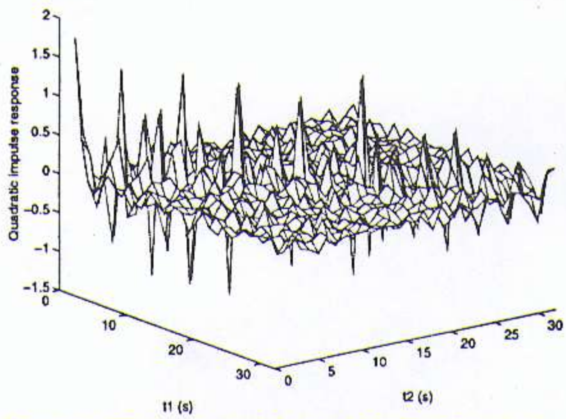


**Figure 4** *Estimate of Second Order Impulse Response*

## 6. Conclusions

This paper has demonstrated that HOS allow the analysis of scattering data in such a way as to allow a characterisation of any non-linearities. This characterisation can form the basis of a screening procedure which permits the rejection of signals scattered from non-bubble targets.

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## 7. References

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